A General Formulation of the Flexible Manipulator Dynamics

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Abstract

In this paper a new non-linear formulation of the flexible-link manipulator dynamics is presented. The proposed formulation is specially suitable for the solution of the inverse dynamics problem by successive approximations. In particular the path of the nominal rigid-link manipulator is modified by adding the elastic displacements and rotations that are obtained from the proposed mathematical model. Moreover the proposed method allows to use the current notations of the rigid-link manipulator kinematics and the calculation schemes familiar in the finite element method to model the links and the loads.

Keywords: dynamics, kinematic chain, manipulator, link, flexibility.

1 Introduction

In order to simulate the manipulator dynamics, rigid-link models were used till the late sixties. The high speed performances and the requirement of reduced weight links have shown the limitations of these models. In particular, the vibrations during the motion and the residual vibrations of the end-effector greatly affect the positioning precision and can be foreseen only by means of flexible-link models of the manipulator.

The study of the manipulator dynamics addresses the following three problems:

- forward dynamics: i.e., to calculate the end-effector path and the manipulator configurations during the motion when the external forces and torques are given;
- inverse dynamics: i.e., to calculate the forces and torques applied to the manipulator links by the actuators when the manipulator performs a given motion;
- control system design: i.e., to find a simplified manipulator mathematical model, suitable for real time computation, to be used in the control system.

The first two problems deal with the manipulator dynamics simulation and their solutions are the starting points to address the last problem.

When the link flexibility is considered, the manipulator dynamics simulation involves the solution of a non-linear differential equation system because of rigid-motion and elastic-motion coupling terms (Turcic and Midha, 1984) and just small deformations lend to general formulations. In many applications (Sunada, 1981; Sandor and Erdman, 1984;

Huang and Lee, 1988; Di Gregorio, 1992) which involve angular speeds of the links lower than their first natural frequency these terms can be neglected. In general they have to be considered at least by means of first approximation methods.

The flexible manipulator equations of motion have been written by using different techniques that can be grouped in two categories: Lagrangian techniques and Newton-Eulerian techniques.

In the first category (Shabana and Wehage, 1983a, 1983b; Agrawal and Shabana, 1985; Shabana, 1986; Khulief and Shabana, 1986; Arteaga, 1998) a mathematical system is obtained by using the Lagrange's equations and a discretized model of the flexible-link behaviour. The link discretization is accomplished by using a reduced set of the link modal shapes (assumed modes) to represent the link deformations or by using a finite element model of the link. The assumed mode discretization involves some problems concerning the number and type of modes to be used. Indeed the frequency response bandwidth of the model depends on the mode number and the simulation accuracy depends on the boundary conditions that have been used to obtain the assumed modes. The finite element discretization shows problems in defining a truncation criterion that states how many nodes have to be used for each link. In (Theodore and Ghosal,1995; Moulin and Bayo, 1997) a comparison between the two discretization has been presented.

Book (1983) extended to the flexible manipulators with only revolute pairs the Denavit-Hartenberg notation used in the rigid-link manipulator models by using assumed modes.

The Newton-Eulerian techniques (Naganathan and Sony, 1986; Huang and Lee, 1988; Shabana, 1990; Damaren and Sharf, 1995; Gawronski et al., 1995; Yoshikawa and Hosoda, 1996; Boyer and Coiffet, 1996a, 1996b; Boyer and Khalil, 1998) consider, at first, the motion equations of the links as they were not constrained and, then, assemble them to get the dynamic model of the manipulator. Each flexible-link is transformed into a system of elasticly connected rigid bodies by means of various techniques and assumed modes or finite element model are used to introduce the link deformation into the model.

In this paper a new non-linear formulation of the flexible-link manipulator dynamics is presented. The basic ideas this paper moves from are a generalization of the simplified models presented in (Di Gregorio, 1992; Gawronski et al., 1995; Yoshikawa and Hosoda, 1996). In (Di Gregorio, 1992) for general links and joints and, independently, in (Gawronski et al., 1995) for revolute joints the manipulator actual motion is considered as a sum of rigid motion plus deformed motion and a finite element model of the manipulator is generated by using the small deformation approximation and neglecting (Di Gregorio, 1992) or partly considering (Gawronski et al.,1995) the Coriolis and centrifugal terms. In (Yoshikawa and Hosoda, 1996) a lumped parameter model is presented for serial manipulator in which the flexible-link has been transformed into a system of virtual rigid-links serially joined.

The formulation presented in this paper is specially suitable for the solution of the inverse dynamics problem by successive approximations. In particular the path of the nominal rigid-link manipulator is modified by adding the elastic displacements and rotations that are obtained from the proposed mathematical model.

The proposed method allows to use the current notations of the rigid-link manipulator kinematics and the calculation schemes familiar in the finite element method to model the links and the loads.

Moreover the presented formulation can be implemented as an add-on package in computer codes that simulate the rigid-link manipulator dynamics.

2 Background

The conceptual scheme that will be used to obtain the motion equations of a generic flexible-link manipulator is based on two partitions of the links in finite elements. The first partition is used to calculate the stiffness and damping matrices of the links by means of standard finite element method (FEM) shape functions that relate the generalized displacements inside each element to the generalized displacements of the element nodes. The second partition cuts the generic j-th link into a number, n_j, of rigid bodies that is equal to the node number of the first partition through the rule "each rigid body has to contain one node". The last partition is used for associating to each node the external loads and the inertial characteristics (mass and inertia tensor) of the rigid body it falls in.

The link model (Fig.1) that comes out is a system of rigid bodies connected each other by massless springs and dampers. Thus the model is a lumped parameter model in which the stiffness and damping matrices are computed by FEM techniques and the mass matrix is computed by a non-consistent technique that takes completely into account the inertial loads and does not generate diagonal matrices.

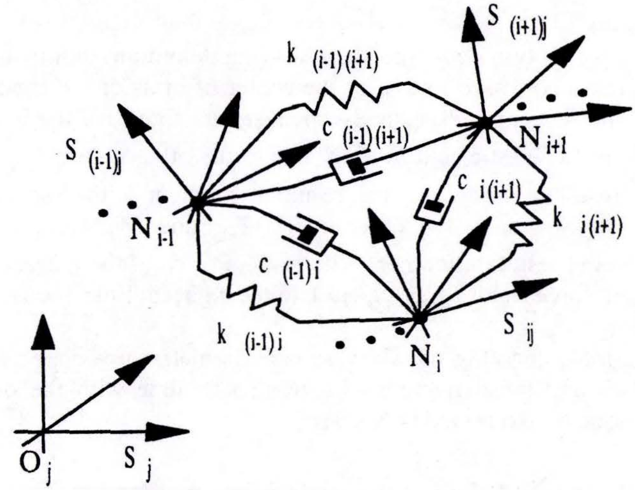

Fig.1: Model of the flexible link

For the generic j-th link (Fig.1) the following references are introduced: one Cartesian reference system, S_j, fixed to the undeformed link and n_j Cartesian reference systems, S_{ij} ($i=1,...,n_j$) with origin in the i-th node, N_i, and embedded in the rigid body that contains N_i.

The small deformation hypothesis is used during the assemblage of the node mass matrix and of the element stiffness and damping matrices to form the correspondent link matrices. This operation is carried out as in the FEM, i. e., by assuming that the nodes, N_i, and the node reference, S_{ij}, are in their undeformed location and that the S_{ij} rotations with respect to S_j, due to the link flexibility, can be considered as elementary rotations.

3 Link Motion Equations

If the j-th link is considered ideally separated from the remaining manipulator, the link motion equations are the set of the node motion equations written in an inertial reference system, S'_j, which coincides with the rigid link reference system, S_j, at the considered instant.

In order to write this equation system, at first, the N_i node motion equations will be written in an inertial reference system, S'_{ij}, that coincides with the node reference system, S_{ij}, at the considered instant. Then, by using the small deformation hypothesis, that implies the system $S'_{ij} \equiv S_{ij}$ is in the undeformed location of the node N_i, a reference system change from S'_{ij} to S'_j will be carried out in the N_i node motion equations.

The resulting mathematical model is a system of $6n_j$ scalar equations that are all expressed in S'_j.

3.1 Node Equations

In deriving the node motion equations the following definitions will be used (Fig.2): G_i is the position vector, expressed in S'_{ij}, of the center of mass of the rigid body fixed to the node N_i; F_{el_i} and M_{el_i} are, respectively, the resultant force and the resultant moment with respect to N_i of the elastic actions on N_i due to the other nodes; F_{sm_i} and M_{sm_i} are, respectively, the resultant force and the resultant moment with respect to N_i of the damping actions on N_i due to the other nodes; F_{ex_i} and M_{ex_i} are, respectively, the resultant force and the resultant moment with respect to N_i of the external actions on N_i due, for instance, to force fields (gravity, etc.), to the adjacent links (constraint reactions) and to the actuator.

With these notations, denoting the skew symmetric matrix associated to the vector **a** with the symbol $\tilde{a}$ and the derivative with respect to time with the dot, the motion equations of the node N_i, expressed in S'_{ij}, are

$$F_{ex_i} + F_{el_i} + F_{sm_i} = m_{N_i}(\ddot{N}_i - \tilde{G}_i\dot{\omega}_{N_i} - \tilde{\omega}_{N_i}\tilde{G}_i\omega_{N_i}) \quad (1.1)$$

$$M_{ex_i} + M_{el_i} + M_{sm_i} = m_{N_i}\tilde{G}_i\ddot{N}_i + I_{N_i}\dot{\omega}_{N_i} + \tilde{\omega}_{N_i}I_{N_i}\omega_{N_i} \quad (1.2)$$

where m_{N_i} is the total mass of the rigid body fixed to N_i, I_{N_i} is the inertia tensor of the same body calculated in S_{ij}, ω_{N_i} and $\dot{\omega}_{N_i}$ are, respectively, angular velocity and angular acceleration of N_i expressed in S'_{ij}.

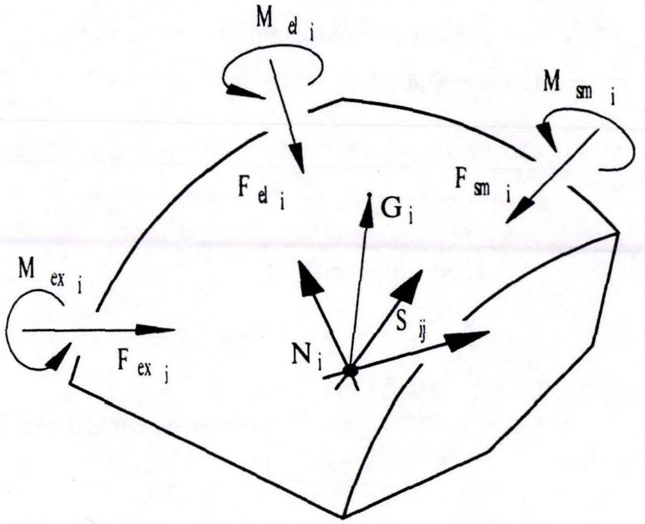

Fig. 2: Rigid body fixed to the node N_i

The actual motion of the reference system S_{ij} can be obtained by composing a nominal rigid body motion of the j-th link with the deformation motion, supposed small.

If N_{ir} indicates the position vector that locates the nominal position of N_i, due to the rigid body motion, and δ_{N_i} indicates the displacement vector (N_i-N_{ir}), then the following relations hold

$$\ddot{N}_i = \ddot{N}_{ir} + \dot{\tilde{\omega}}_j\delta_{N_i} + \tilde{\omega}_j\tilde{\omega}_j\delta_{N_i} + \ddot{\delta}_{N_i} + 2\,\tilde{\omega}_j\dot{\delta}_{N_i} \quad (2.1)$$

$$\omega_{N_i} = \omega_j + \gamma_{N_i} \quad (2.2)$$

where ω_j and $\dot{\omega}_j$ are, respectively, angular velocity and angular acceleration of the reference S_j, fixed to the undeformed j-th link, and γ_{N_i} is the contribution to the angular velocity, ω_{N_i}, of S_{ij} due to the deformation of the j-th link.

Substituting (2) into (1) yields

$$F_{ex_i} + F_{el_i} + F_{sm_i} = m_{N_i}\{\ddot{N}_{ir} - \tilde{G}_i\dot{\omega}_j - \tilde{\omega}_j\tilde{G}_i\omega_j + \ddot{\delta}_{N_i} - \tilde{G}_i\dot{\gamma}_{N_i} - \tilde{\gamma}_{N_i}\tilde{G}_i\gamma_{N_i} + 2\,\tilde{\omega}_j\dot{\delta}_{N_i} - (2\,\tilde{\omega}_j\tilde{G}_i - \tilde{G}_i\tilde{\omega}_j)\gamma_{N_i} + (\tilde{\dot{\omega}}_j + \tilde{\omega}_j\tilde{\omega}_j)\delta_{N_i}\} \tag{3.1}$$

$$M_{ex_i} + M_{el_i} + M_{sm_i} = m_{N_i}\tilde{G}_i\ddot{N}_{ir} + I_{N_i}\dot{\omega}_j + \tilde{\omega}_j I_{N_i}\omega_j + m_{N_i}\tilde{G}_i\ddot{\delta}_{N_i} + I_{N_i}\dot{\gamma}_{N_i} + \tilde{\gamma}_{N_i} I_{N_i}\gamma_{N_i} + 2\,m_{N_i}\tilde{G}_i\tilde{\omega}_j\dot{\delta}_{N_i} + [\tilde{\omega}_j I_{N_i} - (\widetilde{I_{N_i}\omega_j})]\gamma_{N_i} + m_{N_i}\tilde{G}_i(\tilde{\dot{\omega}}_j + \tilde{\omega}_j\tilde{\omega}_j)\delta_{N_i} \tag{3.2}$$

Vector equations (3) can be transformed into a compact matrix form by using the following definitions

$$M_{N_i} = \begin{bmatrix} m_{N_i} 1_{3\times3} & -m_{N_i}\tilde{G}_i \\ m_{N_i}\tilde{G}_i & I_{N_i} \end{bmatrix} \text{ (nodal mass matrix)} \tag{4.1}$$

$$C_{N_i}(\omega_j) = \begin{bmatrix} 2\,m_{N_i}\tilde{\omega}_j & -m_{N_i}(2\tilde{\omega}_j\tilde{G}_i - \tilde{G}_i\tilde{\omega}_j) \\ 2\,m_{N_i}\tilde{G}_i\tilde{\omega}_j & [\tilde{\omega}_j I_{N_i} - (\widetilde{I_{N_i}\omega_j})] \end{bmatrix} \text{ (nodal dynamic damping matrix)} \tag{4.2}$$

$$K_{N_i}(\omega_j) = \begin{bmatrix} m_{N_i}(\tilde{\dot{\omega}}_j + \tilde{\omega}_j\tilde{\omega}_j) & 0_{3\times3} \\ m_{N_i}\tilde{G}_i(\tilde{\dot{\omega}}_j + \tilde{\omega}_j\tilde{\omega}_j) & 0_{3\times3} \end{bmatrix} \text{ (nodal dynamic stiffness matrix)} \tag{4.3}$$

$$F_{CN_i}(\omega_j) = \begin{Bmatrix} -m_{N_i}\tilde{\omega}_j\tilde{G}_i\omega_j \\ \tilde{\omega}_j I_{N_i}\omega_j \end{Bmatrix}; F_{CN_i}(\gamma_{N_i}) = \begin{Bmatrix} -m_{N_i}\tilde{\gamma}_{N_i}\tilde{G}_i\gamma_{N_i} \\ \tilde{\gamma}_{N_i} I_{N_i}\gamma_{N_i} \end{Bmatrix}; \text{ (centrifugal wrenches)} \tag{4.4}$$

$$S_{ex_i} = \begin{Bmatrix} F_{ex_i} \\ M_{ex_i} \end{Bmatrix} \text{ (nodal external wrench); } S_{el_i} = \begin{Bmatrix} F_{el_i} \\ M_{el_i} \end{Bmatrix} \text{ (nodal elastic wrench);}$$
$$S_{sm_i} = \begin{Bmatrix} F_{sm_i} \\ M_{sm_i} \end{Bmatrix} \text{ (nodal damping wrench);} \tag{4.5}$$

$$v_{N_i} = \begin{Bmatrix} \delta_{N_i} \\ \alpha_{N_i} \end{Bmatrix} \quad \text{(with } \dot{\alpha}_{N_i} = \gamma_{N_i}\text{) (nodal generalized displacement 6×1 vector)} \tag{4.6}$$

$$\dot{\mu}_{N_i} = \begin{Bmatrix} \dot{N}_{ir} \\ \omega_j \end{Bmatrix} \text{ (nodal rigid twist)} \tag{4.7}$$

where $1_{3\times3}$ and $0_{3\times3}$ are the 3×3 identity and null matrices, respectively.

Introducing (4) into (3) yields the following 6-equation system in matrix form

$$\begin{aligned} S_{ex_i} &= M_{N_i}\ddot{\mu}_{N_i} + M_{N_i}\ddot{v}_{N_i} + C_{N_i}(\omega_j)\dot{v}_{N_i} + K_{N_i}(\omega_j)v_{N_i} + \\ &\quad F_{CN_i}(\omega_j) + F_{CN_i}(\gamma_{N_i}) - S_{el_i} - S_{sm_i} \end{aligned} \tag{5}$$

Equation (5) is the vector motion equation of the i-th node of the j-th link expressed in the inertial reference S'_{ij} and n_j system (5) can be written: one for each node of the j-th link. In order to obtain a system of $6n_j$ scalar equations expressed in the same reference system, the reference system change from S'_{ij} to S'_j has to be carried out in (5). Indeed S'_j is the inertial reference that coincides with S_j at the considered instant and it is unique for the link j (Fig.1). In this reference change the elastic deformations of the j-th link are considered small with respect to the link dimensions so that any S_{ij} can be treated as it was in its undeformed location and the 3×1 rotation arrays α_{N_i}, present in the 6×1 displacement vectors v_{N_i}, are treated as they were elementary rotation vectors, i.e., they behave like 3×1 vectors in the reference changes. This way of introducing the small deformation hypothesis is the one used in the finite element method (FEM).

These assumptions allow to treat the rotation matrix ${}^jR_{N_i}$ that transform the 3×1 vectors expressed in S_{ij} into the correspondent 3×1 vectors expressed in S_j as it was a constant matrix uniquely depending on the undeformed j-th link shape.

If ${}^jT_{N_i}$ is the following 6×6 transformation matrix

$${}^jT_{N_i} = \begin{bmatrix} {}^jR_{N_i} & 0_{3\times3} \\ 0_{3\times3} & {}^jR_{N_i} \end{bmatrix}$$

then system (5) expressed in S'_j becomes

$$\begin{aligned} {}^jS_{ex_i} &= {}^jM_{N_i}\,{}^j\ddot{\mu}_{N_i} + {}^jM_{N_i}\,{}^j\ddot{v}_{N_i} + {}^jC_{N_i}(\omega_j)\,{}^j\dot{v}_{N_i} + {}^jK_{N_i}(\omega_j)\,{}^jv_{N_i} + \\ &\quad {}^jF_{CN_i}(\omega_j) + {}^jF_{CN_i}(\gamma_{N_i}) - {}^jS_{el_i} - {}^jS_{sm_i} \end{aligned} \tag{6}$$

where the generic 6×6 matrix A_{N_i} and the generic 6×1 vector b_{N_i} expressed in S'_{ij} are respectively transformed into the matrix ${}^jA_{N_i}$ and the vector ${}^jb_{N_i}$ expressed in S'_j by using the following transformation rules

$$\begin{aligned} {}^jA_{N_i} &= {}^jT_{N_i} A_{N_i}\,{}^jT_{N_i}^T \\ {}^jb_{N_i} &= {}^jT_{N_i} b_{N_i} \end{aligned}$$

where the right superscript (^{T}) denotes the transpose.

3.2 Link Equations

If the following $6n_j \times 6n_j$ matrices and $6n_j \times 1$ vectors are defined

$$ {}^{j}M = \begin{bmatrix} {}^{j}M_{N_1} & \cdots & 0_{6\times6} \\ \vdots & \ddots & \vdots \\ 0_{6\times6} & \cdots & {}^{j}M_{N_{n_j}} \end{bmatrix} \tag{7.1} $$

$$ {}^{j}C(\omega_j) = \begin{bmatrix} {}^{j}C_{N_1} & \cdots & 0_{6\times6} \\ \vdots & \ddots & \vdots \\ 0_{6\times6} & \cdots & {}^{j}C_{N_{n_j}} \end{bmatrix} \tag{7.2} $$

$$ {}^{j}K(\omega_j) = \begin{bmatrix} {}^{j}K_{N_1} & \cdots & 0_{6\times6} \\ \vdots & \ddots & \vdots \\ 0_{6\times6} & \cdots & {}^{j}K_{N_{n_j}} \end{bmatrix} \tag{7.3} $$

$$ {}^{j}\mathbf{f}_{ex} = \begin{Bmatrix} {}^{j}S_{ex_1} \\ \vdots \\ {}^{j}S_{ex_{n_j}} \end{Bmatrix}; \quad {}^{j}\mathbf{f}_{el} = \begin{Bmatrix} {}^{j}S_{el_1} \\ \vdots \\ {}^{j}S_{el_{n_j}} \end{Bmatrix}; \quad {}^{j}\mathbf{f}_{sm} = \begin{Bmatrix} {}^{j}S_{sm_1} \\ \vdots \\ {}^{j}S_{sm_{n_j}} \end{Bmatrix} \tag{7.4} $$

$$ {}^{j}F_C(\omega_j) = \begin{Bmatrix} {}^{j}F_{CN_1}(\omega_j) \\ \vdots \\ {}^{j}F_{CN_{n_j}}(\omega_j) \end{Bmatrix}; \quad {}^{j}F_C(\gamma) = \begin{Bmatrix} {}^{j}F_{CN_1}(\gamma_{N_1}) \\ \vdots \\ {}^{j}F_{CN_{n_j}}(\gamma_{N_{n_j}}) \end{Bmatrix} \tag{7.5} $$

$$ {}^{j}\mathbf{v} = \begin{Bmatrix} {}^{j}\mathbf{v}_{N_1} \\ \vdots \\ {}^{j}\mathbf{v}_{N_{n_j}} \end{Bmatrix}; \quad {}^{j}\ddot{\mu} = \begin{Bmatrix} {}^{j}\ddot{\mu}_{N_1} \\ \vdots \\ {}^{j}\ddot{\mu}_{N_{n_j}} \end{Bmatrix} \tag{7.6} $$

the set of all the vector equations (6) correspondent to all the nodes of the j-th link can be grouped into the following vector equation

$$ {}^{j}\mathbf{f}_{ex} = {}^{j}M\,{}^{j}\ddot{\mu} + {}^{j}M\,{}^{j}\ddot{\mathbf{v}} + {}^{j}C(\omega_j)\,{}^{j}\dot{\mathbf{v}} + {}^{j}K(\omega_j)\,{}^{j}\mathbf{v} + {}^{j}F_C(\omega_j) + {}^{j}F_C(\gamma) - {}^{j}\mathbf{f}_{el} - {}^{j}\mathbf{f}_{sm} \tag{8} $$

System (8) is the vector motion equation of the j-th link expressed in the reference system S'_j.

By using the FEM techniques the $6n_j \times 1$ vectors ${}^j\mathbf{f}_{el}$ and ${}^j\mathbf{f}_{sm}$ can be written as follows

$$ {}^j\mathbf{f}_{el} = -{}^jK_0\, {}^j\mathbf{v} \tag{9.1} $$
$$ {}^j\mathbf{f}_{sm} = -{}^jC_0\, {}^j\dot{\mathbf{v}} \tag{9.2} $$

where ${}^jK_0$ and ${}^jC_0$ are, respectively, the stiffness matrix and the damping matrix expressed in S_j. These matrices do not depend on the link motion, i.e., they are constant.

Substituting (9) into (8) yields

$$ {}^j\mathbf{f}_{ex} - {}^jF_C(\omega_j) - {}^jM\, {}^j\ddot{\mu} = {}^jM\, {}^j\ddot{\mathbf{v}} + [{}^jC_0 + {}^jC(\omega_j)]\, {}^j\dot{\mathbf{v}} + [{}^jK_0 + {}^jK(\omega_j)]\, {}^j\mathbf{v} + {}^jF_C(\gamma) \tag{10} $$

If m+1 is the number of the manipulator links, m vector equations like (10) are written: one for each link. Each vector equation is referred to the reference system of the link it belongs to. In order to get all the link motion equations expressed in the same reference system, (10) has to be expressed in a reference system, S_0, fixed to the manipulator frame, i.e., in (10) a reference system change from S'_j to S_0 has to be carried out.

The rotation matrix ${}^0R_j$ that transforms 3×1 vectors expressed in S'_j into correspondent 3×1 vectors expressed in S_0 depends only on the coordinates of the joints that are met by moving from the frame to the j-th link through adjacent links. Moreover S'_j coincides, at the considered instant, with S_j that is fixed to the j-th undeformed link, thus the joint coordinates involved in ${}^0R_j$ are those which define the nominal rigid-link motion of the manipulator.

If the $6n_j \times 6n_j$ transformation matrix ${}^0T_j$ is defined as follows

$$ {}^0T_j = \begin{bmatrix} {}^0R_j & \cdots & 0_{3\times3} \\ \vdots & \ddots & \vdots \\ 0_{3\times3} & \cdots & {}^0R_j \end{bmatrix} \tag{11} $$

the rules that transform, respectively, the generic $6n_j \times 6n_j$ matrix ${}^jA$ and the generic $6n_j \times 1$ vector ${}^jb$ expressed in S_j into the correspondent matrix ${}^0A_j$ and vector ${}^0b_j$ expressed in S_0 are

$$ {}^0A_j = {}^0T_j\, {}^jA\, {}^0T_j^T \tag{12.1} $$
$$ {}^0b_j = {}^0T_j\, {}^jb \tag{12.2} $$

By applying (12) to the matrices and vectors that appear in (10) the following vector motion equation is obtained

$$ {}^0\mathbf{f}_{ex|j} - {}^0F_{C|j}(\omega_j) - {}^0M_j\, {}^0\ddot{\mu}_j = {}^0M_j\, {}^0\ddot{\mathbf{v}}_j + [{}^0C_{0|j} + {}^0C_j(\omega_j)]\, {}^0\dot{\mathbf{v}}_j + [{}^0K_{0|j} + {}^0K_j(\omega_j)]\, {}^0\mathbf{v}_j + {}^0F_{C|j}(\gamma) \tag{13} $$

4 Dynamic Model of the Flexible-Link Manipulator

For every link, a $6n_j$-dimensioned vector equation, like (13), is written. By introducing the following matrix and vector definitions

$$M_f = \begin{bmatrix} {}^0M_1 & \cdots & 0_{6n_1\times 6n_1} \\ \vdots & \ddots & \vdots \\ 0_{6n_m\times 6n_m} & \cdots & {}^0M_m \end{bmatrix} \tag{14.1}$$

$$C_f(\omega) = \begin{bmatrix} {}^0C_{0|1}+{}^0C_1(\omega_1) & \cdots & 0_{6n_1\times 6n_1} \\ \vdots & \ddots & \vdots \\ 0_{6n_m\times 6n_m} & \cdots & {}^0C_{0|m}+{}^0C_m(\omega_m) \end{bmatrix} \tag{14.2}$$

$$K_f(\omega) = \begin{bmatrix} {}^0K_{0|1}+{}^0K_1(\omega_1) & \cdots & 0_{6n_1\times 6n_1} \\ \vdots & \ddots & \vdots \\ 0_{6n_m\times 6n_m} & \cdots & {}^0K_{0|m}+{}^0K_m(\omega_m) \end{bmatrix} \tag{14.3}$$

$$\mathbf{f}_{ex|f} = \begin{Bmatrix} {}^0\mathbf{f}_{ex|1} \\ \vdots \\ {}^0\mathbf{f}_{ex|m} \end{Bmatrix}; \tag{14.4}$$

$$F_{C|f}(\omega) = \begin{Bmatrix} F_{C|1}(\omega_1) \\ \vdots \\ F_{C|m}(\omega_m) \end{Bmatrix}; \quad F_{C|f}(\gamma) = \begin{Bmatrix} F_{C|1}(\gamma_1) \\ \vdots \\ F_{C|m}(\gamma_m) \end{Bmatrix} \tag{14.5}$$

$$\mathbf{v}_f = \begin{Bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{Bmatrix}; \quad \ddot{\boldsymbol{\mu}}_f = \begin{Bmatrix} \ddot{\boldsymbol{\mu}}_1 \\ \vdots \\ \ddot{\boldsymbol{\mu}}_m \end{Bmatrix} \tag{14.6}$$

the set of all the vector link equations of motion can be collected in a compact matrix form as follows

$$\mathbf{f}_{ex|f} - F_{C|f}(\omega) - M_f\ddot{\boldsymbol{\mu}}_f = M_f\ddot{\mathbf{v}}_f + C_f(\omega)\dot{\mathbf{v}}_f + K_f(\omega)\mathbf{v}_f + F_{C|f}(\gamma) \tag{15}$$

Relation (15) is a system composed by a number n_f of scalar equations equal to

$$n_f = 6\sum_{j=1}^{m} n_j \tag{16}$$

If g is the number of joint coordinates that define the nominal rigid-link manipulator configuration, the vector $\boldsymbol{\mu}_f$ depends on g parameters.

By considering the joints as rigid joints the generalized displacements of two adjacent link nodes that coincide because of the joint between the two links are the same. Therefore the independent components of the vector $\mathbf{v}_f$ are only $(n_f - 6k)$ where k is the number of the node couples that coincides in the joints.

Moreover in the external force vector $\mathbf{f}_{ex|f}$ there are $(6k - g)$ passive constraint reaction components and g active constraint reaction components due to the actuators.

Thus system (15) is composed by n_f ordinary differential equations in $(n_f - 6k + g)$ unknown state variables (variables whose derivatives are present) plus 6k unknown constraint reaction components ($(6k - g)$ internal variables plus g control variables). The last ones are present in linear form and can be eliminated by summing two at a time the vector equations of the node couples that coincides in the joints.

These operations can be carried out at once by multiplying (15) by a suitable $(n_f-6k)\times n_f$ Boolean matrix B (Atzori, 1985). Moreover the n_f components of the vectors $\mathbf{v}_f$ and $\boldsymbol{\mu}_f$ are reduced to the to the only (n_f-6k) independent components collected in the vectors $\mathbf{v}_c$ and $\boldsymbol{\mu}_c$, respectively, by means of the following rules (Atzori, 1985)

$$\mathbf{v}_f = B^T \mathbf{v}_c \tag{17.1}$$

$$\boldsymbol{\mu}_f = B^T \boldsymbol{\mu}_c \tag{17.2}$$

In this way the flexible-link manipulator motion equations in matrix form become

$$\mathbf{f}_{ex|c} - F_{C|c}(\omega) - M_c\ddot{\boldsymbol{\mu}}_c = M_c\ddot{\mathbf{v}}_c + C_c(\omega)\dot{\mathbf{v}}_c + K_c(\omega)\mathbf{v}_c + F_{C|c}(\gamma) \tag{18}$$

where

$$M_c = BM_fB^T;\ C_c(\omega) = BC_f(\omega)B^T;\ K_c(\omega) = BK_f(\omega)B^T;$$
$$\mathbf{f}_{ex|c} = B\mathbf{f}_{ex|f};\ F_{C|c} = BF_{C|f};$$

System (18) is composed by $(n_f - 6k)$ scalar ordinary differential equations in the (n_f-6k+g) state variables ($(n_f - 6k)$ generalized displacements plus g joint coordinates) in which the passive constraint reactions and the actuator actions are not present.

(18) has been obtained without any hypothesis about the architecture of the kinematic chain the manipulator is constituted by, thus it applies to both open and closed kinematic chain. Moreover the presence of flexible joints can be introduced by adding as many virtual flexible-links as the number of the flexible joints and by treating the resulting kinematic chain as a rigid-joint manipulator.

In conclusion, the only restriction to the validity of dynamic model (18) is the hypothesis of small elastic deformations.

5 Solution Schemes

The manipulator dynamic model that has been presented is highly non-linear both in the joint coordinates and in the elastic displacements of the links. Nevertheless, if the nominal rigid-link motion of the manipulator is known from other computations, system (18) is non-linear only in the term $F_{Ck}(\gamma)$ that is a second order term in the deformation angular velocities and because of the small deformation assumption it could be also neglected in a first approximation computation.

System (18) can be used in the solution of both the inverse and forward dynamics problems.

5.1 Inverse Dynamics Problem Solution

The inverse dynamics problem solution implies the computation of the forces and torques to be applied by the actuators once the end-effector path is known.

Starting from the end-effector path, the nominal rigid-link motion of the manipulator can be calculated by solving the inverse kinematics problem via standard packages for rigid-link manipulators. Once the joint coordinates that define the nominal rigid-link motion are known at any time instant, the terms of system (18) that depend only on the nominal rigid-link motion can be calculated and system (18) becomes an ordinary differential equation system in the unknown vector $\mathbf{v}_c$. Thus it can be solved via standard numerical integration algorithms.

So far $\boldsymbol{\mu}_c$ and $\mathbf{v}_c$ have been computed, hence $\boldsymbol{\mu}_f$ and $\mathbf{v}_f$ can be calculated by relations (17). In the end the 6k active and passive constraint reaction components can be computed by introducing $\boldsymbol{\mu}_f$ and $\mathbf{v}_f$ into (15) and solving a linear system. Therefore the original problem has been solved.

5.2 Forward Dynamics Problem Solution

The forward dynamics problem solution implies the computation of the joint coordinates and of the elastic deformations once the g components of the forces and/or torques applied by the actuators are known.

By introducing into (15) the given values of the g force and/or torque components applied by the actuators, system (15) become a system of n_f non-linear differential equations in n_f unknown functions ((n_f–6k+g) position variables plus (6k–g) passive constraint reaction components) and it can be integrated by standard numeric algorithms provided that it has been put in some canonical forms. This method requires further algebraic manipulations of (15) and it will be treated in another paper.

Instead the most direct solution technique is the following: by using the given data of the actuation forces and/or torques the rigid-link forward dynamics problem is solved so that the g joint coordinates of the nominal rigid-link motion are computed; next, the computed data of the nominal rigid-link motion are substituted into (18) and system (18) is numerically integrated to obtain $\mathbf{v}_c$. Even if this technique is faster than the first one, it solves the flexible-link forward dynamics problem in approximate way. The approximation size can be evaluated by computing again the actuator actions through system (15) and comparing the results with the given data.

6 Considerations

In order that the presented dynamic model is actually built, the node numbers to be used for each link has to be considered: the higher the node numbers the higher the frequency bandwidth the model applies in and, on the other hand, the computation burden.

By considering the highest harmonic component to be modeled and the natural frequencies of the links the most suitable node number for each link can be valued.

This problem is present in all the methods that discretize the links through finite elements and it is the dual problem of choosing the mode type and number in the assumed mode techniques.

Another important issues to be stressed is the presence in (13), the link model, of a stiffness matrix that depends on the angular velocity and acceleration of the link. This implies that the natural frequencies of the links are shifted when they rotate and it is in accordance with what is reported in the literature.

7 Conclusion

In this paper a general formulation of the flexible-link manipulator dynamics has been presented. As far as the author is aware the reported formulation is new.

Further, solution methods have been proposed for both the inverse and forward dynamics problems. The solution schemes lend to be easily implemented through add-on modules in standard programs that solve the rigid-link manipulator dynamics. Moreover they use finite element data available from commercial FEM programs.

The proposed formulation needs the only condition that the elastic deformations of the links have to be small with respect to the link dimensions. Moreover it models both closed loop and open chain manipulators. In the end it can be extended to flexible-joint manipulators, even if it has been presented in the case of rigid-joints.

Acknowledgments

This paper has been supported by Italian MURST funds.

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