Ordering Role of Additive Noise in Extended Media

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Abstract
We consider two examples of extended media under the influence of additive noise: a coupled stochastic oscillators and spatial nonlinear lattice which contains overdamped oscillators. In both systems role of the additive noise is crucial. In the first system additive noise increases signal to noise ratio, resulting in stochastic resonance. In the second system additive noise causes phase transition manifesting itself in the formation of ordered spatial patterns. Surprisingly, we find parallels between two phenomena considered.

Keywords: Noise-induced phase transitions, stochastic resonance, pattern formation.

Introduction

As usual noise is understood as a reason of disorder in physical systems. However investigations of the last two decades have shown that noise may cause ordering in nonlinear systems. As an examples let us note such effects as noise-induced transitions (Horsthemke and Lefever, 1984), noise induced directed transport (Hänggi and Bartussek, 1996) or coherence resonance (Pikovski and Kurths, 1997). Also stochastic resonance (Gammaitoni et al, 1998) and noise induced nonequilibrium phase transitions (Van den Broeck et al, 1994a,1994b) are examples where noise sustains order in nonlinear systems rather that to destroy it.

Most of investigations on the effect of noise in nonlinear nonequilibrium systems concerns with theoretical models. There are only few papers on experimental verifications reporting noise induced phenomena (Guderian et al, 1996; S. Kádár et al, 1998;Löcher et al, 1998).

Nevertheless, the importance of noise on small length scales is without doubt. At scales comparable with cell lengths or smaller noise (heat) is one of the most important sources of energy which is best manifested by presence of Brownian motion.

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At levels of Brownian systems there is a lack of directed work and the mesoscopic objects are obliged to use noise to sustain order inside the system.

This circumstance motivates investigations of extended noisy media. Surprisingly, very simple model systems with a wide range of applicability enables us to show the ordering role also of additive noise in extended media. The present paper reviews our study about the role of the additive noise in nonlinear extended systems with noise. We investigate systems where two control parameters are of the most importance: intensity of the additive noise and strength of the coupling. To our knowledge two examples meet these requirements: coupled stochastic oscillators under the action of an external force and nonlinear lattices which contain coupled overdamped oscillators under the influence of additive and multiplicative noise.

We consider and compare two examples: in the Sec.1. we address the problem of stochastic resonance in the system of coupled stochastic resonators and in the Sec.2. we study formation of inhomogeneous spatial patterns induced by the additive noise in nonlinear lattice. As stated above, two control parameter are crucial for the systems under consideration: coupling and intensity of the additive noise. Variation of these two parameters results in the most ordered phase. For the phenomenon of the stochastic resonance it corresponds to the maximum in the spectral power amplification (SPA) and the signal-to-noise ratio (SNR), for the formation of the spatial patterns to the maximum in the structure function.

Another interesting finding are similarities between two phenomena considered: in both system the ordering is of nonmonotonous character with respect to the intensity of the additive noise. To our surprise thorough analysis has shown that there are in-depth reasons for the parallels noted. We discuss these similarities and summarize results obtained in the last Sec.

1 Stochastic Resonance in Ensemble of Stochastic Oscillators

1.1 Stochastic Resonance in Single Systems

Stochastic resonance is a wide spread phenomenon ranging from ice ages to neuronal systems (Benzi et al, 1982, Nicolis, 1982; Gammaitoni et al, 1998). The common secret of all this systems is a internal time scale which can be tuned by varying internal or external noise. In most cases it is the time to reach a certain threshold in the nonlinear system. This stochastically occurring event may be identified with the output of the system.

The second ingredient represents a small signal inputting into the systems and changing temporarily the system state or, for example, the distance to the threshold. It consists of some ordered time sequence, periodic or random with some finite correlation time. The signal may be also contaminated with the noise originating the escapes in the threshold dynamics.

The effect of the signal may be expressed in terms of a modulation of a potential barrier which should be escaped to reach the threshold. Thereby, generally the inputting signal value is sufficiently small. It never decreases the barrier so far that the system would reach the threshold without the action of the noise.

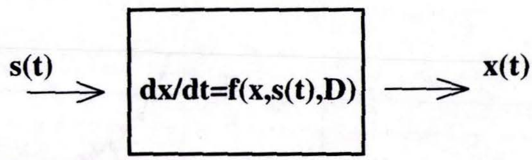

Fig. 1: General scheme of a stochastic resonator: A nonlinear noisy system with time scale depending on the noise intensity D is forced by an ordered signal $s(t)$. The output $x(t)$ becomes similar (ordered) to the input for optimally selected noise values.

For small noise in the system there is no ordered reaction of the system, i.e. the system rarely reaches the threshold at moments without reflecting the signal. Large noise also overwhelms the signal totally and the output of the system becomes random, too. Otherwise, for optimally selected noise the dynamics reaches the threshold in times comparable with the signal times. In that case the induced variations of the escape time caused by the small signal are sufficient that the output will follow strictly the structured values of the signal (with some phase shift). Hence, the output becomes similar to the inputting signal but amplified by some magnitudes dependent of the value of the threshold.

In Fig. 2 we compare the output with the input by determining the Kullback-entropy (Schimansky-Geier et al, 1998)

$$K[P^0, P] := \sum_i p_i \log \frac{p_i}{p_i^o} \ , \tag{1}$$

which measures the distance between the two distributions P^0 and P. It is always nonnegative and it vanishes if and only if P^0 and P are identical. We identify $p_i^0 := p_n^{in}(\mathbf{i_n})$ and $p_i := p_n^{out}(\mathbf{i_n})$ being the distributions of binary sub sequences of length n of the input and the output, respectively. The input is simply a periodical sequence with half period about 6. The distribution of the output were estimated by counting the occurrence of a certain sequence from time series of a periodically driven stochastic Schmitt-trigger with additive noise and noise intensity D. The

Kullback entropy takes smallest values for finite noise in the system. In this region of noise intensities the output is nearly converging to the input.For lower D the intermittent output is far from being periodic. For large intensities the periodic structure gets lost due to hoppings without reference to the signal.

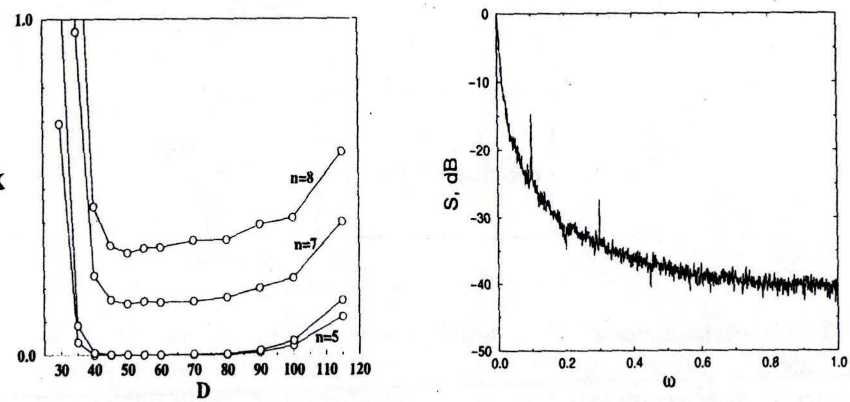

Fig. 2: Left: Kullback entropy vs noise intensity for different word lengths n. The curves quantify the distance between input and output statistics; here, the input consists of the periodic signal; Right: Spectrum of a periodically modulated bistable system. Peaks are seen at odd multiples of the signal frequencies and dips at even multiples.

Stochastic resonance (SR) is well investigated in bistable systems. In Fig. 3 a periodically modulated bistable potential $U(x) = -x^2/2 + x^4/4$ is shown. The dynamics of a bistable overdamped systems with signal and additive noise reads

$$\dot{x} = -\frac{\partial U(x)}{\partial x} + Asin(\Omega t + \phi) + \sqrt{2D}\xi(t) \tag{2}$$

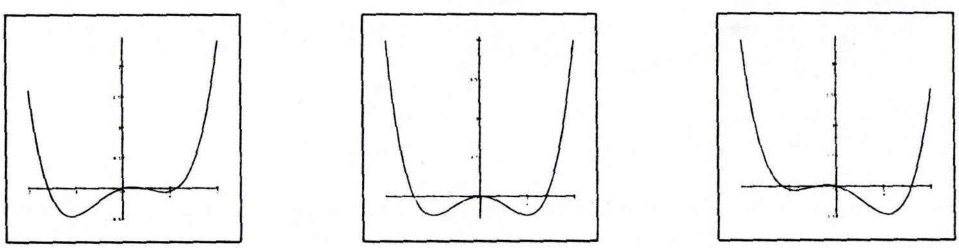

Fig. 3: Potential of a periodically modulated bistable systems. In case the noise is optimally selected the system dynamics follows the applied signal.

Best performance for small signal amplitudes A is achieved for noise intensities D_{opt} where the mean time for firstly passing the unstable state $x = 0$ equals the half

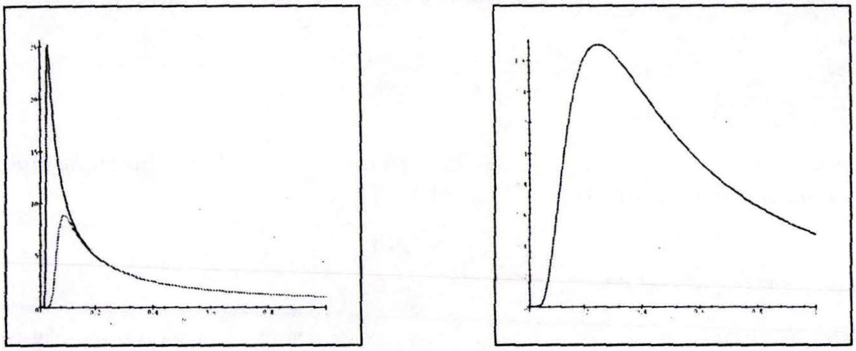

Fig. 4: SPA (left) and SNR (right) of a single two-state-resonator with respect to noise intensity. The SPA curve with the higher maximum is for smaller frequency Ω whereas the SNR is independent on the frequency.

period of the driving signal, i.e. $T(D_{opt}) = T_s/2$. It is the noisy time scale which is given approximately as for the considered potential $T(D) = \sqrt{2}\pi \exp 1/(4D)$.

Two relevant parameters for characterizing SR can be extracted from the spectrum of the output if averaged over the initial phase ϕ. A typical shape of the spectrum for a bistable system is presented in Fig. 2 as well. For small signal amplitudes this spectrum decomposes in a amplitude independent noise part and a signal part having a delta peak above the signal frequency and weighted by a factor ρ

$$S_{x,x}(\omega) = S^0_{x,x}(\omega) + A^2\rho\delta(\Omega - \omega) \tag{3}$$

This factor is named spectral power amplification (SPA) and can be defined by linear response theory through the Fourier transform of the response function $\rho = |\chi(\omega_s, D)|^2$. The second parameter is the signal-to-noise-ratio R. It relates the full weight of the signal to the noise level at the signal frequency, i.e. $R = \pi A^2\rho/S^0_{x,x}(\Omega)$.

For the purpose to study lateron coupled stochastic resonators it will be sufficient to reduce the dynamics to a theory of two states neglecting the dynamics within the wells. Such an approach was proposed by McNamara and Wiesenfeld (1989) who considered a periodically modulated telegraph process. The probabilities $p(\sigma)$ to find the process in state σ satisfy $p(\sigma) + p(-\sigma) = 1$ and their time-evolution is governed by

$$\dot{p}(\sigma) = -\dot{p}(-\sigma) = W(-\sigma)p(-\sigma) - W(\sigma)p(\sigma), \tag{4}$$

where $W(\sigma)$ denotes the rate of the transition $\sigma \to -\sigma$ given by (McNamara and Wiesenfeld, 1989)

$$W(\sigma) = \frac{\alpha}{2}[1 - \sigma\delta\cos(\omega t + \phi)]. \tag{5}$$

For the bistable system with small amplitudes the rate α is inverse to escape time $T(D)$

$$\alpha(D) = \alpha_0 \exp\left(-\frac{1}{4D}\right) \tag{6}$$

and $\delta = A/D$ in linear response. We lateron set $\alpha_0 = 1$. With this input one is able to calculate the SPA and the SNR as well. They read

$$\rho = \frac{1}{D^2(1 + \Omega^2/\alpha(D)^2)} \quad ; \quad R_0 = \frac{\pi}{4}\frac{A^2}{D^2}\alpha(D). \tag{7}$$

The dependency on noise of this two expressions is depicted in Fig.(4). We mention that the peak of the SNR does not reflect the dynamics of the system under consideration. Also the SPA achieves maximal values for noise intensities only near the earlier introduced D_{opt} by the matching condition. This condition can be mathematically extracted by the consideration of two limits, the low frequency SPA for $\Omega \ll \alpha(D)$ which is simply $\rho_{LF} = 1/D^2$ and alternatively the high frequency limit $\rho_{HF} = \alpha^2/(D^2\Omega^2)$. Both limits envelopes the real SPA and converges near its maximal value. From $\rho_{LF} = \rho_{HF}$ one immediately gets $\alpha = \omega$.

1.2 Coupled Two State Resonators

When stochastic resonators are coupled in parallel way the SPA and the SNR can be enhanced (Jung et al, 1992, Bulsara et al, 1993; Neiman and Schimansky-Geier, 1995; Morillo et al, 1995 ; Marchesoni et al, 1996; Dikshtein et al, 1998). Lindner et al. (1995, 1996) introduced the notion of 'array enhanced stochastic resonance' for the coupling-induced increase of the SNR. However, if the coupling becomes too strong the SPA and the SNR fall off again. Therefore, apart from the noise strength in a single stochastic resonator, the coupling strength has optimal values as well.

As an easy way to illustrate this topic we recently have investigated a system of coupled two state resonators (Schimansky-Geier and Siewert, 1997; Siewert and Schimansky-Geier, 1998) with temperature T employing Glauber's model of a stochastic Ising model (Glauber 1963). Indeed a coupled magnetic spin system represents a good candidate for SR as was shown by Brey and Prados (1995) and by Néda (1995). Coupling originates barriers for the spin flippings. Therefore, if a periodic force is applied for a given coupling strength the temperature has to be chosen optimally to achieve a best periodic response of the system. But an increase of the coupling strength weakens monotonously the value of the peak in the spectrum in this case and weak coupling shows the best performance.

Alternatively, a connected chain of two state resonators models features of coupled bistable elements where a barrier still exist for the single uncoupled element. We used the periodically modulated Arrhenius-like expressions for the local dynamics (Eq.5) of each element of the chain and coupling as introduced by Glauber which favours with $\gamma > 0$ a parallel ($\gamma < 0$: antiparallel) alignment of the states . The rates for a transition $\sigma_i \to -\sigma_i$ of the ith spin in a chain reads

$$W_i(\sigma_i) = \frac{\alpha(T)}{2}\left[1 - \sigma_i \frac{A}{T}\cos(\omega t + \phi)\right]\left[1 - \frac{\gamma}{2}(\sigma_{i-1} + \sigma_{i+1})\sigma_i\right]. \quad (8)$$

This rates defines the dynamics by the chain and should be inserted in a master equation for the probability function $p(\bar{\sigma}, t)$ to find the chain in a particular configuration $\bar{\sigma} = (\ldots\ldots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \ldots\ldots)$ at time t. We further introduce the spin coupling parameter J by $\gamma = \tanh 2J/T$.

Numerically generated realizations are presented in Fig. 5 (Ruszczynski, 1997). A best periodic response of the chain is obtained near $T \propto 0.5$ and $J \propto 0.6$.

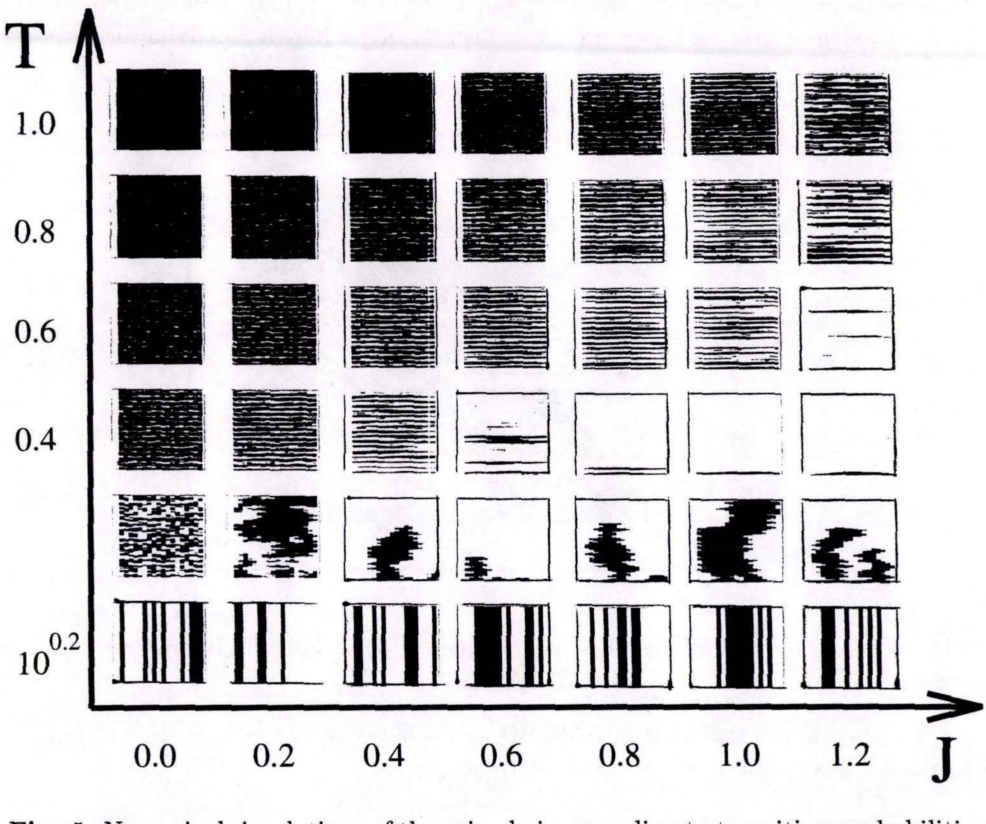

Fig. 5: Numerical simulations of the spin chain according to transition probabilities (eq. 8). In each panel a chain of hundred elements with running time bottom-up is presented. Dark and white regions correspond to spin up and down, respectively.

From the master equation for small amplitudes A the SPA can be determined. The correlation function in the unperturbed case was given by Glauber (1963). Hence, all measures for a discussion of SR in the coupled chain are known.

The SPA is presented in Fig. 6 and can be brought into the shape

$$\rho = \rho_s \left(1 + \frac{\omega^2 \exp\left(\frac{2}{T}\right)}{\left(1 - \tanh\left(\frac{2J}{T}\right)\right)^2}\right)^{-1}, \tag{9}$$

with

$$\rho_s = \frac{1}{T^2} \exp\left(\frac{4J}{T}\right), \tag{10}$$

being the static response of the chain on a constant force. With increasing coupling J this static response grows whereas the second factor in (9) displays the dynamic inability to follow the signal. The larger the frequency the smaller the weight of the delta peak in the spectrum. Both dependencies result in bell shaped curve for the SPA, since the static prefactor increases linearly in γ whereas the dynamic response decrease with γ^{-2}.

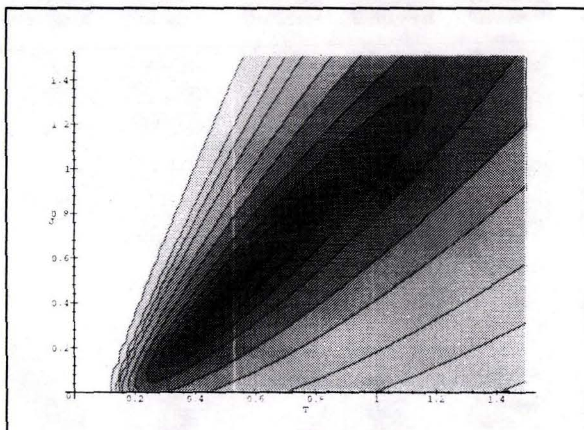

Fig. 6: Counterplot of the SPA in dependence on T and J. Increasing darkness corresponds to larger values of the SPA.

A discussion of the SNR should distinguish between two principal arrangements of the output (see Fig. 7). For the summed output

$$M(t) = \sum_i^N \sigma_i(t). \tag{11}$$

of all elements the noise part of the spectrum can be calculated explicitly (Glauber, 1963) in the limit of a infinitely long chain, $N \to \infty$. The SNR monotonously decreases with increasing coupling

$$R_M = \frac{\pi}{4} \alpha \sqrt{1 - \gamma^2 \delta^2}, \tag{12}$$

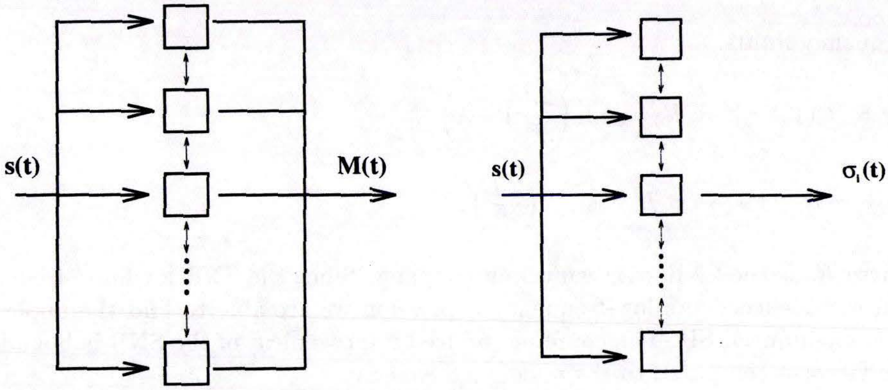

Fig. 7: Different schemes of coupled resonators. Whereas in the left scheme for the global output the SNR decreases monotonously with increasing coupling the response of a single element in the coupled chain exhibits optimal coupling for the best SNR

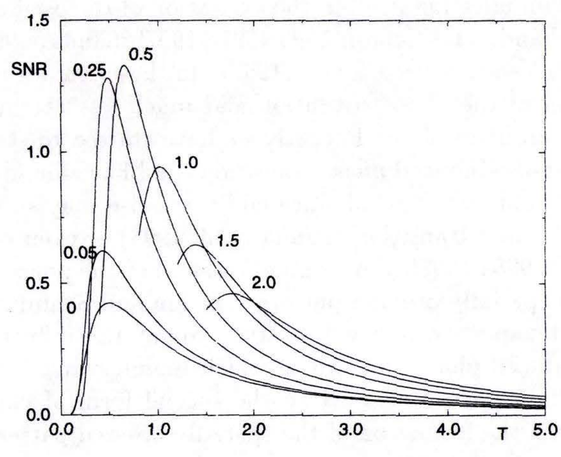

Fig. 8: SNR of a single element embedded in a infinitely long chain for different couplings J in dependence on T.

Differently, in Fig. 8 the SNR of a single element with the coupled chain is presented for different couplings. Array-enhanced response is seen. With moderate coupling a single element inside the chain exhibits a larger SNR compared to the uncoupled resonator $J = 0$ which is the result of the McNamara-Wiesenfeld-theory(7).

Estimates of the SNR of a single element can be given for the low- and high-frequency limits.

$$R_s = R_0(1+\gamma)^2 = R_0\left(\tanh\left(\frac{2J}{T}\right)+1\right)^2, \tag{13}$$

$$R_{HF} = R_0\sqrt{1-\gamma^2} = R_0\cosh^{-1}\left(\frac{2J}{T}\right), \tag{14}$$

where R_0 is the SNR with vanishing coupling. Since the SNR for finite frequencies can never exceed the low-frequency expression we are able to find the upper limit of array induced SR. The coupling induced improvement of the SNR is bounded by the factor 4 compared to the uncoupled element.

2 Spatial Inhomogenious Patterns Induced by the Additive Noise

A numerous models have been reported to demonstrate nonequilibrium noise-induced phase transitions manifesting itself in the excitation of the oscillations or appearing of "mean field" (Landa and Zaikin, 1996,1997a,1997b;Landa, 1996;Van den Broeck et al, 1994a,1994b;Van den Broeck et al 1997). In these studies multiplicative noise has been a reason of the phase transition and much less attention has been paid to the role of the additive noise. Recently we have started to study an influence of additive noise on noise-induced phase transitions and have found that this influence can be important and even crucial since additive noise may shift the boudaries of the noise-induced phase transition (Landa et al, 1998) or even cause such a transition (Landa et al, 1998a,1998b). As a manifestation of the phase transition additive noise may induce spatially ordered patterns (Zaikin and Schimansky-Geier, 1998).

In the present paper we review our study about the influence of the additive noise on noise-induced phase transitions which manifest itself in the formation of the "mean field". As a consequence of the special form of coupling these phase transitions result in the formation of the spatially ordered patterns.

2.1 The Model

We consider a scalar filed $x_{\mathbf{r}}$ defined on a spatial lattice with points $\mathbf{r}$ and described by a set of Langevin equations (Parrondo et al, 1996):

$$\dot{x}_{\mathbf{r}} = f(x_{\mathbf{r}}) + g(x_{\mathbf{r}})\xi_{\mathbf{r}} + \mathcal{L}x_{\mathbf{r}} + \zeta_{\mathbf{r}} \tag{15}$$

with f and g defined as

$$f(x) = -x(1+x^2)^2 \quad g(x) = a^2 + x^2 \tag{16}$$

and $\xi_{\mathbf{r}}$, $\zeta_{\mathbf{r}}$ are independent zero-mean-value Gaussian white noise sources:

$$\langle \xi_{\mathbf{r}}(t)\xi_{\mathbf{r}'}(t')\rangle = \sigma_\xi^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t') \quad (17)$$

$$\langle \zeta_{\mathbf{r}}(t)\zeta_{\mathbf{r}'}(t')\rangle = \sigma_\zeta^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t') \quad (18)$$

The spatial coupling in the model is described by the coupling operator $\mathcal{L}$, which is a discretized version of the Swift-Hohenberg coupling term $-D(q_0^2+\nabla^2)^2$:

$$\mathcal{L}x_{\mathbf{r}} = -D\left\{q_0^2 - \frac{1}{\Delta^2}\sum_{i=1}^{2d}\left[1 - exp\left(\Delta e_i \cdot \frac{\partial}{\partial \mathbf{r}}\right)\right]\right\}^2 x_{\mathbf{r}}. \quad (19)$$

Here e_i represents the unit vectors of the cubic lattice. The lattice is of the dimension d, and Δ is the lattice space.

Such form of the function $g(x)$ implies that the parameter a is responsible for an additive noise strongly correlated with the multiplicative one. To investigate the influence of additive noise on the noise-induced phase transition we study two different problems. First the constant contribution a^2 of the multiplicative noise $\xi_{\mathbf{r}}$ is changed, setting $\sigma_\zeta^2 = 0$. The origin one could see, for instance, in a decomposition of the multiplicative noise into two parts $g(x)\xi_{\mathbf{r}} = a^2\xi_{\mathbf{r}}^1 + x^2\xi_{\mathbf{r}}^2$. Changing the parameter a would imply an increase or a decrease of additive noise $a^2\xi_{\mathbf{r}}^1$ strongly correlated to the multiplicative one. This constant contribution of noise is essential for the nonequilibrium phase transition under consideration. Only in the presence of the additive component with an optimally selected value the system exhibits spatial disordered states.

A different situation is the variation of the noise intensity σ_ζ^2. It models additive noise independent of the multiplicative one. In that case we set $a = 0$. Again we will find a strong influence of the additive noise ζ.

Using generalized Weiss' mean field theory (Parrondo et al, 1996) the conditions of phase transition can be found. According to this theory we replace the value of the scalar variable $x_{\mathbf{r}'}$ at the sites coupled to $x_{\mathbf{r}}$ by its averaged value, assuming the following specific non-uniform average field:

$$\langle x_{\mathbf{r}'}\rangle = \langle x\rangle cos\left[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right]. \quad (20)$$

Substituting eq.(20) in eq. (19) we get for $x_{\mathbf{r}}$:

$$\dot{x} = f(x) + g(x)\xi + D\omega(\mathbf{k})x - D_{\text{eff}}(x - \langle x\rangle) + \zeta \quad (21)$$

where

$$D_{\text{eff}} = \left[\left(\frac{2d}{\Delta^2} - q_0^2\right)^2 + \frac{2d}{\Delta^2} + \omega(\mathbf{k})\right] D \quad (22)$$

and

$$\omega(\mathbf{k}) = -D\left[q_0^2 - \frac{2}{\Delta^2}(2 - cosk_x\Delta - cosk_y\Delta)\right]^2 \quad (23)$$

The expression $\omega(\mathbf{k})$ is the dispersion relation which is defined for the case of a two-dimensional lattice as:

$$\mathcal{L}e^{i\mathbf{kr}} = \omega(\mathbf{k})e^{i\mathbf{kr}}, \tag{24}$$

where $\mathcal{L}$ acts on a plane wave $e^{i\mathbf{kr}}$.

For $|\mathbf{k}| << 2\pi/\Delta$ the dispersion relation $\omega(\mathbf{k})$ reduces to the relation for the continuous Swift-Hohenberg model: $-D(q_0^2 - |\mathbf{k}|^2)^2$. For the most unstable mode in the discrete case $\omega(\mathbf{k}) = 0$ (see. (Parrondo et al, 1996)).

Note that the value $\langle x \rangle$ plays the role of the amplitude of the spatial patterns with an effective diffusion coefficient D_{eff}.

The Fokker-Planck eq. corresponding to the eq.(21) in the case $\omega(\mathbf{k}) = 0$ is:

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}\left(\Big(f(x) - D_{\text{eff}}\,(x - \langle x \rangle)\Big)w - \frac{\sigma_\xi^2}{2}\left(g(x)\frac{\partial}{\partial x}\big(g(x)w\big)\right) - \frac{\sigma_\zeta^2}{2}\frac{\partial w}{\partial x}\right)$$

According to this this equation the exact steady state probability parametrically depends on $\langle x \rangle$:

$$w_{st}(x) = \frac{C(\langle x \rangle)}{\sqrt{\sigma_\xi^2 g^2(x) + \sigma_\zeta^2}} \exp\left(2\int_0^x \frac{f(y) - D_{\text{eff}}(y - \langle x \rangle)}{\sigma_\xi^2 g^2(y) + \sigma_\zeta^2}\,dy\right), \tag{25}$$

where $C(\langle x \rangle)$ is the normalization constant determined by the following expression:

$$C^{-1}(\langle x \rangle) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_\xi^2 g^2(x) + \sigma_\zeta^2}} \exp\left(2\int_0^x \frac{f(y) - D_{\text{eff}}(y - \langle x \rangle)}{\sigma_\xi^2 g^2(y) + \sigma_\zeta^2}\,dy\right) dx. \tag{26}$$

For the value $\langle x \rangle$ we obtain:

$$\langle x \rangle = \int x w_{st}(x, \langle x \rangle)dx. \tag{27}$$

This equation is a nonlinear equation for the unknown value $\langle x \rangle$ and closes the system of equations.

Solving eq.(27) we can calculate boundaries of phases with $\langle x \rangle \neq 0$ (order) and $\langle x \rangle = 0$ (disorder) for specific $\mathbf{k}$ which modes are excited first. Non-zero solution of eq.27 means excitation of the corresponding mode and hence existence of the phase transition. The special form of the spatial coupling is responsible for the fact that the transition manifests itself in a formation of ordered spatial patterns with the wave number defined by the parameter q_0.

Computing Eq. (27) one can find that the condition for the existence of nonzero solutions is

$$\left|\frac{dF}{dm}\right|_{m=0} \geq 1. \tag{28}$$

We note that for rather large D four non-zero roots (two stable and two unstable) of the eq.27 may be observed. From this we expect that additionally noise-induced first-order phase transition may be also found in this model (to this point see also (Muller et al, 1997;Kim et al, 1997)).

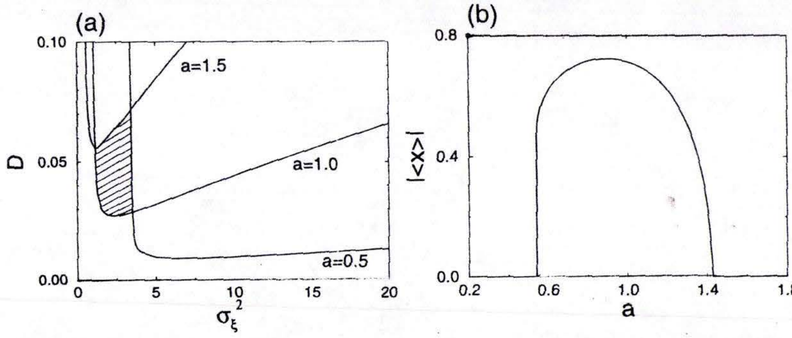

Fig. 9: (a): The boundaries of the phase transition on the plane $(\sigma_\xi^2; D)$ are shown for the case of correlated additive noise. The values of parameter a are shown in the picture. (b): the corresponding dependence of the order parameter $|\langle x\rangle|$ vs the control parameter a for $D = 0.06$, $\sigma_\xi^2 = 3.0$.

2.2 Influence of Additive Noise on Noise-induced Transition

First let us consider the case if an additive noise is strongly correlated with multiplicative one (it means $\sigma_\zeta^2 = 0$). For different values of a the boundary of the phase transition on the plane (σ_ξ^2, D) is shown in Fig. 9. As it is seen from this plotting the reentrant phase transition occurs for the specific value of a with the increase of σ_ξ^2 (Parrondo et al, 1996). Solving the eq. 27 for others values of a we find that as a decreases the boundary of the phase transition significantly dropped and right shifted (see Fig. 9). Hence there is a set of parameters (σ_ξ^2, D) for which the reentrant phase transition occurs with the increase of a (dashed region in the Fig. 9). It means that for fixed values of σ_ξ^2 and D an increase of additive noise intensity will first induce the spatial patterns and then destroy them. We note that this phase transition is possible only in the presence of multiplicative noise. Corresponding dependence of the order parameter $|\langle x\rangle|$ on control parameter a is shown in Fig. 9.

Now we consider the case where the additive noise is uncorrelated (independent) with the multiplicative one ($a = 0$, $\sigma_\zeta^2 \neq 0$). In Fig. 10 it can be clearly seen that in this case the behaviour of the system is qualitatively the same: for fixed parameters (D, σ_ζ^2) an increase of the multiplicative noise intensity σ_ξ^2 causes the noise-induced phase transition. Hence for large enough coupling D one expects the formation of the spatially ordered patterns if σ_ξ^2 exceeds it critical value. As to influence of the additive noise on the transition, an amplification of the additive noise intensity shifts the transition boundaries and therefore causes the reentrant disorder-order-disorder nonequilibrium phase transition. To illustrate it let us take a point with

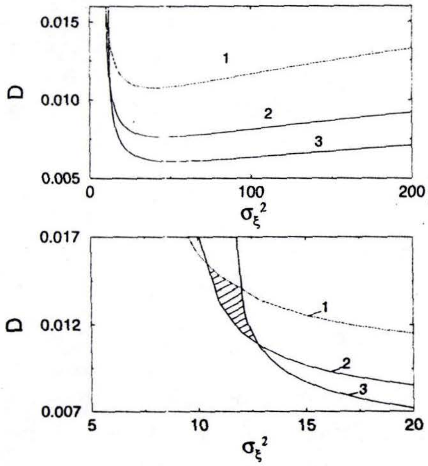

Fig. 10: The case of uncorrelated additive noise: the boundaries of the phase transition on the plane $(\sigma_\xi; D)$. The parameter σ_ζ^2 is equal to 1.1 (label 1), 0.5 (label 2), and 0.3 (label 3).

fixed parameters (D, σ_ξ^2) from the dashed region in the Fig. 10: with an increase of σ_ζ^2 this point first belongs to the disordered phase, then to the ordered one, and then again to the disordered phase.

2.3 Numerical Simulations

Now we compare predictions of the theory considered above with results of numerical simulations of the initial eqs. (15). We use an Euler scheme for stochastic differential equations interpreted in the Stratonovich sense (Raminez-Piscina et al, 1993;Sancho et al, 1982). The time step has been set $\Delta t = 5 \cdot 10^{-4}$. For simulations we integrate the scalar field $x_{\mathbf{r}}(t)$ on a two-dimensional square lattice 128×128 with conditions $x_{\mathbf{r}} = 0$ and $\mathbf{n} \cdot \nabla x_{\mathbf{r}} = 0$ at the boundaries. Here $\mathbf{n}$ is the vector normal to the boundary.

First we set $\sigma_\zeta^2 = 0$ and $a \neq 0$. The remaining parameters are $D = 1$, $q_0 = 0.7$, $\Delta = 0.5$ and $\sigma_\xi^2 = 1.8$. For these values the mean field theory predicts the existence of spatial patterns of the most unstable mode $|k| = 1.0478$ for $a = 1$. For additive noise intensities significantly larger than this value, for example $a = 10.0$, or significantly smaller, $a = 0.1$, according to the mean field theory no spatial patterns will be exhibitted.

In Fig. 11 the picture of the field after 100 time units has been plotted for three different noise intensities. Clearly one can see the appearance of the spatial patterns

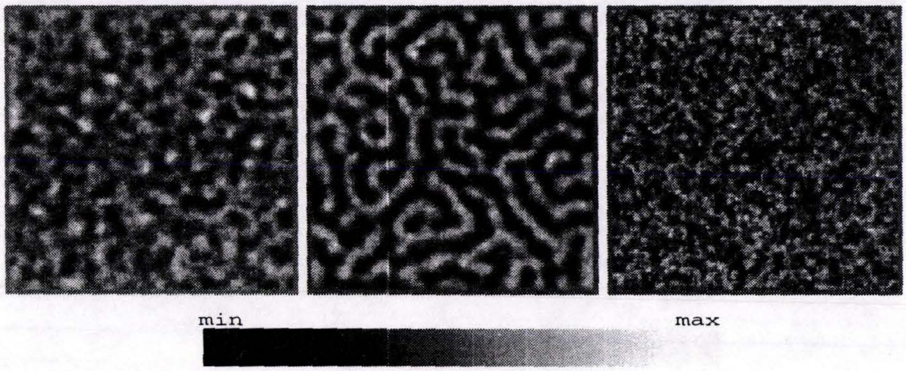

Fig. 11: Reentrant noise-induced phase transition: snapshots of the field for $D = 1.0$, $\sigma_\xi^2 = 1.8$, and $\sigma_\zeta^2 = 0$. The parameter a is equal to 0.1, 1.0, and 10.0 (from left to the right). The increase of the additive noise induces spatial patterns. The color code is shown in the same figure.

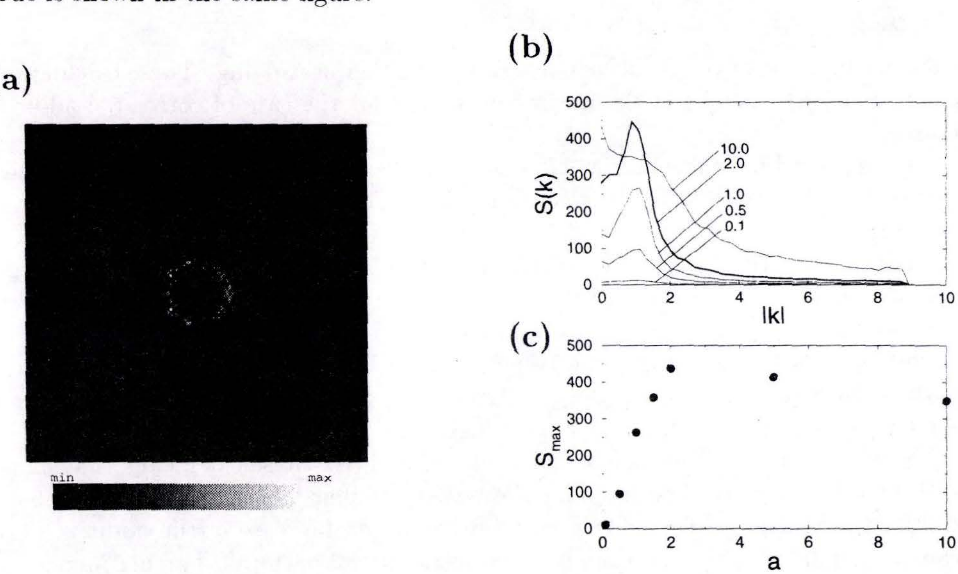

Fig. 12: (a): Rotational symmetry of ordered patterns can be observed in 2D–fourier transform of the pattern shown in Fig.11(case $a = 1.0$). (Max;Min) values are (1337;0.1). (b): Fourier transform averaged over angles for $D = 1.0$, $\sigma_\xi^2 = 1.8$. Values of parameter a are shown in the figure. (c): The dependence of S_{max} on a.

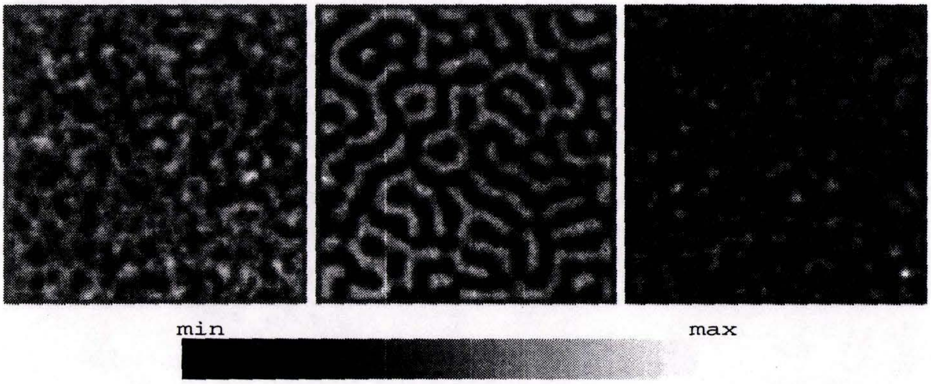

Fig. 13: Results of numerical simulations in the case of the uncorrelated additive noise: snapshots of the field for the parameter σ_ζ^2 equal to 0.001, 0.7, and 10.0 (from left to the right). Remaining parameters are $D = 3.5$, $\sigma_\xi^2 = 13$, $a = 0$, and $\Delta t = 10^{-7}$. (Max;Min) values are (0.0072:-0.0075), (7.14:-6.33), and (1.07:-0.61).

with the increase of the additive noise and its further destroying. These calculations confirm the predictions of the mean-field theory for the case of correlated additive noise.

The ordered patterns in Fig. 11 (case $a = 1.0$) have rotational symmetry, which can be clearly observed in the two-dimensional Fourier transform of the field (see Fig. 12).

To make the transition more evident we have plotted the Fourier transform of the field averaged over the angles of the wave vector. It is shown in Fig. 12. for different values of a. With an increase of a a maximum in this structure function is found. It corresponds to the dominating value $|k|_{max}$ indicating the appearance of spatial pattern with wave length $2\pi/|k|_{max}$. After an optimal value of a the maximum of the structure function disappears again signalling the destruction of the order.

Next we consider the case of uncorrelated additive noise. It means that $a = 0$ and $\sigma_\zeta^2 \neq 0$. Numerical simulations show that the behaviour of the model is quite similar to the case of the correlated additive noise. Increase of the additive noise causes formation of the rotationally-symmetric spatial patterns. Further increase of the additive noise destroys this pattern (see Fig. 13). These results are also in a good agreement with predictions of the mean field theory.

Now let us discuss the mechanism responsible for the appearance of the ordered spatially patterns with the increase of the additive noise and further its destroying. As it is stated above the ordered phase is a result of the noise-induced phase transition, so our aim is to understand why increase of the additive noise leads to this

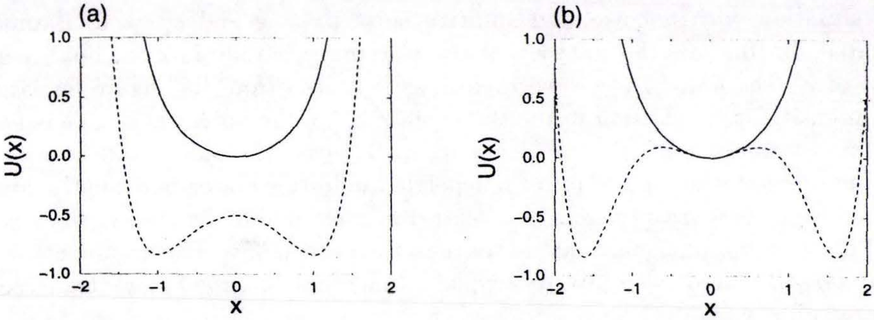

Fig. 14: Short time evolution of the average value. Potential $U(x)$ for the parameters: (a) $\sigma_\xi^2 = 2$; solid line: $a^2 = 0.1$, dashed line: $a^2 = 1.0$; (b) a=0, solid line: σ_ξ^2=2, dashed line: σ_ξ^2=5.

transition.

The drift part in the Fokker-Planck operator (Stratonovich case) determines the time evolution of the first moment for a single element of the lattice

$$\langle \dot{x} \rangle = \langle f(x) \rangle + \frac{\sigma_\xi^2}{2} \langle g(x) g'(x) \rangle. \tag{29}$$

As it was argued in (Van den Broeck et al, 1997) the evolution over short times of an initial Delta function is well approximated by a Gaussian which extremum obeys

$$\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma_\xi^2}{2} g(\bar{x}) g'(\bar{x}). \tag{30}$$

Here $\bar{x} = \langle x \rangle$ is the maximum of the probability which is the average value in this approximation. For this dynamics one is able to introduce a potential $U(x) = U_0(x) + U_{noise} = -\int f(x)dx - \sigma_\xi^2 g^2(x)/4$ where $U_0(x)$ is the unperturbed potential and $U_{noise} < 0$ describes the action of the noise. In the case under consideration $U_0(x) = x^2(1 + x^2 + x^4/3)/2$ which is monostable with a minimum at $x_0 = 0$.

Now we consider how additive noise modifies the potential $U(x)$. We start with the case if $\sigma_\zeta^2 = 0$ and additive noise is included in the equations through $g(x) = a^2 + x^2$ by the constant a. For small a the potential $U(x)$ remains monostable and there is no possibility of the phase transition in the system. If we increase a i.e. the intensity of the correlated additive noise the potential $U(x)$ becomes bistable if $a > a_{crit} = 1/\sqrt{\sigma_\xi^2}$ (see Fig. 14 *a*). For sufficiently strong coupling this occurred bistability will be the reason of the local ordered regions at short time scales, which coarsen and grow with the time. Hence the additive part of the noise in the function g is essential for occurrence of the nonequilibrium phase transition.

The situation with uncorrelated additive noise ($a = 0$ and $\sigma_\zeta^2 \neq 0$) is more complicated. In this case the state $x = 0$ always remains stable since the noisy part $U_{noise}(x) \propto x^4$ (see Fig. 14 *b*). Nevertheless, as it is seen from this figure for large enough intensity σ_ξ^2 in addition to the stable state $x = 0$ the potential $U(x)$ has two minima more, precisely if $\sigma_\xi^2 > 4$. Therefore in this case the phase transition is a result of hard excitation and requires independent additive noise. Sufficiently large additive noise causes escapes from the central minimum and the system does not return if the new minimal states are lower than the central one. The argumentation given can be considered as an intuitive explaination of the observed phase transition induced by additive noise.

3 Discussion and Conclusions

In the discussion we want to trace parallels between the behaviour of the SNR in SR-phenomena and the structure function in the reentrant phase transitions in dependence on the additive noise. Both phenomena prove the ordering role of additive noise in nonlinear systems far from equilibrium. The influence of additive noise results in an ordered response of the system which is manifested by the increasing SNR and structure function if increasing the intensity of the noise. Moreover, both characteristics depend nonmonotonically on this intensity and hence, an optimally selected value exhibits the most ordered behaviour.

Let us consider possible reasons of this similarity. For that purpose we reformulate the noise induced phase transition as a situation typically occurring in SR. The influence of the neighbours supplied by the coupling serves as a driving force for the single system in the lattice with a bistable potential. Under this influence every single system is trying to obey the rules of the whole system, for example to choose the proper minimum of a potential. It replaces in a self-consistent way the action of the periodic input in SR where the system is forced to follow the periodic stimulus.

Accordance to stochastic resonance becomes evident since this information is best transmitted to the single system if the intensity of an additive noise is optimally selected. For smaller and larger values of noise intensity the ordering process is not effective like in stochastic resonance. As a result and quite analogously to the shape of the SNR the maximum of structure function behaves nonmonotonously in dependence on the parameter a. The similarities are obviously bounded since in SR the input is independent from the reaction of the system. In our case it differs due to the mutual interaction between the elements of the lattice. It determines the structure of the output which plays the role of the input for another element.

In conclusion, we have shown on behalf of two nonlinear coupled noisy systems that an increase of the additive noise may surprisingly induce order. A further increase of the additive noise destroys the ordered structures again. In both cases coupling plays an important role. In SR-phenomena it improves significantly the response of a single element embedded in a chain. For the noise induced phase transition the coupling is crucial in synchronizing the elements of the lattice.

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