Eigenbehaviour in Deterministic Systems

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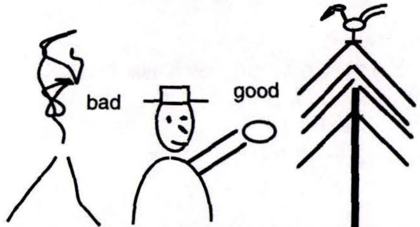
Abstract
The meaning of "Eigenbehaviour" is discussed under a philosophical and a mathematical point of view. We show that Eigenbehaviour characterises the turbulent behaviour of a fluid modelled by the Navier Stokes equations. Eigenbehaviour is therefore a concept that can be understood in the frame of deterministic systems.
Keywords: Eigenbehaviour, Navier-Stokes equations, Turbulence, Granulation.

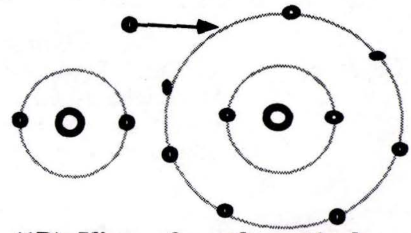
1 Introduction

An understanding in social sciences needs concepts that can not be realised in the mathematical theories used in natural sciences (Weiss, 1992). A closed framework for wider concepts is provided by constructivism (Glaserfeld, 1996), where fundamental ideas like selfreference and autopoiesis can be discussed. It is an open question which of the fundamental concepts of social sciences can also be realised into a mathematical framework (Puntel, 1991). This question will be answered here for one of the most basic concepts in social sciences, the idea of an "Eigenbehaviour". A behaviour of a system is called Eigenbehaviour, if it is independent from its design and from external influences. The Eigenbehaviour of a system impedes that an outer observer can forecast its future behaviour. It is a reason that makes anticipation difficult. Eigenbehaviour is therefore an important concept in a theory of anticipatory systems (Dubois, 1997). We discuss this concept in the view of constructivism and we recall that also practical engineers had used it in hydromechanics to distinguish between turbulent and laminar fluids. By an observation of the fluid around a profile, engineers have noted that the fluid in the turbulent region is largely separated from the main stream. A precise mathematical definition for this separation will be given and with a new solution method for the Navier-Stokes equations, an expression will be obtained that assigns to every point $(\vec{r}, t)$ a Fuzzy membership degree to the region of separation. Our discussion yields a new understanding of turbulence. We show that turbulent behaviour can be understood as the emergence of an Eigenbehaviour in the fluid that can not be anticipated.

2 Eigenbehaviour in the View of Constructivism

Constructivism enables an understanding of selfreference and autopoiesis, concepts that characterise living systems and their differences to machines. Selfrefering systems are able to change the language and the basic laws that define the relation with their environment. Selfrefering systems question the frame that defines their live (Figure 1A). In contrast to the constructivistic point of view, the foundation on a fixed language and on fixed basic definitions is indispensable for a mathematical theory. This basic foundations of the theory can not be discussed in the theory itself. (Goedel proofed that some questions concerning the wholeness of a theory can not be answered in the theory itself). For the formulation of a mathematical theory, a fixed frame is necessary, containing the description language and the definitions of the basic elements for the





(1A) Constructivistic view: Reality is formed by the environment and the meaning assigned to this environment by the beings.

(1B) View of mathematical systems theory: The world is constructed with elementary elements.

construction of the complex objects. Objects in a mathematical theory are determined by the elementary elements and the laws describing the relations between this elementary elements (Figure 1B). The fixations made in the beginning of the construction of a mathematical theory are decisive for the whole theory. (It is one of the main goals of Fuzzy Logic, to make this fixations as little engaging as possible. But also Fuzzy Logic is a mathematical theory with basic definitions (Kiendl, 1997).)
Self determination is one of the fundamental principles that marks the difference between artificial systems that can be described with mathematical systems theory and living systems that need wider concepts (of constructivism) to make them understandable and investigable. The self determination of the knowledge of the beings in the constructive viewpoint has been formulated by Luhmann (1997):

(LU1): "A constructivistic epistemology understands knowledge no longer as representation of environmental states of affairs (in any symbolic form whatsoever) but as 'Eigenbehaviour' of a selfreferential system."

The objects are therefore, in the constructive viewpoint, not only determined by the initial configurations and the dates, coming from the environment to the objects, but also by their 'Eigenbehaviour' that gives them some independence from the environment. We know from Hegel, that the faculty to discriminate gives us the capacity to build ideas, fixed in the words of our language. If this discriminations are not fixed but discussed by the beings in the time development of the world, then the world will be undetermined. Lumann formulates this effect:

(LU2): "The result of a re-entry into the system of the distinction between system and environment is that such systems operate in the mode of selfproduced indeterminacy."

An important difference between social sciences and natural science is the impossibility in social sciences to fix a priori the basic description elements over which a general theory can be constructed. This point had been stressed clearly in the book of J.Weiss over the work of Max Weber. Weiss shows that Weber does not offer a theory for social sciences but a framework to find methods to deduce a scientific understanding of social phenomena (Weiss, 1992).

3 Emergence of an 'Eigenbehaviour' in Deterministic Systems

The understanding of the relation between the wider concepts, necessary in social sciences, and the methods offered by mathematical systems theory is one of the most important problems. It is the challenges of mathematical systems theory to explain with its own methods as many phenomena as possible from constructivism.
The objective of this article is to show that the emergence of an 'Eigenbehaviour' independent of fixed initial conditions and of the exterior influences on the system, is also possible in well-defined deterministic mathematical systems (compare statement (LU1)).

Due to the statement (LU2) this effect is only possible in systems with a re-entry of data produced by the system itself, or spoken in a mathematical language, by systems defined by **nonlinear** differential equations.
The examined system is a fluid outside a boundary, defined by the Navier-Stokes equations: (1)

$$\frac{\partial \vec{u}(t,x,y,z)}{\partial t}+(\vec{u}(t,x,y,z)\cdot\nabla)\vec{u}(t,x,y,z)-\frac{\eta}{\rho}\Delta\vec{u}(t,x,y,z)=\vec{F}(t,x,y,z)-\frac{1}{\rho}\nabla p(t,x,y,z)$$

and the continuum equation: $\nabla\,\vec{u}(t,x,y,z)=0$ (2)

($\vec{u}$ denotes the velocity of the fluid at time t in $\vec{r}=(x,y,z)$, $\nabla := \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right)^T$,
$\Delta := \nabla^T\cdot\nabla$, $\vec{F}$ are the external forces, p is the pressure and η, ρ are parameters.)

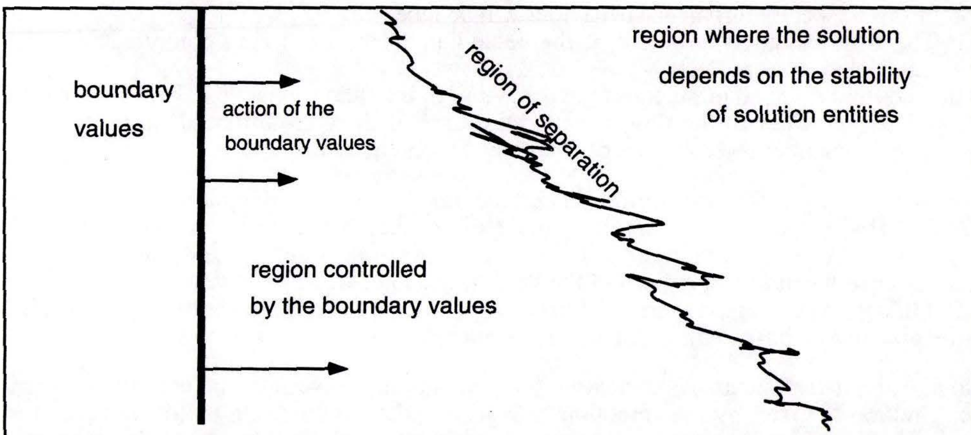
By an observation of the fluid, two different behaviours can easily be distinguished. In the laminar case, the behaviour of the fluid is similar to the behaviour of a solution of a linear partial differential equation, but in the turbulent case, great instabilities in the behaviour of the fluid will be observed. It is an open question, how to give an exact definition of turbulence in a mathematical framework. In the following table, some of the proposals to specify turbulence behaviour will be repeated:
(I) The fluid is called turbulent, if the behaviour of the fluid is to complicated to be calculated.(Oseen, 1930)
(II) The fluid is called turbulent, if the behaviour of the fluid is chaotic.(Chaos theory)
(III) The behaviour of the fluid is called turbulent if there are bifurcations in the time development of the solutions of equation (1).(Kirchgässner, 1975)

But as it is very difficult to verify this effects for the solution of equation (1) (Ansorge Sonar, 1997), we examine here a simpler effect that is accompanied with turbulent behaviour. From a practical engineering viewpoint, Greiner and Stock (1978) characterise the turbulent regions of the fluid by the following properties:
(IV) In a turbulent region there exist many vortexes and the fluid in the region is **largely separated** from the main stream of the fluid.

In a mathematical language, a measure for the coupling between the parts of the solution in a region is given by the smoothness-degree of the solution. On the other hand, for smooth Eigenfunctions, the smoothness degree of a solution can be defined by the decrease of the coefficients, for the Eigenfunctions of large Eigenvalues, in the Eigenfunction expansion of the solution $\vec{u}$. A measure for the separation of the fluid is therefore provided by the increase of the coefficients, for Eigenfunctions of large Eigenvalues, in the Eigenfunction expansion of the solution $\vec{u}$.
The Theorem postulates that this measure can be deduced from equation (1) for linear boundaries. The proof of the Theorem depends on a new solution method.

Theorem: From equation (1) and boundary values on linear boundaries a mathematical expression $\Phi(t,\vec{r})\in[0,1]$ can be derived that characterises the separation of the solution in a region from the main stream or from the influence of the boundary values. This expression $\Phi(t,\vec{r})\in[0,1]$ is a **membership degree** of a space point $\vec{r}\in\mathbb{R}^3$ at time t to a region of separation. $\Phi(t,\vec{r})$ defines regions where the parts of the solution are disconnecting.

Consequences of the Theorem:
(A) The function $\Phi(t,\vec{r})$ characterises a typical behaviour of the solutions of nonlinear partial differential equations, because (as shown in the proof) for linear partial differential equations, the function $\Phi(t,\vec{r})$ would be identical zero.
(B) The region of separation, characterised by high values of the membership function $\Phi(t,\vec{r})$ ($\Phi(t,\vec{r}) \approx 1$), forms a frontier for the influence from the boundary values (and from other parts of the fluid). Behind this frontier, the solution is no longer controlled by the boundary values. The fluid shows an **Eigenbehaviour** and exact anticipation is now impossible. The form of the solution behind the frontier is selected by the stability of the solution form under the action of the Navier Stokes equations (Figure 2). Stable parts of the solution are parts that are preserved (in a time interval) under the action of the Navier Stokes equations. The frequency of the observation of a solution form (the probability of this form) is proportional to its stability. There exist different methods to examine the stability (in this sense) of solutions of the Navier Stokes equations (Hopf, 1942), (Kirchgässner, 1975), (Heisenberg, 1948).
(C) The connected regions in the complement of the region of separation form the **granules** in the fluid.



(2) Different behaviours of a solution of equation (1): The membership function $\Phi(t,\vec{r})$ defines parts of the fluid that are no longer controlled and formed by the boundary values but by their own 'Eigenbehaviour'.

(D) By the solution algorithm used in the proof of the Theorem, the smoothest solution of (1) is selected. It can be shown that this principle signifies a maximisation of the entropy of the solution. Our selection principle corresponds to the second theorem of the theory of heat, whereas in functional analysis the relevant solution is selected claiming the limitation of the energy of the solution and therefore uses the first theorem of the theory of heat (Hopf, 1951).

4 Conclusion

"Eigenbehaviour" characterises a turbulent fluid modelled by the Navier-Stokes equations. Eigenbehaviour is therefore not a property that characterises social systems and distinguishes this systems from deterministic systems.

Proof of the Theorem
(I) Approximation of solutions of the Navier Stokes equations:
The solution algorithm will be presented in four steps:

Step 1: Elimination of the pressure term p:

With the denotations: $\Xi(\vec{u}) := -\frac{\rho}{\eta}\vec{u}\nabla + \Delta\vec{u}$, $\vec{f} := \vec{F} - \frac{1}{\rho}\nabla p$

equation (1) can be written in the form:

$$\frac{\partial}{\partial t}\vec{u} = \frac{\eta}{\rho}\Xi(\vec{u})\vec{u} + \vec{f} \tag{3a}$$

A solution of equation (3a) will provide us a solution of the system ((1)&(2)) due to arguments given by Ladyshenskaya (1965):
The function space V of all possible solutions $\vec{u}$ of ((1)&(2)) can be represented by the sum of two spaces: $V = V_\nabla \oplus V_\nabla^\perp$ with

$$V_\nabla := \{\vec{u} \in V | \text{rot}\,\vec{u} = 0, \text{where rot}\,\vec{u} \text{ is interpreted as a distribution in } D'\}$$

and $$V_\nabla^\perp := \left\{\vec{u} \in V \middle| \forall \vec{v} \in V_\nabla : \int_{\mathbb{R}^3} \vec{v}(\vec{r})\vec{u}(\vec{r})d\vec{r} = 0\right\}.$$

Representing $\vec{F}$ as a sum $\vec{F} = \vec{F}_{V_\nabla} + \vec{F}_{V_\nabla^\perp}$, with $\vec{F}_{V_\nabla} \in V_\nabla$ and $\vec{F}_{V_\nabla^\perp} \in V_\nabla^\perp$,

and defining p such that : $\frac{1}{\rho}\nabla p = \vec{F}_{V_\nabla}$ (p exists because of rot $\vec{F}_{V_\nabla} = 0$)

then equation (1) is reduced to $\frac{\partial}{\partial t}\vec{u} = \frac{\eta}{\rho}\Xi(\vec{u})\vec{u} + \vec{F} - \frac{1}{\rho}\nabla p = \frac{\eta}{\rho}\Xi(\vec{u})\vec{u} + \vec{F}_{V_\nabla^\perp}$.

The projection $P_{V_\nabla^\perp} : V \twoheadrightarrow V_{\nabla\perp}$ is a continuos operator that does not produce any separation of the parts of the flow $\vec{u}$ or any chaotic effects.
The following discussion can therefore be restricted to equation (3a).

Step 2: Discretisation of the problem: To solve equation (3a), the problem will be restricted to very special boundary and initial conditions.

$$\vec{u}(0,x,y,z) = 0 \text{ for } t < 0 \tag{3b}$$

$$\vec{u}(t,0,y,z) = \vec{u}_0(t,y,z) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \tag{3c}$$

$$\vec{f}(t,x,y,z) = 0 \tag{3d}$$

For this boundary conditions, the Navier Stokes equations have the form for $\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$:

$$\frac{\eta}{\rho}\cdot\frac{\partial^2}{\partial x^2}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix} - u\cdot\frac{\partial}{\partial x}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{\eta}{\rho}\cdot\frac{\partial^2}{\partial y^2}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix} - v\cdot\frac{\partial}{\partial y}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{\eta}{\rho}\cdot\frac{\partial^2}{\partial z^2}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix} - w\cdot\frac{\partial}{\partial z}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

The substitutions $A(\vec{u}) := \frac{\rho}{\eta}\cdot u$ and $B(\vec{u}) := \left[\frac{\partial^2}{\partial y^2} - \frac{\rho}{\eta}\cdot v\cdot\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} - \frac{\rho}{\eta}\cdot w\cdot\frac{\partial}{\partial z}\right]$ yield:

$$\frac{\partial^2}{\partial x^2}\cdot\begin{pmatrix}u\\v\\w\end{pmatrix} - A(\vec{u})\frac{\partial}{\partial x}\cdot\begin{pmatrix}u\\v\\w\end{pmatrix} + B(\vec{u})\cdot\begin{pmatrix}u\\v\\w\end{pmatrix} = \frac{\rho}{\eta}\cdot\frac{\partial}{\partial t}\begin{pmatrix}u\\v\\w\end{pmatrix} \tag{4}$$

and for the operator $\Xi(\vec{u})$: $\Xi(\vec{u}) = \frac{\partial^2}{\partial x^2} - A(\vec{u})\frac{\partial}{\partial x} + B(\vec{u})$.

To calculate an approximation of the solution of equation (3), a representation of the functions $u(t, x, y)$ by sequences of step functions will be used.

Let $u_\Delta : \mathbf{Z}^4 \to \mathbb{R}$ $(\Delta \in \mathbb{R}_+ := \{r \in \mathbb{R} | r < 0\})$ denote a family of functions that satisfies the approximation property (5):

$$u(t, x, y, z) = u_\Delta\left(i_t, i_x, i_y, i_z\right) + o(\Delta) \tag{5}$$

for $i_t, i_x, i_y, i_z \in \mathbf{Z}$ with $i_t := \left[\frac{t}{\Delta}\right], i_x := \left[\frac{x}{\Delta}\right], i_y := \left[\frac{y}{\Delta}\right], i_z := \left[\frac{z}{\Delta}\right]$, $[r] := \mathit{sup}\{z \in \mathbf{Z} \;|z \le r\}$.

The operators $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are replaced by discrete convolution operators

$s_{\Delta t}*$, $s_{\Delta x}*$, $s_{\Delta y}*$ and $s_{\Delta z}*$ with: $s_{\Delta\xi}(i) := \begin{cases} 1/\Delta & \text{for } i = 0 \\ -1/\Delta & \text{for } i = 1 \\ 0 & \text{else} \end{cases}$ $\quad (\xi = t, x, y, z)$, (6)

$k(i) = l(i) * h(i)$ is defined by: $k(i) := \left\{\begin{matrix} \sum_{j=-\infty}^{\infty} h(i-j)\cdot l(j) \text{ if this expression exists} \\ \text{undefined} \qquad\qquad \text{else} \end{matrix}\right\}$

and $k^{*2} := k * k$.

The operator $\Xi(\vec{u}(t, x, y, z))$ is represented by the six dimensional matrix:

$$\Xi_\Delta(\vec{u}_\Delta)(i_x, i'_x, i_y, i'_y, i_z, i'_z) := \tag{7}$$

$$s_{\Delta x}^{*2}(i_x, i'_x, i_y, i'_y, i_z, i'_z) + A(\vec{u})(i_x, i'_x, i_y, i'_y, i_z, i'_z)\cdot s_{\Delta x}(i_x, i'_x, i_y, i'_y, i_z, i'_z) + B(\vec{u})(i_x, i'_x, i_y, i'_y, i_z, i'_z)$$

with $A(\vec{u})(i_x, i'_x, i_y, i'_y, i_z, i'_z) = \frac{\rho}{\eta}u(i_t, i_x, i_y, i_z)$,

$$B(\vec{u})(i_x, i'_x, i_y, i'_y, i_z, i'_z) = s_{\Delta y}{}^{*2}(i_x, i'_x, i_y, i'_y, i_z, i'_z) - V(i_x, i'_x, i_y, i'_y, i_z, i'_z)\cdot s_{\Delta y}(i_x, i'_x, i_y, i'_y, i_z, i'_z)$$
$$+ s_{\Delta z}{}^{*2}(i_x, i'_x, i_y, i'_y, i_z, i'_z) - W(i_x, i'_x, i_y, i'_y, i_z, i'_z)\cdot s_{\Delta z}(i_x, i'_x, i_y, i'_y, i_z, i'_z) ,$$

$s_{\Delta\xi}(i_x, i'_x, i_y, i'_y, i_z, i'_z) := s_{\Delta\xi}(i_\xi - i'_\xi)$ for $\xi = x, y, z$,

$V(i_x, i'_x, i_y, i'_y, i_z, i'_z) = \frac{\rho}{\eta}v(i_t, i_x, i_y, i_z)$, $W(i_x, i'_x, i_y, i'_y, i_z, i'_z) = \frac{\rho}{\eta}w(i_t, i_x, i_y, i_z)$.

The matrix $\Xi_\Delta(\vec{u}_\Delta)(i_x, i'_x, i_y, i'_y, i_z, i'_z)$ is a block triangular matrix because of

$\Xi_\Delta(\vec{u}_\Delta)(i_x, i'_x, i_y, i'_y, i_z, i'_z) = 0 \quad \text{for} \quad i_x < i'_x \quad \text{or} \quad i_y < i'_y \quad \text{or} \quad i_z < i'_z.$

For block triangular matrices, the root and the inverse in the space of block triangular matrices can be calculated. This calculations are explained in Example 1 for the simple case of a two dimensional block matrix (Sommer, 1977).

Example 1: Calculation of the root of a block triangular matrix.

$$\Xi(x,y) := a(x,y)\frac{\partial^2}{\partial y^2} + b(x,y)\frac{\partial}{\partial y} + c(x,y) \leftrightarrow \Xi_\Delta(i_x, i_y) := \begin{pmatrix} \ddots \text{ main diagonal} & & & \\ a_{10} & a_{11} & a_{12} & 0 \\ 0 & a_{00} & a_{01} & a_{02} \\ 0 & 0 & a_{-10} & a_{-11} \\ 0 & 0 & 0 & a_{-20} \\ & & & \ddots \end{pmatrix}.$$

$$\Omega_\Delta(i_x, i_y) := \begin{pmatrix} \ldots b_{10} & b_{11} & b_{12} & b_{22}\ldots \\ \ldots 0 & b_{00} & b_{01} & b_{02}\ldots \\ \ldots 0 & 0 & b_{-10} & b_{-11}\ldots \\ \ldots 0 & 0 & 0 & b_{-20}\ldots \end{pmatrix} \text{ satisfies } \Omega_\Delta(i_x, i_y) \circ \Omega_\Delta(i_x, i_y) = \Xi_\Delta(i_x, i_y)$$

if the elements of the matrices $\Xi_\Delta(i_x, i_y)$ and $\Omega_\Delta(i_x, i_y)^2$ are equal or if the following equations hold:

(0) $\quad \forall j \in \mathbf{Z} : \quad a_{j0} = \pm b_{j0}^2$

(1) $\quad \forall j \in \mathbf{Z} : \quad a_{j1} = b_{j0} \cdot b_{j1} + b_{j1} \cdot b_{(j-1)0}$

(2) $\quad \forall j \in \mathbf{Z} : \quad a_{j2} = b_{j0} \cdot b_{j2} + b_{j1} \cdot b_{(j-1)1} + b_{j2} \cdot b_{(j-2)0}$

....

From equation (0) calculate b_{j0} for $j \in \mathbb{Z}$ and then from equation (1) b_{j1} for $j \in \mathbb{Z}$ and then from equation (2) b_{j2} for $j \in \mathbb{Z}$ ...and then from equation (k) b_{jk} for $j \in \mathbb{Z}$.

Step 3: Reduction of problem (3) to a homogenious difference equation for the operator $\Xi_\Delta(\vec{u}_\Delta)(i_x, i'_x, i_y, i'_y, i_z, i'_z)$:

The arguments given in this step show that the representation of $\frac{\partial}{\partial t}$ by a slightly modified convolution operator $\tilde{s}_{\Delta t}$ * reduces problem (3) to a homogenious difference equation for the operator $\Xi_\Delta(\vec{u}_\Delta)(i_x, i'_x, i_y, i'_y, i_z, i'_z)$.

For an approximation of $\vec{u}$ by the step functions $\vec{u}_\Delta$, we obtain from (3) the equations:

$$\Xi_\Delta(\vec{u}_\Delta)\vec{u}_\Delta = \frac{\rho}{\eta}\tilde{S}_{\Delta t} * \vec{u}_\Delta \tag{8a}$$

$$\tilde{S}_{\Delta t} * \vec{u}_\Delta(0, i_x, i_y, i_z) = 0 \tag{8b}$$

$$\vec{u}_\Delta\left(i_t, 0, i_y, i_z\right) = \vec{u}_{0\Delta}\left(i_t, i_y, i_z\right) \tag{8c}$$

where $\vec{u}_{\Delta 0}\left(i_t, i_y, i_z\right)$ is an approximation of $\vec{u}_0(t, y, z) \in C^\infty\left(\mathbb{R}^3, \mathbb{R}^3\right)$ by a step function with step length Δ and $\tilde{S}_{\Delta t}\, *$ is an approximation of the operator $S_{\Delta t}\, *$ defined:

$$\tilde{S}_{\Delta t}(i) := \begin{cases} \frac{1}{\Delta} & \text{for } i = 1 \\ -\frac{1}{\Delta} & \text{for } i = 2. \\ 0 & \text{else} \end{cases}$$

The discrete problem (8) will be solved following the order of the time instants i_t. Because of (3b), in the first step we have to solve the equation:

$$\Xi_\Delta\left(\vec{u}_\Delta\left(0, i_x, i_y, i_z\right)\right)\vec{u}_\Delta\left(1, i_x, i_y, i_z\right) = 0 \tag{9a}$$

with the boundary condition : $\vec{u}_\Delta\left(1, 0, i_y, i_z\right) = \vec{u}_{0\Delta}\left(1, i_y, i_z\right)$. (9b)

In step $i_t + 1$ $\left(i_t \in \mathbf{N}\right)$ an inhomogenious difference equation has to be solved:

$$\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)\vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right) = \frac{\rho}{\eta} \underbrace{\tilde{S}_{\Delta t} * \vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right)}_{\text{depends on } \vec{u}_\Delta\left(i_t', i_x, i_y, i_z\right) \text{ for } i_t' = i_t, i_t - 1}$$

$\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)^{-1}$ is a convolution operator that can be calculated with the methods of Example 1 from the operator $\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)$ and therefore the last equation can be transformed into the equation: (10a)

$$\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)\left(\vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right) - \underbrace{\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)^{-1} \frac{\rho}{\eta} \tilde{S}_{\Delta t} * \vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right)}_{\text{depends on } \vec{u}_\Delta\left(i_t', i_x, i_y, i_z\right) \text{ for } i_t' = i_t, i_t - 1}\right) = 0$$

where the boundary condition (10b) had to be satisfied:

$$\vec{u}_\Delta\left(i_t + 1, 0, i_y, i_z\right) = \vec{u}_{0\Delta}\left(i_t + 1, i_y, i_z\right) \tag{10b}$$

As the function $\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)^{-1} \frac{\rho}{\eta} \tilde{S}_{\Delta t} * \vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right)$ can be calculated from values $\vec{u}_\Delta\left(i_t', i_x, i_y, i_z\right)$ for $i_t' = i_t, i_t - 1$ obtained in the steps before, it remains to solve an equation:

$$\Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)\left(\vec{v}_\Delta\left(i_t + 1, i_x, i_y, i_z\right)\right) = 0 \tag{11a}$$

with: (11b)

$$\vec{v}_\Delta\left(i_t + 1, i_x, i_y, i_z\right) := \left(\vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right) - \Xi_\Delta\left(\vec{u}_\Delta\left(i_t, i_x, i_y, i_z\right)\right)^{-1} \frac{\rho}{\eta} \tilde{S}_{\Delta t} * \vec{u}_\Delta\left(i_t + 1, i_x, i_y, i_z\right)\right)$$

Step 4: The solution of the homogeneous difference equation (11) for the operator $\Xi_\Delta(\vec{u}_\Delta)$. Factorisation of the operator $\Xi_\Delta(\vec{u}_\Delta)$:

The equation (11a) is the discretisation of a homogeneous partial differential equation of second order. As shown in Example 2, the formal solution of such a differential equation is composed by two solution families. To transmit the solution method of Example 2 to the problem (3), the operator $\Xi_\Delta(\vec{u}_\Delta)$ has to be factorised:

$$\frac{\partial^2}{\partial x^2}-A(\vec{u})\frac{\partial}{\partial x}+B(\vec{u})=\left(\frac{\partial}{\partial x}+c_1(\vec{u})\right)\left(\frac{\partial}{\partial x}+c_2(\vec{u})\right) \tag{12}$$

where $c_1(\vec{u})=\begin{pmatrix} c_{1x}(\vec{u}) & 0 & 0 \\ 0 & c_{1y}(\vec{u}) & 0 \\ 0 & 0 & c_{1z}(\vec{u}) \end{pmatrix}$ and $c_2(\vec{u})=\begin{pmatrix} c_{2x}(\vec{u}) & 0 & 0 \\ 0 & c_{2y}(\vec{u}) & 0 \\ 0 & 0 & c_{2z}(\vec{u}) \end{pmatrix}$ are matrices with elements in $\mathbb{C}$ for fixed $i_x, i_y \in \mathbf{Z}$.

The explicit form of (12) is:

$$\frac{\partial^2}{\partial x^2}-A(\vec{u})\frac{\partial}{\partial x}+B(\vec{u})=\frac{\partial^2}{\partial x^2}+\left(c_1(\vec{u})+c_2(\vec{u})\right)\frac{\partial}{\partial x}+\frac{\partial c_2(\vec{u})}{\partial x}+c_1(\vec{u})\cdot c_2(\vec{u})$$

The expression $\frac{\partial c_2(\vec{u})}{\partial x}\cdot\vec{u}$ can be transformed: $\frac{\partial c_2(\vec{u})}{\partial x}\cdot\vec{u}=\frac{\partial c_2(\vec{u})}{\partial \vec{u}}\cdot\frac{\partial \vec{u}}{\partial x}\cdot\vec{u}=$

$$\begin{pmatrix} \left(\frac{\partial c_{2x}(\vec{u})}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial c_{2x}(\vec{u})}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial c_{2x}(\vec{u})}{\partial w}\cdot\frac{\partial w}{\partial x}\right) & 0 & 0 \\ 0 & \left(\frac{\partial c_{2y}(\vec{u})}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial c_{2y}(\vec{u})}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial c_{2y}(\vec{u})}{\partial w}\cdot\frac{\partial w}{\partial x}\right) & 0 \\ 0 & 0 & \left(\frac{\partial c_{2z}(\vec{u})}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial c_{2z}(\vec{u})}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial c_{2z}(\vec{u})}{\partial w}\cdot\frac{\partial w}{\partial x}\right) \end{pmatrix}\cdot\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial c_{2x}(\vec{u})}{\partial u}\cdot u & \frac{\partial c_{2x}(\vec{u})}{\partial v}\cdot u & \frac{\partial c_{2x}(\vec{u})}{\partial w}\cdot u \\ \frac{\partial c_{2y}(\vec{u})}{\partial u}\cdot v & \frac{\partial c_{2y}(\vec{u})}{\partial v}\cdot v & \frac{\partial c_{2y}(\vec{u})}{\partial w}\cdot v \\ \frac{\partial c_{2z}(\vec{u})}{\partial u}\cdot w & \frac{\partial c_{2z}(\vec{u})}{\partial v}\cdot w & \frac{\partial c_{2z}(\vec{u})}{\partial w}\cdot w \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Example 2. Solution of the homogeneous partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x,y,z)=0 \tag{E1}$$

A discretisation of the differential equation (E1) provides the difference equation (E2):

$$\left(s^{*2}_{\Delta x}+s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)u_\Delta\left(i_x,i_y,i_z\right)=0 \tag{E2}$$

As shown in Example 1, the square root $\sqrt{\left(s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)}$ of the operator $\left(s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)$ can be calculated and we obtain from equation (E2): (E3)

$$\left(s_{\Delta x}-\sqrt{-1}\sqrt{\left(s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)}\right)\cdot\left(s_{\Delta x}+\sqrt{-1}\sqrt{\left(s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)}\right)u_\Delta=\left(s_{\Delta x}+\sqrt{-1}\sqrt{\left(s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)}\right)\cdot\left(s_{\Delta x}-\sqrt{-1}\sqrt{\left(s^{*2}_{\Delta y}+s^{*2}_{\Delta z}\right)}\right)u_\Delta=0$$

Equation (E3) can be split into two equations:

$$\left(s_{\Delta x}-\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)u_{\Delta}=0 \quad \text{(E4+)} \quad \text{and} \quad \left(s_{\Delta x}+\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)u_{\Delta}=0 \quad \text{(E4-)}$$

The definition of $s_{\Delta x}$ together with (E4) yields:

$$u_{\Delta+}\left(i_x+1,i_y,i_z\right)=\left(id+\Delta\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)u_{\Delta+}\left(i_x,i_y,i_z\right) \qquad \text{(E5+)}$$

and $$u_{\Delta-}\left(i_x+1,i_y,i_z\right)=\left(id-\Delta\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)u_{\Delta-}\left(i_x,i_y,i_z\right) \qquad \text{(E5-)}$$

with the solution families:

$$u_{\Delta+}\left(i_x,i_y,i_z\right)=\left(id+\Delta\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)^{i_x}u_{\Delta+}\left(0,i_y,i_z\right) \qquad \text{(E6+)}$$

and $$u_{\Delta+}\left(i_x,i_y,i_z\right)=\left(id+\Delta\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)^{i_x}u_{\Delta+}\left(0,i_y,i_z\right) \qquad \text{(E6-)}$$

For $$V_{\Delta+}:=\operatorname{span}\left\{e_\lambda \,\middle|\, \begin{array}{l} e_\lambda \text{ is an Eigenvector of } \sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)} \\ \text{with Eigenvalue } \lambda \text{ and } \mathrm{Re}\lambda\le 0 \end{array}\right\}$$

and $$V_{\Delta-}:=\operatorname{span}\left\{e_\lambda \,\middle|\, \begin{array}{l} e_\lambda \text{ is an Eigenvector of } \sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)} \\ \text{with Eigenvalue } \lambda \text{ and } \mathrm{Re}\lambda> 0 \end{array}\right\}$$

(E5+) provides a smooth solution $u_+(x,y,z)$ for boundary values approximated by step functions in $V_{\Delta+}$ and (E5-) provides a smooth solution $u_-(x,y,z)$ for boundary values approximated by step functions in $V_{\Delta-}$. This statement is due to the following facts:

(a) All possible boundary is values can be approximated by step functions from $V_{\Delta+}\oplus V_{\Delta-}$ for $\Delta\to 0$, $x=\lim\limits_{\Delta\to 0} i_x\cdot\Delta$, $y=\lim\limits_{\Delta\to 0} i_y\cdot\Delta$, and. $z=\lim\limits_{\Delta\to 0} i_z\cdot\Delta$.

(b) For an Eigenfunction e_λ of the operator $\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}$ with the Eigenvalue λ we obtain from equation (E5$\pm$) with $\Delta=1/n$, $i_x=[n\cdot x]$ for $n\to\infty$::

$$u(x,y,z)=\lim_{n\to\infty}\left(id\pm \tfrac{1}{n}\sqrt{-1}\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}\right)^{n\cdot x}e_\lambda(y,z)=exp\left(\pm\lambda\sqrt{-1}\cdot x\right)\cdot e_\lambda(y,z)$$

(c) From (b) follows that for the selected solution families and for the representation of $u(x,y,z)$ by series of Eigenfunctions of the operator $\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}$, the coefficients are exponentially decreasing respective to x. As the Eigenfunctions of $\sqrt{\left(s_{\Delta y}^{*2}+s_{\Delta z}^{*2}\right)}$ are exponential functions the relation between the smoothness of a function and the decrease of the coefficients in their Eigenfunction-expansion can be deduced from a wellknown theorem from Fourier-Transformation F :

$f(x)$ n-times differentiable $\Rightarrow \lim\limits_{\omega\to\infty} F(f(x),\omega)/\omega^n=0 \quad \Rightarrow f(x)$ (n-2)-times differentiable

With the notation: $\frac{\partial \overset{\leftrightarrow}{c_2}(\vec{u})}{\partial \vec{u}} := \begin{pmatrix} \frac{\partial c_{2x}(\vec{u})}{\partial u}\cdot u & \frac{\partial c_{2x}(\vec{u})}{\partial v}\cdot u & \frac{\partial c_{2x}(\vec{u})}{\partial w}\cdot u \\ \frac{\partial c_{2y}(\vec{u})}{\partial u}\cdot v & \frac{\partial c_{2y}(\vec{u})}{\partial v}\cdot v & \frac{\partial c_{2y}(\vec{u})}{\partial w}\cdot v \\ \frac{\partial c_{2z}(\vec{u})}{\partial u}\cdot w & \frac{\partial c_{2z}(\vec{u})}{\partial v}\cdot w & \frac{\partial c_{2z}(\vec{u})}{\partial w}\cdot w \end{pmatrix}$ the factorisation (12)

is equivalent to the equation:

$$\frac{\partial^2}{\partial x^2} - A(\vec{u})\frac{\partial}{\partial x} + B(\vec{u}) = \frac{\partial^2}{\partial x^2} + \left(c_1(\vec{u}) + c_2(\vec{u}) + \frac{\partial \overset{\leftrightarrow}{c_2}(\vec{u})}{\partial \vec{u}} \right)\frac{\partial}{\partial x} + c_1(\vec{u})\cdot c_2(\vec{u})$$

and a comparison of the coefficients yields ($Id := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$):

$$-A(\vec{u})Id = c_1(\vec{u}) + c_2(\vec{u}) + \frac{\partial \overset{\leftrightarrow}{c_2}(\vec{u})}{\partial \vec{u}} \tag{13a}$$

$$B(\vec{u})Id = c_1(\vec{u})\cdot c_2(\vec{u}) \tag{13b}$$

or

$$\frac{\partial \overset{\leftrightarrow}{c_2}(\vec{u})}{\partial \vec{u}} = -A(\vec{u})Id - c_2(\vec{u}) - B(\vec{u})Id\cdot c_2(\vec{u})^{-1} \tag{14a}$$

The initial conditions of equation (14a) for $\vec{u} = 0$ are deduced from equation (3b):

$$A(\vec{0}) = 0 \text{ and } B(\vec{0}) = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{14b}$$

The operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ has the factorisation: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} =$

$$= \left(\frac{\partial}{\partial x} + \sqrt{-1}\sqrt{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}\right)\left(\frac{\partial}{\partial x} - \sqrt{-1}\sqrt{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}\right) = \left(\frac{\partial}{\partial x} - \sqrt{-1}\sqrt{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}\right)\left(\frac{\partial}{\partial x} + \sqrt{-1}\sqrt{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}\right)$$

Therefore from equation (14b), we deduce the initial conditions for the operator

$c_2(\vec{u})$ for $\vec{u} = 0$:

$$c_{2+}(\vec{0}) = \left(\frac{\partial}{\partial x} + \sqrt{-1}\sqrt{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}\right) \tag{14c+}$$

$$c_{2-}(\vec{0}) = \left(\frac{\partial}{\partial x} - \sqrt{-1}\sqrt{\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}\right) \tag{14c-}$$

Using this two initial conditions (14c), two solutions for $c_2(\vec{u})$ can be deduced from equation (14a). This solutions are called $c_{2+}(\vec{u})$ and $c_{2-}(\vec{u})$.

In correspondence to the discretisation used for $\vec{u}$,the solutions $c_{2+}(\vec{u})$ and $c_{2-}(\vec{u})$ can be approximated by step functions $c_{\Delta 2+}(\vec{u})$ and $c_{\Delta 2-}(\vec{u})$ with step size Δ depending on integer values $i_u, i_v, i_w \in \mathbf{Z}$. For this step functions, we obtain from equation (14a) the difference equation (with $e_u := \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_v := \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_w := \begin{pmatrix}0\\0\\1\end{pmatrix}$):

$$\begin{pmatrix} \frac{c_{\Delta 2x}(\vec{u}+\Delta e_u)-c_{\Delta 2x}(\vec{u})}{\Delta}\cdot u & \frac{c_{\Delta 2x}(\vec{u}+\Delta e_v)-c_{\Delta 2x}(\vec{u})}{\Delta}\cdot u & \frac{c_{2x}(\vec{u}+\Delta e_w)-c_{2x}(\vec{u})}{\Delta}\cdot u \\ \frac{c_{\Delta 2y}(\vec{u}+\Delta e_u)-c_{\Delta 2y}(\vec{u})}{\Delta}\cdot v & \frac{c_{\Delta 2y}(\vec{u}+\Delta e_v)-c_{\Delta 2y}(\vec{u})}{\Delta}\cdot v & \frac{c_{\Delta 2y}(\vec{u}+\Delta e_w)-c_{\Delta 2y}(\vec{u})}{\Delta}\cdot v \\ \frac{c_{\Delta 2z}(\vec{u}+\Delta e_u)-c_{\Delta 2z}(\vec{u})}{\Delta}\cdot w & \frac{c_{\Delta 2z}(\vec{u}+\Delta e_v)-c_{\Delta 2z}(\vec{u})}{\Delta}\cdot w & \frac{c_{\Delta 2z}(\vec{u}+\Delta e_w)-c_{\Delta 2z}(\vec{u})}{\Delta}\cdot w \end{pmatrix} =$$

$$= -A_\Delta(\vec{u})\mathrm{Id} - c_{\Delta 2}(\vec{u}) - B_\Delta(\vec{u})\mathrm{Id}\cdot c_{\Delta 2}(\vec{u})^{-1}$$

or

$$\begin{pmatrix} c_{\Delta 2x}(\vec{u}+\Delta e_u)\cdot u & c_{\Delta 2x}(\vec{u}+\Delta e_v)\cdot u & c_{2x}(\vec{u}+\Delta e_w)\cdot u \\ c_{\Delta 2y}(\vec{u}+\Delta e_u)\cdot v & c_{\Delta 2y}(\vec{u}+\Delta e_v)\cdot v & c_{\Delta 2y}(\vec{u}+\Delta e_w)\cdot v \\ c_{\Delta 2z}(\vec{u}+\Delta e_u)\cdot w & c_{\Delta 2z}(\vec{u}+\Delta e_v)\cdot w & c_{\Delta 2z}(\vec{u}+\Delta e_w)\cdot w \end{pmatrix} = \tag{15}$$

$$= \begin{pmatrix} c_{\Delta 2x}(\vec{u})\cdot u & c_{\Delta 2x}(\vec{u})\cdot u & c_{2x}(\vec{u})\cdot u \\ c_{\Delta 2y}(\vec{u})\cdot v & c_{\Delta 2y}(\vec{u})\cdot v & c_{\Delta 2y}(\vec{u})\cdot v \\ c_{\Delta 2z}(\vec{u})\cdot w & c_{\Delta 2z}(\vec{u})\cdot w & c_{\Delta 2z}(\vec{u})\cdot w \end{pmatrix} - \Delta\cdot\left(A_\Delta(\vec{u})\mathrm{Id} + c_{\Delta 2}(\vec{u}) + B_\Delta(\vec{u})\mathrm{Id}\cdot c_{\Delta 2}(\vec{u})^{-1}\right).$$

From equation (15) and the initial conditions (14c), we have the following consequences:

$$c_{2x}(\vec{u}) = c_{2x}(u),\quad c_{2y}(\vec{u}) = c_{2y}(v),\quad c_{2z}(\vec{u}) = c_{2z}(w) \tag{16}$$

and

$$c_{\Delta 2x}(u+\Delta) = c_{\Delta 2x}(u) - \Delta\cdot\left(A_\Delta(u) + c_{\Delta 2x}(u) + B_\Delta(\vec{u})\cdot c_{\Delta 2x}(u)^{-1}\right) \tag{17a}$$

$$c_{\Delta 2y}(v+\Delta) = c_{\Delta 2y}(v) - \Delta\cdot\left(A_\Delta(u) + c_{\Delta 2y}(v) + B_\Delta(\vec{u})\cdot c_{\Delta 2y}(u)^{-1}\right) \tag{17b}$$

$$c_{\Delta 2z}(w+\Delta) = c_{\Delta 2z}(w) - \Delta\cdot\left(A_\Delta(u) + c_{\Delta 2z}(w) + B_\Delta(\vec{u})\cdot c_{\Delta 2z}(w)^{-1}\right) \tag{17c}$$

The equations (17) are (for i_t and i_x fixed) equations for infinite dimensional triangular four-dimensional matrices $c_{\Delta 2x}(u+\Delta)$, $c_{\Delta 2y}(u+\Delta)$, $c_{\Delta 2z}(u+\Delta)$ with:

$$B(\vec{u})(i_y, i'_y, i_z, i'_z) = \begin{cases} \frac{1}{\Delta^2} - \frac{\rho}{\eta\cdot\Delta} v(i_t, i_x, i_y, i_z) + \frac{1}{\Delta^2} - \frac{\rho}{\eta\cdot\Delta} w(i_t, i_x, i_y, i_z) & \text{for } i_y = i'_y, i_z = i'_z \\ -\frac{2}{\Delta^2} + \frac{\rho}{\eta\cdot\Delta} v(i_t, i_x, i_y, i_z) & \text{for } i_y + 1 = i'_y, i_z = i'_z \\ -\frac{2}{\Delta^2} + \frac{\rho}{\eta\cdot\Delta} w(i_t, i_x, i_y, i_z) & \text{for } i_y = i'_y, i_z + 1 = i'_z \\ \frac{1}{\Delta^2} & \text{for } i_y + 2 = i'_y, i_z = i'_z \\ \frac{1}{\Delta^2} & \text{for } i_y = i'_y, i_z + 2 = i'_z \\ 0 & \text{else} \end{cases}$$

and

$$A(u)(i_y, i'_y, i_z, i'_z) = \begin{cases} u(i_t, i_x, i_y, i_z) & \text{for } i_y = i'_y, i_z = i'_z \\ 0 & \text{else} \end{cases}$$

Equation (17) can be solved beginning with the main diagonal $i_y = i'_y, i_z = i'$ and then calculating the first diagonals on the right of the main diagonal $i_y + 1 = i'_y, i_z = i'$ and $i_y = i'_y, i_z + 1 = i'$ and so on, using a similar procedure as shown for the calculation of $\Omega_\Delta(i_x, i_y)$ in Example 1. This algorithm proves the existence of a solution of (17) and provides an approximation of this solution defined in bounded regions. For the two initial conditions (14c+) and (14c-) two solutions $c_{2+}(\vec{u})$ and $c_{2-}(\vec{u})$ are obtained.

Remark: The solutions $c_{2+}(\vec{u})$ and $c_{2-}(\vec{u})$ of equation (17) may depend on the path from $\vec{0}$ to $\vec{u}$.This path dependence does not destroy the solution method. The operators $c_{2+}(\vec{u})$, $c_{2-}(\vec{u})$, and $c_{1+}(\vec{u})$ $c_{1-}(\vec{u})$ are not commutative.

Using the operators $c_{\Delta 2+}(\vec{u})$ and $c_{\Delta 2-}(\vec{u})$, equation (11a) can be divided into the two equations:

$$\left(S_{\Delta x} + c_{\Delta 2+}(\vec{u})\right)\vec{v}\left(i_t + 1, i_x, i_y, i_z\right) = 0 \tag{18a}$$

and

$$\left(S_{\Delta x} + c_{\Delta 2-}(\vec{u})\right)\vec{v}\left(i_t + 1, i_x, i_y, i_z\right) = 0 \tag{18b}$$

or equivalently:

$$\vec{v}\left(i_t + 1, i_x + 1, i_y, i_z\right) = \left(\delta + c_{\Delta 2+}(\vec{u})\right)\vec{v}\left(i_t + 1, i_x, i_y, i_z\right) \tag{19a}$$

$$\vec{v}\left(i_t + 1, i_x + 1, i_y, i_z\right) = \left(\delta + c_{\Delta 2-}(\vec{u})\right)\vec{v}\left(i_t + 1, i_x, i_y, i_z\right) \tag{19b}$$

The equations (18) correspond to the equations (E4) of Example2. The following Lemma shows that the solution decomposition method can also be applied for the equations (18). The only difference is that the operator $\Xi(\vec{u})$ depends on $\vec{u}$ and this entails the dependence of the space partition $V_\Delta = V_{\Delta+}(\vec{u}) \oplus V_{\Delta-}(\vec{u})$ from $\vec{u}$. As $\vec{u}$ is changing in time and in the parameter x , in each solution step (for each pair $i_t + 1, i_x$), a new partition has to be calculated.

Lemma: The space of all summable step functions V can be decomposed into two subspaces $V_\Delta = V_{\Delta+}(\vec{u}) \oplus V_{\Delta-}(\vec{u})$ such that:

$V_{\Delta+}(\vec{u}) \subseteq \text{span}\left\{e_\lambda \middle| e_\lambda \text{ is an eigenvector for the operator } c_{\Delta 2+}(\vec{u}) \text{ and the eigenvalue } \lambda \text{ with } \text{Re}(\lambda) \le \theta\right\}$

$V_{\Delta-}(\vec{u}) \subseteq \text{span}\left\{e_\lambda \middle| e_\lambda \text{ is an eigenvector for the operator } c_{\Delta 2-}(\vec{u}) \text{ and the eigenvalue } \lambda \text{ with } \text{Re}(\lambda) \le \theta\right\}$

where the minimal value that can be chosen for θ is bounded ($\theta < \infty$).

Proof: The lemma is a consequence of the following facts:

(A) A change of $\vec{u}$ does not imply a change of the type of the differential operator $\Xi(\vec{u})$ ($\vec{u}$ does not affect the coefficients of the second derivations (Tijonov 1983 p.24)).

(B) The eigenfunctions of the operators $c_{2+}(\vec{u})$ and $c_{2-}(\vec{u})$ form a basis in the space V_Δ. This property is deduced from the arguments given in Example 2 for equation (6).

Therefore $V_\Delta = V_{\Delta+}(\vec{u}) \oplus V_{\Delta-}(\vec{u})$ holds for some θ. $\theta < \infty$ can be deduced from the statements:

(C) For $|\lambda| \to \infty$ the eigenvector e_λ of $c_{2+}(\vec{u})$ $(c_{2-}(\vec{u}))$ and the eigenvalue λ tends to an eigenvector $\tilde{e}_\lambda$ of $\sqrt{\partial^2/\partial y^2 + \partial^2/\partial z^2}$ $(-\sqrt{\partial^2/\partial y^2 + \partial^2/\partial z^2})$ and the eigenvalue λ .

For the operators $\sqrt{\partial^2/\partial y^2 + \partial^2/\partial z^2}$ and $-\sqrt{\partial^2/\partial y^2 + \partial^2/\partial z^2}$ the Lemma had been proved in Example 2 for $\theta=0$ (compare the definition of V_+ and V_-).

(D) A very small change of the vectors of a basis does not destroy their linear independence. //

Let $P_{\Delta+}(\vec{u}(i_t, i_x, i_y, i_z))$, $P_{\Delta-}(\vec{u}(i_t, i_x, i_y, i_z))$ denote the projections

$P_{\Delta+}(\vec{u}(i_t, i_x, i_y, i_z)): V_\Delta \twoheadrightarrow V_{\Delta+}(\vec{u}(i_t, i_x, i_y, i_z)), P_{\Delta-}(\vec{u}(i_t, i_x, i_y, i_z)): V_\Delta \twoheadrightarrow V_{\Delta-}(\vec{u}(i_t, i_x, i_y, i_z))$

Using equation (19), with the solution decomposition method, we find the equations (20)

$$\vec{v}_{\Delta+}(i_t + 1, i_x + 1, i_y, i_z) = (\delta + c_{\Delta 2+}(\vec{u}))P_{\Delta+}(\vec{u}_\Delta(i_t, i_x, i_y, i_z))\vec{v}_\Delta(i_t + 1, i_x, i_y, i_z) \quad (20a)$$

$$\vec{v}_{\Delta-}(i_t + 1, i_x + 1, i_y, i_z) = (\delta + c_{\Delta 2-}(\vec{u}))P_{\Delta-}(\vec{u}_\Delta(i_t, i_x, i_y, i_z))\vec{v}_\Delta(i_t + 1, i_x, i_y, i_z) \quad (20b)$$

$$\vec{v}_\Delta(i_t + 1, i_x + 1, i_y, i_z) = \vec{v}_{\Delta+}(i_t + 1, i_x + 1, i_y, i_z) + \vec{v}_{\Delta-}(i_t + 1, i_x + 1, i_y, i_z) \quad (20c)$$

From equation (10b) and equation (11b) we deduce the initial condition: (20d)

$$\vec{v}_\Delta(i_t + 1, 0, i_y, i_z) = \vec{u}_{0\Delta}(i_t + 1, i_y, i_z) - \Xi_\Delta(\vec{u}_\Delta(i_t, 0, i_y, i_z))^{-1} \tilde{S}_{\Delta t} * \vec{u}_\Delta(i_t + 1, i_x, i_y, i_z)$$

Equation (11b) provides the sequence $\left\{\vec{u}_\Delta(i_t + 1, i_x, i_y, i_z)\right\}_{\Delta=\frac{1}{n}(n\in\mathbb{N})}$ by the equation:

$$\vec{u}_\Delta(i_t + 1, i_x, i_y, i_z) := \left(\vec{v}_\Delta(i_t + 1, i_x, i_y, i_z) - \Xi_\Delta(\vec{u}_\Delta(i_t, i_x, i_y, i_z))^{-1} \tilde{S}_{\Delta t} * \vec{u}_\Delta(i_t + 1, i_x, i_y, i_z)\right)$$

The sequence $\left\{\vec{u}_\Delta(i_t + 1, i_x, i_y, i_z)\right\}_{\Delta=\frac{1}{n}(n\in\mathbb{N})}$ is a solution of (3) in the sense of Non-standard Analysis (Laugwitz, 1977) or can be interpreted as an approximation of the solution by step functions.

(II) Definition of the membership degree to the region of separation:
The solution of the Navier Stokes equation defined in (I), had been obtained in complete analogy to the solution decomposition method presented in Example 2. But there is one important difference. In the linear case, the decomposition of the space $V = V_+ \oplus V_-$ could be chosen fixed for the complete solution algorithm. In the nonlinear case, in each solution step $i_x \mapsto i_x + 1$ the operator $\Xi_\Delta(\vec{u}_\Delta(i_t, i_x, i_y, i_z))$ changes and it is therefore necessary to calculate a new decomposition $V_\Delta = V_{\Delta+}(\vec{u}) \oplus V_{\Delta-}(\vec{u})$ in each step. By reason of this change of the decomposition, the smoothness of the solution $\vec{u}(t, x, y, z)$ can no longer be deduced from the solution decomposition method as in the linear case.

The reason is that in each step, because of the change of the basis vectors, solution parts represented by eigenvectors of small eigenvalues can be transformed to parts represented by eigenvectors of large eigenvalues in the changed basis. The decrease for $i_x \to \infty$ of the coefficients of the eigenvectors of large eigenvalues in the series can no longer be proved in the nonlinear case. This is the reason, why in general the smoothness of the solution of (3) can not be proved.

From the previous discussion we deduce an idea for the definition of unsmoothness or of **the membership degree to a region of separation**:

(RS): *The strength of the transfer of solution parts represented by eigenvectors of* $c_{\Delta 2+}\left(\bar{u}\left(i_t+1, i_x, i_y, i_z\right)\right)$ $\left(c_{\Delta 2-}\left(\bar{u}\left(i_t+1, i_x, i_y, i_z\right)\right)\right)$ *with small eigenvalues to eigenvectors of* $c_{\Delta 2+}\left(\bar{u}\left(i_t+1, i_x+1, i_y, i_z\right)\right)$ $\left(c_{\Delta 2-}\left(\bar{u}\left(i_t+1, i_x+1, i_y, i_z\right)\right)\right)$ *with large eigenvalues, produced by the change of the space decomposition, is a measure for the unsmoothness of the solution.*

A formal expression for this measure will now be deduced. We call the value of this measure in (t, x, y, z) the degree of separation of the solution in the time instant t and the space point (x, y, z).

With a Taylor series expansion of $c_{\Delta 2+}(\bar{u})$ and $c_{\Delta 2-}(\bar{u})$

$$c_{\Delta 2\pm}(\bar{u}+\Delta\bar{u}) = c_{\Delta 2\pm}(\bar{u}) + \frac{\partial c_{\Delta 2\pm}(\bar{u})}{\partial \bar{u}} \cdot \Delta\bar{u} + \text{terms of higher order}$$

and the eigenvectors e_λ, $\tilde{e}_\lambda$ to the eigenvalue λ for the operator $c_{\Delta 2\pm}(\bar{u})$ res. $c_{\Delta 2\pm}(\bar{u}+\Delta\bar{u})$ we obtain the equations:

$$\lambda\tilde{e}_\lambda = c_{\Delta 2\pm}(\bar{u}+\Delta\bar{u})\tilde{e}_\lambda = c_{\Delta 2\pm}(\bar{u})\tilde{e}_\lambda + \frac{\partial c_{\Delta 2\pm}(\bar{u})}{\partial \bar{u}} \cdot \Delta\bar{u} \cdot \tilde{e}_\lambda + \text{terms of higher order}$$

and $\quad \lambda e_\lambda = c_{\Delta 2\pm}(\bar{u}) e_\lambda$.

Consequently we have

$$\lambda\tilde{e}_\lambda - \lambda e_\lambda = c_{\Delta 2\pm}(\bar{u})\left(\tilde{e}_\lambda - e_\lambda\right) + \frac{\partial c_{\Delta 2\pm}(\bar{u})}{\partial \bar{u}} \cdot \Delta\bar{u} \cdot \tilde{e}_\lambda + \text{terms of higher order}$$

or $\quad \Delta e_\lambda := \tilde{e}_\lambda - e_\lambda = \left(\lambda \mathrm{Id} - c_{\Delta 2\pm}(\bar{u})\right)^{-1} \frac{\partial c_{\Delta 2\pm}(\bar{u})}{\partial \bar{u}} \cdot \Delta\bar{u} \cdot \tilde{e}_\lambda + \text{terms of higher order}.$

(For linear partial differential equations, from $\frac{\partial c_{\Delta 2\pm}(\bar{u})}{\partial \bar{u}} \equiv 0$ we deduce $\Delta e_\lambda \equiv 0$.)

A formal expression that represents the idea formulated in (RS) is therefore:

$$R(t,x) := \max_{\substack{\|\Delta\bar{u}\|=1 \\ e_{\lambda\pm}, e_{\tilde{\lambda}\pm} \text{ eigenvectors of } c_{\Delta 2\pm}(\bar{u}) \\ Re\lambda, Re\tilde{\lambda} \le \theta}} \left| \left\langle \left(\lambda \mathrm{Id} - c_{\Delta 2\pm}(\bar{u})\right)^{-1} \frac{\partial c_{\Delta 2\pm}(\bar{u})}{\partial \bar{u}} \cdot \Delta\bar{u} \cdot e_\lambda, e_{\tilde{\lambda}} \right\rangle \cdot \tilde{\lambda} \right|$$

(θ is defined in the Lemma of (AI))

(The multiplicator $\tilde{\lambda}$ is necessary to guarantee the dominant influence of the transfer to eigenvectors of high eigenvalues. $\langle \bullet, \bullet \rangle$ denotes the scalar-product of $\mathbb{R}^3$)

To obtain also a dependence of y, z, the definition of R can be modified: $\tilde{R}(t, x, y_0, z_0) :=$

$$\max_{\substack{\|\Delta\vec{u}\|=1 \\ e_{\lambda\pm},\, e_{\tilde{\lambda}\pm} \text{ eigenvectors of } c_{\Delta 2\pm}(\vec{u}) \\ Re\,\lambda,\, Re\,\tilde{\lambda} \le \theta \text{ (}\theta\text{ is defined in the Lemma of section 3)}}} \left| \left\langle \left(\lambda Id - c_{\Delta 2\pm}(\vec{u})\right)^{-1} \frac{\partial c_{\Delta 2\pm}(\vec{u})}{\partial \vec{u}} \cdot \Delta\vec{u} \cdot e_{\lambda}, \varphi(y_0 - y, z_0 - z) \cdot e_{\tilde{\lambda}} \right\rangle \cdot \tilde{\lambda} \right|$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is smooth, $\varphi \ge 0$ and $\varphi(x, y) = 0$ for $x^2 + y^2 > \varepsilon_\varphi$.

Let $\psi : [0, \infty] \to [0, 1]$ denote a strict monotone continuous bijection. The **membership degree to a region of separation** is defined: $\phi(t, x, y_0, z_0) := \psi\left(\tilde{R}(t, x, y_0, z_0)\right)$

Remarks: **(A)** The solution algorithm (I) minimises $\phi(t, x, y_0, z_0)$.

(B) For linear partial differential equations from $\Delta e_\lambda \equiv 0$ we deduce $\phi(t, x, y_0, z_0) \equiv 0$.

(C) The function $\phi(t, x, y_0, z_0)$ defines in each time instant the **solution-granules** (Zadeh, 1997) by the connected regions in the complement of the whole space and the region of separation.

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