Some Remarks and Experiments with Fourier Decision Diagrams on Finite Non-Abelian Groups

Radomir S. Stanković, Dejan Milenović, Dragan Janković
Dept. of Computer Sceince, Faculty of Electronics
Beogradska 14, 18 000 Niš, Yugoslavia
Fax: 381 18 531-276, e-mail: stanko@503c1.elfak.ni.ac.yu

Abstract

This paper presents a study of the complexity of Fourier Decision Diagrams on finite Non-Abelian Groups (FNADDs) for representation of discrete functions. FNADDs are introduced as a generalization of Spectral transform decision diagrams. They are offered as a solution for depth reduction problem in DDs representations of discrete functions.

This study is intended to prove experimentally basic features of FNADDs. It is performed through some examples showing complexity of FNADDs for switching and multiple-output switching functions. Comparison with some other decision diagrams is provided. It is shown that FNADDs are very efficient in representation of algebraic functions.

Key words: Discrete functions, Switching functions, Decision diagrams, Fourier decision diagrams, Non-Abelian groups.

1 Discrete Functions

In this paper, it is assumed under the term discrete function a mapping $f : G \to P$, where G is a finite group of order g and P a field that may be the complex field or a finite field. If G is a decomposable group,

$$G = \times_{i=1}^{n} G_i, \quad g = \prod_{i=1}^{n} g_i, \quad g_1 \leq g_2 \leq \ldots \leq g_n. \tag{1}$$

where g_i is order of the constituent subgroup G_i, then f is an n variable function $f(x_1, \ldots, x_n)$, $x_i \in G_i$.

International Journal of Computing Anticipatory Systems, Volume 3, 1999
Edited by D. M. Dubois, CHAOS, Liège, Belgium, ISSN 1373-5411 ISBN 2-9600179-4-3

A function f defined in g points, can be considered as a function of different number of variables, depending on the group G of order g assumed for the domain group for f.

Example 1 *A three-variable switching function $f(x_1, x_2, x_3)$ is usually considered as a function on $G_8 = C_2 \times C_2 \times C_2$, where $C_2 = (\{0,1\}, \oplus)$, and $\oplus$ denotes the componentwise addition modulo 2, EXOR. If this decomposition for G is assumed, f is defined by the vector*

$$\mathbf{F} = [f(000), f(001), f(010), f(011), f(100), f(101), f(110), f(111)]^T.$$

It can be alternatively considered as a two-variable function $f(x_1, X_2)$, with $x_1 \in C_2$, $X_2 \in C_4$, where C_4 is a cyclic group of order 4. Thus, f is considered as a function on $G_8 = C_2 \times C_4$ and can be represented by the vector

$$\mathbf{F} = [f(00), f(01), f(02), f(03), f(10), f(11), f(12), f(13)]^T.$$

The same function can be considered as an one-variable function on $G_8 = Q_2$, where Q_2 is the quaternion group defined in Addendum. In that case, it is represented by the vector

$$\mathbf{F} = [f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7)]^T.$$

2 Decision Diagrams

Decision Diagrams (DD) are convenient data structure for representation of discrete functions. A DD is a rooted acyclic graph consisting of the root node, a set of non-terminal nodes, and a set of constant nodes connected with edges. For a given function f, a DD is designed by the reduction of the corresponding Decision Tree (DT) representing f.

Complexity of a DD is usually characterized by its

1. size - the total number of nodes (non-terminal and constant nodes),
2. width - the maximal number of non-terminal nodes per a level, where a level in a DD consists of nodes to which the same variable in f is assigned.
3. depth - the number of levels in the DD.

Various classes of DDs are defined to represent different classes of discrete functions (Sasao, Fujita, 1996). They are derived as extensions and generalizations of Binary Decision Diagrams (BDDs) (Ackers, 1978), (Bryant, 1986). The introduction of different DDs was directed towards two basic goals:

1. Extension of DDs representations to different classes of discrete functions,

2. Optimization of DDs representations.

Note as examples for the first direction, the Multi-Terminal Binary DDs (MT-BDDs) (Clarke et al., 1993), which are an extension of BDDs to represent integer-valued or complex-valued functions on the dyadic groups. Then, extensions of MT-BDDs to p-adic groups for representation of multiple-valued (MV) functions and complex-valued functions denoted, respectively, as the Multiple-place DDs (MDDs) (Srinivasan et al., 1990), and the Multi-Terminal DDs (MTDDs) (Miller, 1994). Extensions to matrix-valued functions are given through the Fourier DDs on non-Abelian Groups with Preprocessing (FNAPDDs) (Stanković, 1997).

Examples for the other direction of generalization of BDDs are Kronecker DDs (KDDs) and Pseudo-Kronecker DDs (PKDDs) (Sasao, 1996), (Sasao, 1998). Their integer generalizations, the Binary Moment DDs (BMDs) (Bryant, Chen, 1994), are closely related to the Arithmetic Transform DDs (ACDDs) (Stanković, Sasao, Moraga, 1996). To the same class belong Fourier DDs on Abelian (FADDs) and non-Abelian groups (FNADDs) for both MV and complex-valued functions (Stanković, 1996a). Further examples are the Edge-Valued Binary DDs (EVBDDs) (Lai et al., 1994), *BMDs (Bryant, Chen, 1994) and other related DDs with attributed edges (Drechsler, Becker, 1996).

Some classes of DDs are introduced purposely to solve some particular problems in logic design (Hasan-Babu, Sasao, 1997), (Iguchi, Sasao, Matsuura, 1997).

3 Depth Reduction in DDs

Primary optimization goal in DDs representations is reduction of the size of the DDs for a given function f. The reason for that is simple. In many applications related with calculations and realizations based on DDs representations of discrete functions, some calculation subprocedure or a circuit, respectively, is assigned to each non-terminal node. However, in many applications, reduction of the depth of DDs is another very important problem. For example, in DDs based design methods for logic networks, the propagation delay in the produced network is directly proportional to the depth of the DD.

For the depth reduction, the use of nodes with increased number of outgoing edges is proposed (Sasao, Butler, 1994). In Quaternary DDs (QDDs) (Sasao, Butler, 1994), depth reduction is performed through recoding of pairs of variables in f. In this method an n-variable function f, n-even number, is mapped into a function f_q of $n/2$ four-valued variables. It is represented by a QDD whose nodes have four outgoing edges. In that way, a BDD for f consisting of n levels is replaced by QDD for f with $n/2$ levels.

In a group-theoretic approach, the method can be interpreted as optimization of DDs representation by changing the decomposition of the domain group G for f (Stanković, 1996a). In QDDs, the domain group G for f that is the product of n

cyclic subgroups C_2 of order 2 is decomposed into the product of $n/2$ cyclic subgroups C_4 of order 4.

A generalization of the method by coding arbitrary subsets of variables in f is straightforward. It assumes decomposition of the domain group G for f into the product of subgroups of arbitrary orders.

However, the number of outgoing edges of nodes determines the width of the DD. Therefore, in many cases, this method for depth reduction increases the width of the DD for f.

For that reason, the use of non-Abelian subgroups in the decomposition of G on which DT for f is defined was proposed in (Stanković, 1996a), and Fourier DDs on finite non-Abelian groups (FNADDs) were introduced. By extending spectral interpretation of DDs (Stanković, 1995), (Stanković, Sasao, Moraga, 1996), we may say that the FNADDs are defined with respect to a particular basis, the Fourier basis on finite non-Abelian groups. With FNADDs reduction of both depth and width is achieved thanks to the properties of the Fourier transforms on non-Abelian groups.

In this paper, we compare through some experimental results the complexity of FNADDs to the complexity of some other DDs.

In Section 4, we briefly repeat some basic definitions of Fourier transform on finite non-Abelian groups. In Sections 5,6,7, and 8 we present definitions of FNADDs in terms of this transform and discuss their basic features. In Section 9, experimental results estimating complexity of FNADDs are presented. In Section 10, some closing remarks on the efficiency of FNADDs are given.

4 Fourier Transform on Finite Non-Abelian Groups

Denote by P the complex field or a finite field. Henceforth it will be assumed that:

1. $charP = 0$, or $charP$ does not divide g,

2. P is a so-called splitting field for G,

where $charP$ is the characteristic of P. These assumptions ensure existence of the Fourier transform on G over P.

Denote by K the number of equivalence classes of irreducible representations of G over P. Each equivalence class contains just one unitary representation.

Denote the K unitary irreducible representations of G in some fixed order by $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{K-1}$. We denote by $\mathbf{R}_w(x)$ the values of $\mathbf{R}_w$ at $x \in G$. Note that $\mathbf{R}_w(x)$ stands for a non-singular $(r_w \times r_w)$ matrix with elements $R_w^{(i,j)}(x)$, $i, j = 11, \ldots, r_w$ in P.

If G is representable in the form (1), then its unitary irreducible representations can be obtained as the Kronecker product of the unitary irreducible representations of subgroups G_i, $i = 1, \ldots, n$. Therefore, the number K of unitary irreducible representations of G can be expressed as

$$K = \prod_{i=1}^{n} K_i,$$

where K_i is the number of unitary irreducible representations of the i-th subgroup G_i.

The functions $\mathbf{R}_w^{(i,j)}(x)$, $w = 0, 1, \ldots, K-1$, $x \in G$, $i, j = 1, \ldots, r_w$ form an orthogonal system in $P(G)$.

Definition 1 *The direct and inverse Fourier transforms of a function $f \in P(G)$ are defined respectively by,*

$$\mathbf{S}_f(w) = r_w g^{-1} \sum_{u=0}^{g-1} f(u)\mathbf{R}_w(u^{-1}), \tag{2}$$

$$f(x) = \sum_{w=0}^{K-1} Tr(\mathbf{S}_f(w)\mathbf{R}_w(x)), \tag{3}$$

where for a matrix $\mathbf{Q}$, $Tr(\mathbf{Q})$ denotes the trace of $\mathbf{Q}$, i.e., the sum of elements on the main diagonal of $\mathbf{Q}$.

Here and in the sequel we shall assume, without explicitly saying so, that all arithmetical operations in (2) and (3) are carried out in the field P.

5 Fourier Decision Trees

Decision trees are defined by using some function expansions (Sasao, 1996). For example, the Shannon tree is defined by using the Shannon expansion rule $f = \overline{x}_i f_0 \oplus x_i f_1$, where x_i is a switching variable, $\overline{x}_i$ is its logic complement, and $f_0 = f(x_i = 0)$ and $f_1 = f(x_i = 1)$. This expansion rule is applied recursively to all the variables in f. Nodes corresponding to the same variable x_i form the i-th level in the Shannon DT. Thus, the depth of the Shannon tree is equal to n - the number of variables in f.

The function expansion determined by the Fourier transform on finite groups was used to define the Fourier transform decision trees (Stanković, 1996a). For the differences in the properties of the Fourier transform on Abelian and non-Abelian groups (Stanković, 1990), the distinction between these two cases has been made.

From FFT theory, Fourier transform on a finite group G decomposable in the form (1) may be considered as the n-dimensional Fourier transform on the constituent subgroups G_i (Stanković, Stojić, Stanković, 1996). The Fourier transform on G_i is used to define an expansion for f with respect to the i-th variable. This mapping is performed at the nodes at the i-th level in the Fourier decision tree to associate f to this DT in the same way as the Shannon decomposition is used to associate f to a BDD (Stanković, 1996a).

Definition 2 *Fourier decision tree on G is defined as the decision tree whose nodes a the i-th level represent functions*

$$f(x_i) = \sum_{w_i \in \Gamma_i} Tr(\mathbf{S}_f(w_i)\mathbf{R}_{w_i}(x_i)), \quad x_i \in G_i, \tag{4}$$

where Γ_i is the dual object of G_i, and $\mathbf{R}_{w_i}$ are the unitary irreducible representations of G_i.

Expansion rule for f defined by (4) is denoted as the Fourier expansion for f with respect to x_i. With this definition, the Fourier decision trees are the decision trees on G in which each path from the root node up to the constant nodes corresponds to an unitary irreducible representation of G. A node at the i-th level has K_i outgoing edges denoted by $\mathbf{R}_{w_i}(u), u \in \Gamma_i$.

In Fourier DT for a given f, the values of constant nodes are the Fourier coefficients of f. The function f is determined from its Fourier DT by using the inverse Fourier transform on G_i defined as in (3), and by following the labels at the edges in the Fourier DT in the same way as in any other DT. This statement is a direct generalization of the considerations in (Stanković, 1995) and (Stanković, Sasao, Moraga, 1996) to DTs on arbitrary, not necessarily Abelian, groups.

The Fourier coefficients corresponding to the group representations with orders $r_w > 1$, $w = 1, \dots K-1$, are $(r_w \times r_w)$ matrices. Therefore, Fourier DTs on finite non-Abelian groups are matrix-valued DTs, since some of their constant nodes are $(r_w \times r_w)$ matrices. The matrix-valued coefficients may be also represented by the DTs by using the method of representation of matrices by the decision diagrams (Clarke et al., 1993). In that way, we derive the number-valued Fourier DTs on finite non-Abelian groups that may be the integer-valued or complex-valued DTs depending on the field P over which the group representations for G are taken. The number of constant nodes in number-valued Fourier DT on a non-Abelian group is equal to the order g of G. Thus, it is the same as in the Fourier DT, or any other DT on the Abelian group of the same order. Comparing to the matrix-valued Fourier DTs, the number of levels in an integer-valued or complex-valued Fourier DT may be increased. The representation of matrix-valued constant nodes by DTs may introduce some levels, depending of the order of the matrix in the matrix-valued node. However, some non-terminal nodes still may be saved, since not all the Fourier coefficients are the matrix-valued (Stanković, 1996b).

The non-terminal nodes in Fourier decision trees are denoted by FA_i for Abelian and FNA_i for non-Abelian groups. The index i denotes the number of outgoing edges, and it is equal to the cardinality K_i of the dual object Γ_i of G_i. The nodes in which the Shannon decomposition or its integer counterpart on cyclic groups is performed are denoted by S_i. In the figures of DTs we use the short notation w_i^j for $\mathbf{R}_{w_i}(j)$.

6 Fourier Decision Diagrams

Fourier decision diagrams are derived by the reduction of the Fourier decision trees. Therefore, the Fourier decision diagrams on Abelian (FADDs) and non-Abelian groups (FNADDs) are distinguished (Stanković, 1996a), (Stanković, 1996b).

In a decision tree, the reduction is possible if there are some isomorphic subtrees. Such subtrees correspond to the equal subvectors in the vector representing values of constant nodes in the DT. In FNADT, the values of constant nodes are Fourier coefficients for f. Thus, in FNADTs, the reduction is possible if there are some equal subvectors in the vector representing the Fourier spectrum $\mathbf{S}_f$ for f.

If for a given assignment of values of a variable $x_i = p_i$, $p_i \in \{0, \ldots, g_i\}$, there are equal subvectors of orders $g_k, k < i$ in the vector $[\mathbf{S}_f]$ of the Fourier coefficients of f, then in the Fourier DT for f the corresponding nodes at the k-th level may be joined. Thus, the redundant nodes may be deleted. If the equal subvectors of orders $g_k, k < i$ in $[\mathbf{S}_f]$ correspond to different assignments of values for x_i, the corresponding nodes may be shared.

Therefore, the generalized BDD reduction rules introduced for the reduction of the Walsh transform DDs (WDDs) (Stanković, Sasao, Moraga, 1996) will be used for the reduction of Fourier decision trees on both Abelian and non-Abelian groups. Recall that the Walsh transform used in definition of WDDs is the Fourier transform on finite dyadic groups (Stanković, Sasao, Moraga, 1996). Therefore, WDDs are a particular example of Fourier DDs on Abelian groups.

Definition 3 *Fourier decision diagrams are DDs derived from the Fourier decision trees by using the generalized BDD reduction rules. A Fourier decision diagram is reduced if further reduction with the same rules is impossible.*

7 Optimization of DDs Representations

The optimization of DDs representations by using FNADDs is explained and illustrated by the following example.

Example 2 *Fig. 1 shows BDT for $f(x_1, x_2, x_3)$ in Example 1. In this DTs representation, f is considered as a function on $G_8 = C_2 \times C_2 \times C_2$. Fig. 2 shows the MTDT for f if it is considered as a function on $G_8 = C_2 \times C_4$. Fig. 3 shows FNADT for f on $G_8 = Q_2$. In this FNADD, the values of constant nodes are the Fourier coefficients for f on Q_2. The Fourier transform on this group is defined in terms of group representations for Q_2 over the complex filed C given in Table 2. From this table, the dual object Γ for Q_2 consists of five elements. Thus, the Fourier spectrum on Q_2 consists of five spectral coefficients $S_f(0), S_f(1), S_f(2), S_f(3)$ and $\mathbf{S}_f(4)$, where $\mathbf{S}_f(4)$ is a (2×2) matrix. In Fig. 3, the matrix-valued coefficient $\mathbf{S}_f(4)$ is represented by the MTDT on the cyclic group C_4 of order 4.*

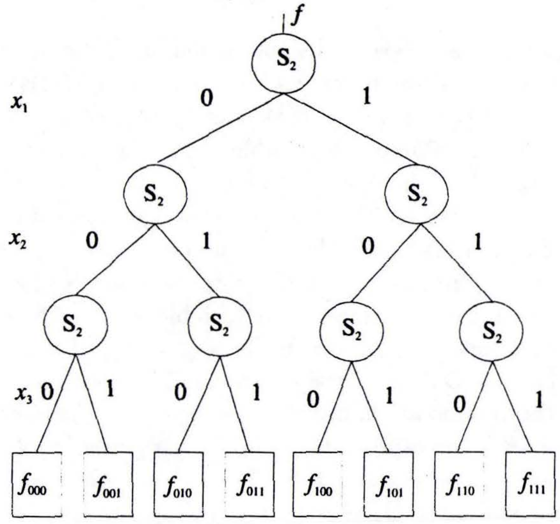

Figure 1: Shannon tree: three-level representation of f on $G_8 = C_2 \times C_2 \times C_2$.

MTDT in Fig. 2 permits reduction of the depth compared to BDD. FNADT in Fig. 3 permits saving of one non-terminal node within the same depth as in MTDT for f.

In general, in any BDT or shared BDT (SBDT) (Minato, 1996), a subtree of the form shown in Fig. 1 can be replaced by the subtree of the form shown in Fig. 2 or Fig. 3. It is clear that the replacement can be done at any level in the DT. As is noted in (Stanković, 1996a), (Stanković, 1996b), the most compact representations are achieved if the non-Abelian groups are used at the nodes just above the constant nodes. That means, in decomposition of G in the form (1), the Abelian groups are used for $g_1, \ldots, g_{r-1}$ and then non-Abelian groups for $g_r \ldots, g_{n-1}$. The value of r, and the orders of the constituent subgroups are chosen such that requirements in (1) are satisfied.

8 Basic Features of FNADDs

The use of non-Abelian groups in the way explained in Example 2, provides depth reduction in DDs representations. Depth reduction is achieved at the price of the increase of the number of outgoing edges of nodes. Their number determines the order of subvectors in the vector of values of constant nodes that should be compared for eventual reduction of the DT. In that way, the number of outgoing edges of nodes

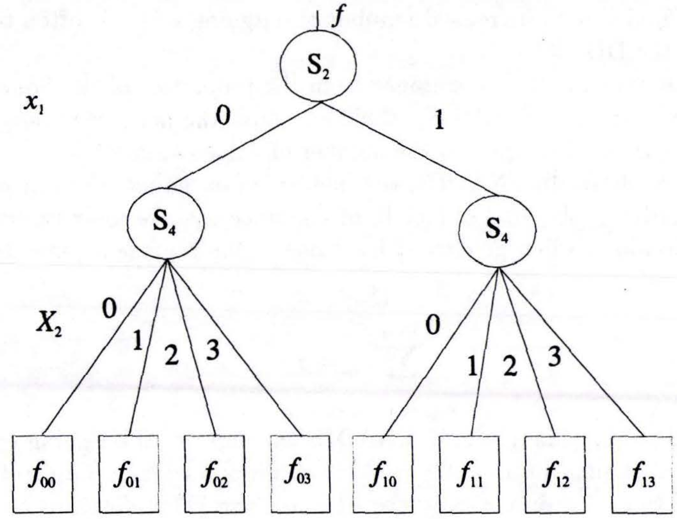

Figure 2: MTDT for f: two-level representation.

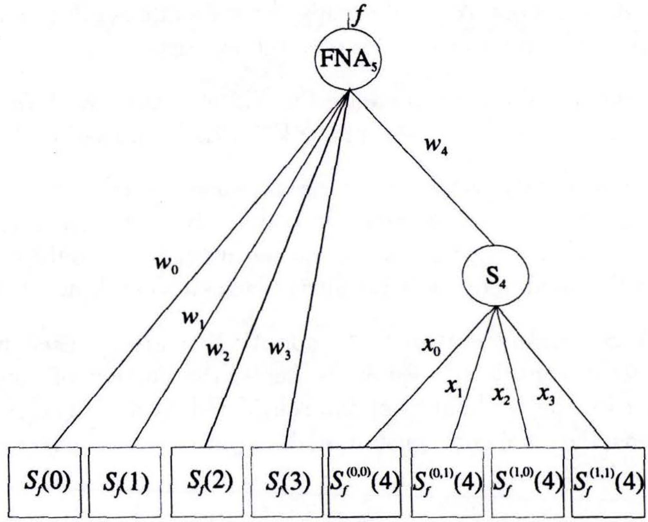

Figure 3: Complex-valued FNADT on Q_2 with MTDT for $\mathbf{S}_f(4)$.

determines the width of the resulting DD. Therefore, the price for the depth reduction by using the nodes with increased number of outgoing edges is often the increase of the width of the DD.

FNADDs take further advantages from the properties of the Fourier transform on non-Abelian groups. In DDs on Abelian groups, the number of outgoing edges of nodes at the i-th level is equal to the number of values x_i can take. Thus, it is equal to the order g_i of G_i. In FNADDs, the number of outgoing edges of nodes is equal to the cardinality of the dual object Γ_i of G_i, since w_i take their values on Γ_i.

Since on non-Abelian groups at least one of the representations $\mathbf{R}_w$ is of order $r_w > 1$, and

$$\sum_{w-1}^{K-1} r_w^2 = g,$$

then always $\Gamma_i \leq g_i$. Therefore, in FNADDs the number of outgoing edges of nodes is always smaller than in any DD on Abelian groups with decomposition of G into subgroups of the same orders as in the Abelian case. Therefore, for that reason, the width of FNADDs is often smaller that the width of DDs on Abelian groups.

Thanks to that FNADDs express the following basic features.

1. The same as QDDs or MTDDs, FNADDs permit reduction of the depth by using nodes with increased number of outgoing edges.

2. Compared to DDs on Abelian groups, for the same depth reduction FNADDs require nodes with fewer number of outgoing edges.

3. Depth reduction does not increase the width of the FNADDs. Moreover, in many cases, the width and size of the FNADD is also reduced.

4. In FNADDs, the number of non-terminal nodes is reduced at the price of the increase of the number of constant nodes. In that property, FNADDs are useful in applications where a calculation procedure should be performed at each non-terminal node, or a circuit is assigned to each non-terminal node.

5. If the Fourier transform on finite non-Abelian groups used in definition of a FNADD is considered over finite fields, the number of constant nodes is restricted by the cardinality of the considered field. Thus, in FNADD over finite fields, the number of constant nodes is not necessarily increased compared to that in MDD.

6. The ratio between the percent of used nodes in a DD compared to the total of nodes in the corresponding DT is better for FNADDs in many cases.

7. The comparison of the depth, width and number of outgoing edges of nodes gives considerable advantages to FNADDs.

Table 1: Group operation for the quaternion group Q_2.

$\circ$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	5	6	7	2	3	0	1
5	5	6	7	4	3	0	1	2
6	6	7	4	5	0	1	2	3
7	7	4	5	6	1	2	3	0

8. FNADDs offer possibility to use negative edges in many cases where that can not be done with MTBDDs and MTDDs.

9 Experimental Results

The experimental results presented in this section confirm and approve basic features of FNADDs. WE used the quaternion group Q_2 of order 8 as the basic non-Abelian group. Thus, we will briefly repeat the basic definition of the Fourier transform on Q_2.

9.1 Quaternion group Q_2

The quaternion group Q_2 of order 8 has two generators a and b, and the group identity is denoted by e. If the group operation is written as abstract multiplication, the following relations hold for the group generators: $b^2 = a^2, bab^{-1} = a^{-1}, a^4 = e$. If the following bijection V is chosen

x	e	a	a^2	a^3	b	ab	a^2b	a^3b
$V(x)$	0	1	2	3	4	5	6	7

then, the group operation Q_2 is defined in Table 1. All the irreducible unitary representations are given in Table ??.

The following example illustrates calculation of the Fourier transform on Q_2.

Example 3 *The dual object Γ of Q_2 is of the cardinality 5, since there are five irreducible unitary representations of this group. Four of representations are 1-dimensional and one is 2-dimensional. The Fourier transform on Q_2 is defined by*

Table 2: Irreducible unitary representations of Q_2 over C.

x	$\mathbf{R}_0$	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{R}_4$
0	1	1	1	1	$\mathbf{I}$
1	1	1	-1	-1	$i\mathbf{A}$
2	1	1	1	1	$-\mathbf{I}$
3	1	1	-1	-1	$i\mathbf{B}$
4	1	-1	1	-1	$\mathbf{C}$
5	1	-1	-1	1	$-i\mathbf{D}$
6	1	-1	1	-1	$\mathbf{E}$
7	1	-1	-1	1	$i\mathbf{D}$
	$r_0 = 1$	$r_1 = 1$	$r_2 = 1$	$r_3 = 1$	$r_4 = 2$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the matrix

$$[\mathbf{Q}_2]^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 2\mathbf{I} & 2i\mathbf{B} & -2\mathbf{I} & 2i\mathbf{A} & 2\mathbf{E} & 2i\mathbf{D} & 2\mathbf{C} & -2i\mathbf{D} \end{bmatrix},$$

where the notation is as in Table ??. Therefore, the Fourier spectrum of a function f on Q_2 consists of five coefficients, four 1-dimensional and one 2-dimensional and can be represented as a vector

$$[\mathbf{S}_f] = \begin{bmatrix} S_f(0) & S_f(1) & S_f(2) & S_f(s) & \mathbf{S}_f(4) \end{bmatrix}^T.$$

For example, the Fourier spectrum of the function f on Q_2 given by the truth-vector $\mathbf{F} = [0\alpha 00\beta\lambda 00]^T$ is given by

$$[\mathbf{S}_f] = \begin{bmatrix} \alpha + \beta + \lambda \\ -\alpha + \beta - \lambda \\ \alpha - \beta - \lambda \\ -\alpha - \beta + \lambda \\ 2\begin{bmatrix} -i\alpha & \beta + i\lambda \\ -\beta + i\lambda & i\alpha \end{bmatrix} \end{bmatrix}.$$

Table 3: Decompositions for the domain groups for FNADDs.

on	4	5	6	7	8	9	10	11	12
G	C_2Q_2	C_4Q_2	Q_2^2	$C_2Q_2^2$	C_4Q_2	Q_2^3	$C_2Q_2^3$	$C_4Q_2^3$	Q_2^4

9.2 Domain groups

We developed algorithms for generation of MTBDDs, MTDDs with different number of outgoing edges, and FNADDs for different decompositions of the domain groups for switching functions of different number of variables. In these experiments, the basic domain group for switching functions G_{2^n} of order 2^n is decomposed into the product of subgroups taken as a suitable combination of C_2, C_4, and Q_2 depending on the value for n. For MTDDs we use different products of groups C_2, C_4, and C_8, and C_2, C_4 and Q_2 for FNADDs. Table 3 shows different decompositions used for FNADDs.

The optimization of FNADDs is performed by freely choosing between the integer extension of the Shannon expansion and Fourier expansion for Abelian subgroups in G. For example, C_2Q_2 denotes that we are using the Shannon expansion for the subgroup of order 2 and the Fourier expansion for the subgroup of order 8. Similar, W_2Q_2 denotes that we are using the Fourier expansion for both subgroups, since the Fourier transform on C_2 is the Walsh transform.

Multiple-output functions with q outputs $f_0 * f_1 * \cdots * f_{q_1}$ are represented by SBDDs, MTBDDs, and MTDDs with nodes having different number of outgoing-edges. For representation by MTBDDs, MTDDs, and FNADDs, they are first represented by the integer-valued functions $f(z)$ through the mapping

$$f(z) = \sum_{i=0}^{q-1} 2^i f_i.$$

9.3 Complexity of FNADDs

Table 4 compares sizes and widths of SBDDs and FNADDs for various benchmark functions. For FNADDs the values of non-terminal nodes (ntn) and constant nodes (cn) are shown separately. Thus, the size of FNADDs is the sum of these two values.

Table 5 shows the sizes of FNADDs for Achille's heal function

$$f = x_1y_1 \vee x_2y_2 \vee \cdots \vee x_{2r}y_{2r},$$

for two different orderings $x_1, x_2, \ldots, x_{2r}, y_1, y_2, \ldots, y_{2r}$ and $x_1, y_1, x_2, y_2, \ldots, x_ny_n$ for different values of n. This example shows that the sizes of FNADDs greatly depends on the initial variables ordering. The depth reduction in FNADDs is achieved at the price of the increase the size of the FNADDs.

Table 4: **SBDDs and FNADDs for various benchmark functions.**

				SBDD		FNADD				
f	in	out	cubes	size	width	ntn	cv	size	width	decomposition
5xp1	7	10	75	90	25	39	128	167	18	$C_2Q_2^2$
bw	5	28	87	116	37	9	25	34	4	C_4Q_2
con1	7	2	9	20	5	13	12	25	6	$C_2Q_2^2$
rd53	5	3	32	25	6	7	14	21	3	C_4Q_2
rd73	7	3	141	45	10	23	30	53	6	$W_2Q_2^2$
xor5	5	1	16	11	2	5	6	11	2	C_4Q_2

Table 5: **Sizes of BDDs and FNADDs for Achilles' heal functions.**

$n = 2r$	BDD		FNADD		
	worst	best	worst	best	decomposition
4	8	6	13	12	C_2Q_2
6	12	8	37	25	Q_2^2
8	14	10	71	40	$C_4Q_2^2$

Table 6 compares sizes of SBDDs and FNADDs for adders, and Table 7 for multipliers. In this example, FNADDs provide the reduction of all three parameters, depth, size, and width and the reduction is considerable compared to SBDDs. The reduction possibilities are compared through the percent of used nodes from the total of nodes in the corresponding DTs. It may be seen that these values are comparable, thus, the possibility to do reduction in SBDDs and FNADDs is comparable.

Table 6: **SBDDs and FNADDs for adders.**

				SBDD					FNADD				
n	in	out	cubes	ntn	cn	size	width	%	ntn	cv	size	width	%
2	4	3	11	190	2	8	21	22.00	4	7	11	2	52.38
3	6	4	31	55	2	57	20	42.86	6	7	13	4	31.70
4	8	5	75	101	2	103	30	19.84	14	14	28	7	33.73
5	10	6	167	224	2	226	62	10.98	18	16	34	7	8.31
6	12	7	355	475	2	477	126	5.81	21	12	33	7	4.00

The effect of the use of non-Abelian groups as is described in Example 1, is studied in Table 8 and Table 9. They compare sizes of MTDDs for different decom-

Table 7: SBDDs and FNADDs for multipliers.

				SBDD					FNADD				
n	in	out	cubes	ntn	cn	size	width	%	ntn	cv	size	width	%
2	4	4	7	17	2	19	5	14.28	4	10	14	2	9.52
3	6	6	32	61	2	63	15	11.27	9	20	29	4	9.75
4	8	8	128	157	2	159	39	7.51	24	42	66	11	13.25
5	10	10	488	471	2	473	114	5.54	37	50	87	17	4.51
6	12	12	939	786	2	788	192	2.34	45	49	94	22	2.68

positions of the domain groups and FNADDs for adders and multipliers. We can observe that for the same depth reduction with MTDDs we have to pay the price of the increased number of outgoing edges. Even in that case, FNADDs have smaller widths and sizes. For adders and multipliers, none of possible decompositions for the domain groups can produce MTDDs that are more compact than the corresponding FNADDs.

Table 10 compares the FNADDs to some other DDs based on spectral transforms. The Arithmetic transform DDs (ACDDs) (Stanković, Sasao, Moraga, 1996), Walsh transform DDs (WDDs) in $(1, -1)$ coding (Stanković, Sasao, Moraga, 1996), and Complex-Hadamard Transform DDs (Falkowski, Rahardja, 1996) are compared with FNADDs for some benchmark functions. ACDDs, WDDs, and CHTDDs are DDs on Abelian groups. Therefore, the depth is equal to n. For rd53 and xor5, the sizes of FNADDs are equal to those of other DDs. However, the number of non-terminal nodes and the width are reduced. In other cases in this example, FNADDs are more efficient with respect to depth, width and size. However, as for other DDs, some examples where FNADDs are not efficient certainly can be found.

9.3.1 Dependency on the number of product terms

To check dependency of parameters of BDDs, MTDDs, and FNADDs on the number of true minterm we generated pseudo-random switching functions of $n = 6$ variables for different number of true minterms s, $(0 < s < 2^6)$ and counted the number of nodes for different DDs. Table 11 shows sizes of BDDs, MTDDs for different decompositions and FNADDs.

In this example, MTDDs with C_8 appear the most efficient. FNADDs are better than BDDs with respect to the number of non-terminal nodes. However, in BDDs and MTDDs the number of constant nodes is always 2, thus, it is natural that their sizes are smaller than the sizes of FNADDs. FNADDs have the better ratio between the number of non-terminal nodes and the size.

In BDDs and MTDDs, the non-terminal nodes take on the average, respectively,

Table 8: Sizes of DDs for adders for different decompositions.

add3	ntn	cn	size	width
SBDD	55	2	57	20
C_2^6	41	15	56	14
C_4^3	17	15	32	12
C_8^2	9	15	24	8
Q_2^2	6	7	13	4

add4	ntn	cn	size	width
SBDD	101	2	103	30
C_2^8	113	31	144	30
C_4^4	49	31	80	28
$C_4C_8^2$	29	31	60	24
$C_4Q_2^2$	14	15	29	7
$W_4Q_2^2$	14	14	28	7

add5	ntn	cn	size	width
SBDD	224	2	226	62
C_2^{10}	289	63	352	62
C_4^5	129	63	192	60
$C_2C_8^3$	75	63	138	56
$C_2Q_2^3$	18	16	34	7
$W_2Q_2^3$	18	16	34	7

add6	ntn	cn	size	width
SBDD	475	2	477	126
C_2^{12}	705	127	832	126
C_4^6	321	127	448	124
C_8^4	193	127	320	120
Q_2^4	21	12	33	7

Table 9: Sizes of DDs for multipliers for different decompositions.

mul3	ntn	cn	size	width
SBDD	61	2	63	15
C_2^6	56	26	82	28
C_4^3	19	26	45	14
C_8^2	8	26	34	7
Q_2^2	9	20	29	4

mul4	ntn	cn	size	width
SBDD	157	2	159	39
C_2^8	240	90	330	120
C_4^4	80	90	170	60
$C_4C_8^2$	35	90	125	30
$C_4Q_2^2$	24	46	70	11
$W_4Q_2^2$	24	42	66	11

mul5	ntn	cn	size	width
SBDD	471	2	473	114
C_2^{10}	992	340	1332	496
C_4^5	331	340	671	248
$C_2C_8^3$	143	340	483	124
$C_2Q_2^3$	46	58	104	23
$W_2Q_2^3$	37	50	87	17

mul6	ntn	cn	size	width
SBDD	786	2	788	192
C_2^{12}	4032	1238	5270	2016
C_4^6	1344	1238	2582	1008
C_8^4	576	1238	1814	504
Q_2^4	45	49	94	22

Table 10: Sizes of ACDDs, WDDs, CHTDDs and FNADDs.

	ACDD				CHTDDs			
f	ntn	cv	size	width	ntn	cv	size	width
5xp1	38	11	49	10	127	128	255	64
bw	31	32	63	16	31	32	63	16
con1	40	5	45	10	115	42	157	56
rd53	15	6	21	5	15	6	21	5
rd73	27	6	33	6	27	8	36	7
xor5	15	6	21	5	9	2	11	2
	WDD(1,-1)				FNADD			
f	ntn	cv	size	width	ntn	cv	size	width
5xp1	37	13	50	9	39	128	167	18
bw	31	32	63	16	9	25	34	4
con1	48	11	59	13	13	12	25	6
rd53	15	5	20	5	7	14	21	3
rd73	28	5	33	7	23	30	53	6
xor5	5	2	7	1	5	6	11	2

90.09%, 81.81%, and 78.41% of the size. In FNADDs, 35.86% of the size are the non-terminal nodes. This experiment proves the corresponding feature of FNADDs which states that in FNADDs the number of non-terminal nodes is reduced at the price of the increase of the number of constant nodes.

10 Conclusion

In word-level DDs, the width can be defined as the maximal number of non-terminal nodes per a level, as that is done in bit-level DDs. In many cases, the number of constant nodes exceeds the number of non-terminal nodes. Therefore, in these cases, the width of word-level DDs is determined by the number of constant nodes. However, in many practical application of DDs, a calculation subprocedure should be performed at each non-terminal node. In many DDs based design methods, a circuit should be assigned to each non-terminal node. The constant nodes are considered as inputs in such subprocedures or circuits. Therefore, DDs with reduced number of non-terminal nodes at the price of constant nodes may be useful in such applications of DDs. In that respect, FNADDs may be very useful and efficient, especially if the FNADDs over finite fields are used in which case the number of constant nodes is restricted to the cardinality of the used finite field. In FNADDs, for the same depth reduction, the number of outgoing edges of nodes is smaller than in other DDs.

In many cases, the ratio between depth, width, size, and number of outgoing

Table 11: **Sizes of BDDs, MTDDs, and FNADDs.**

s	C_2^6			C_4^3			C_8^2			Q_2^2			
	ntn	s	w	ntn	s	w	ntn	s	w	ntn	cn	s	w
4	9	11	3	5	7	3	3	5	2	14	18	32	8
8	18	20	5	10	12	5	6	8	5	15	22	37	9
12	20	22	6	11	13	6	7	9	6	15	29	44	9
16	20	22	6	10	12	5	8	10	7	15	25	40	9
20	24	26	8	13	15	8	9	11	8	15	32	47	9
24	24	26	8	13	15	8	9	11	8	15	33	48	9
28	24	26	8	14	16	9	9	11	8	15	31	46	9
32	21	23	7	12	14	7	8	10	7	15	26	41	9
36	25	27	9	15	17	10	8	10	7	15	33	48	9
40	21	23	7	11	13	6	9	11	8	15	33	48	9
44	22	24	7	11	13	6	8	10	7	15	25	40	9
48	22	24	7	13	15	8	8	10	7	15	31	46	9
52	19	21	6	10	12	5	7	9	6	15	23	38	9
56	18	20	5	9	11	4	6	8	5	13	22	35	7
60	13	15	3	7	9	3	4	6	3	15	14	29	9

edges of nodes, is improved in FNADDs compared to DDs on Abelian groups. In each case, some of these parameters may be reduced, and others will not be considerably increased compared to the corresponding parameters in representation with DDs on Abelian groups.

References

Ackers, S.B. (1978), "Binary decision diagrams", *IEEE Trans. on Computers*, Vol. C-27, No. 6, 509-516.

Hasan-Babu, H.Md., Sasao, T., (1996), "A method to represent multiple-output switching functions by using binary decision diagrams", *Proc. 5th Workshop of Synthesis and System Integration of Mixed Technologies, SASIMI'96*, Nov. 25, 1996, 212-217.

Bryant, R.E., (1986), "Graph-based algorithms for Boolean functions manipulation", *IEEE Trans. Comput.*, Vol. C-35, No. 8, 667-691.

Bryant, R.E., Chen, Y-A., (1994), "Verification of arithmetic functions with binary moment decision diagrams", May 31, 1994, CMU-CS-94-160.

Clarke, E.M., McMillan, K.L., Zhao, X., Fujita, M., (1993), "Spectral transforms for extremely large Boolean functions", in: Kebschull, U., Schubert, E., Rosenstiel, W., Eds., *Proc. IFIP WG 10.5 Workshop on Applications of the Reed-Muller Expansion in Circuit Design*, 16-17.9.1993, Hamburg, Germany, 86-90.

Drechsler, R., Becker, B., (1996), "OKFDDs-Algorithms, applications and extensions", in: Sasao, T., Fujita, M., (Ed.), *Representations of Discrete Functions*, Kluwer Academic Publishers, 163-190.

Falkowski, B.J., Rahardja, S., (1996), "Complex spectral decision diagrams", *Proc. 26th Int. Symp. on Multiple-Valued Logic*, Santiago de Campostela, Spain, May 1996, 255-260.

Iguchi, Y., Sasao, T., Matsuura, M., (1997), " On properties of Kleene TDDs", *Asia and South Pacific Design Automation Conference, ASP-DAC'97*, Jan. 1997, 473-476.

Lai, Y.F., Pedram, M., Vrudhula, S.B.K., (1994), "EVBDD-based algorithms for integer linear programming, spectral transformation, and functional decomposition", *IEEE Trans. Computer-Aided Design of Integrated Circuits and Systems*, Vol. 13, No. 8, 959-975.

Miller, D.M., (1994), "Spectral transformation of multiple-valued decision diagrams", *Proc. 24th Int. Symp. on Multiple-valued Logic*, Boston, Massachusetts, USA, 22.-25.5.1994, 89-96.

Minato, S., (1996), "Graph-based representations of discrete functions", in: Sasao, T., Fujita, M., (Ed.), *Representations of Discrete Functions* Kluwer Academic Publishers, 1-28.

Sasao, T., (1996), "Representations of logic functions by using EXOR operators", in: Sasao, T., Fujita, M., (Ed.), *Representations of Discrete Functions*, Kluwer Academic Publishers, 29-54.

Sasao, T., (1998), *Logic Design: Switching Circuit Theory*, Kindai Kaga-ku, Japan, second edition.

Sasao, T., Butler, J.T., (1994), "A design method for look-up table type FPGA by pseudo-Kronecker expansions", *Proc. 24th Int. Symp. on Multiple-valued Logic*, Boston, Massachusetts, May 25-27, 1994, 97-104.

Sasao, T., Fujita, M., (Eds.), (1996), *Representations of Discrete Functions*, Kluwer Academic Publishers.

Srinivasan, A., Kam, T., Malik, Sh., Brayant, R.K., (1990), "Algorithms for discrete function manipulation", in: *Proc. Inf. Conf. on CAD*, 1990, 92-95.

Stanković, R.S., (1990), "Matrix-interpretation of fast Fourier transforms on finite non-Abelian groups', *Proc. Int. Conf. on Signal Processing, Beijing/90*, October 22-26, 1990, Beijing, China, 1187-1190.

Stanković, R.S., (1995), "Some remarks about spectral transform interpretation of MTBDDs and EVBDDs", *Proc. ASP-DAC'95*, August 29-September 1, 1995, Makuhari Messe, Chiba, Japan, 385-390.

Stanković, R.S., (1996a), "Fourier decision diagrams for optimization of decision diagrams representations of discrete functions", *Proc. Workshop on Post Binary-Ultra Large Scale Integration*, Santiago de Campostela, Spain, 1996, 8-12.

Stanković, R.S., (1996b), *Fourier Decision Diagrams on Finite Non-Abelian Groups in Optimization of Decision Diagrams Representations of Discrete Functions*, Prima Press, Moscow, and Nauka, Belgrade.

Stanković, R.S., (1997), "Fourier decision diagrams on finite non-Abelian groups with preprocessing", *Proc. 27th Int. Symp. on Multiple-Valued Logic*, May 28-30, 1997, Antigonish, Nova Scotia, Canada, 281-286.

Stanković, R.S., Sasao, T., Moraga, C., (1996), "Spectral transform decision diagrams" in: Sasao, T., Fujita, M., (Ed.), *Representations of Discrete Functions*, Kluwer Academic Publishers, 55-92.

Stanković, R.S., Stojić, M.R., Stanković, M.S., (1996), *Recent Developments in Abstract Harmonic Analysis with Applications in Signal Processing*, Nauka, Belgrade and Elektronski fakultet, Niš.