Nonequilibrium Hydrodinamical Processes with Memory and Nonlocality. Mathematical Models and Their Solutions.

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Abstract
The mathematical models for different hierarchical levels of transport processes are discussed. The model equations with memory effects accounting are proposed. Such equations are hyperbolic modifications of the Burgers equation and the Navies- Stokes system. Also the new systems of ordinary differential equations are proposed for investigation memory effects influence on chaos. Some numerical examples of chaotical behaviour in such systems are included.

Keywords: Hierarchical levels, memory, nonlocality, chaos, collapses

1 Introduction

The basic equations for heat transfer and hydrodynamics are usually parabolic heat equation and the Navier-Stokes hydrodynamic equations. But it is known very well that these equations lose their applicability in extended media when characteristic scales of parameters change less then correlation time and correlation length (relaxation or memory and nonlocality effects). Especially many examples of non-applicability were found in turbulence. So more correct equations should be applied in such cases.

There are some well-established facts in theoretical physics on the description of transport processes. The first famous idea is the existence of hierarchy of description levels. If there are N>>1 particles we have many levels for description: N deterministic dynamical laws of Newton for particle movement; Liouville equation for N- particle distribution function, Boltzman equation for one-particle distribution function, hydrodynamic equations for macroparameters (Naveir-Stokes class equations), thermodynamic equations. The choice of the level of description depends on the measure of deviation from equilibrium. The second background idea is the existence of many interrelating relaxation processes and many time and space scales with different relaxation times and lengths. The memory and nonlocality effects are common for all levels. The turbulence is a bright example of such complex phenomena. It should be stressed that each level of description has its own model equations with typical behaviour of solution. Especially interesting is the problem of defining typical 'chaotical behaviour' for a given level. So each level of description has its own specific type of 'chaos', 'autowave' solutions, 'collapses' and so on.

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Thus in this paper it will be compared the levels of description and typical chaos, solitons, collapses for them. Special attention will be devoted to the hydrodynamic level but with accounting memory and nonlocality effects.

2 The Hierarchical Levels of Description in Physics

2.1 Hierarchy of Equations

There are many phenomenological, experimental and mathematical models for hydrodynamic processes. But only theoretical physics can give adequate understanding of transport phenomena in different media under different conditions. So we very briefly describe some main concepts from statistical physics relevant to modeling equations choice.

Since the works of N.Bogoliubov, M.Born, M.Green, Kircvud and Ivon, the canonical approach in theoretical physics is as follows. Let us consider the medium constituted from N independent particles. Then in classical physics we can describe the particle movement precisely by an ordinary differential equations system (Newton equations). But statistical physics considers the ensemble of system by introducing distribution function $f_N(x1, x2, \ldots, xN, t)$ for particles distribution probability at time t. The function f_N is evaluated from the Liouville equation

$$\frac{\partial f_N}{\partial t} + \{f_N, H\} = 0, \qquad (1)$$

where H is the Hamiltonian of the system and {} are the Poisson brackets. But really the function f_N is too informative for the hydrodynamic phenomena description. Usually all necessary parameters are macroparameters (for example temperature, pressure, and velocity: T, P, and V). The main leading principle in such a case is the reduction of the description parameters set. The reduction procedure deals with some hierarchical levels. Firstly by integrating on some variables in phase space we can go to f_1 – 1-particle distribution function with the BBGKI chain of equations for s- particle distribution functions. Remark that for f_1 we can received well known Boltzman equation. These hierarchical levels of description are displayed on the Figure 1 at the end of the paper. These stages with distribution functions named kinetical. Further averaging with one-particle distribution function leads to macroparameters T, U, P above. Usual procedures leads to well known equations of hydrodynamic type: parabolic heat equation, Navier-Stokes equations and so on. But more correct description leads to more difficult equations with memory effects accounting. The reason of memory effects origin under reduction processes is very well described in theoretical physics since the works of Mory, Zwanzig, Picirelly, Zubarev and many others, see review in (Makarenko et al, 1993; Makarenko, 1996, Makarenko et al, 1997). In such approach we receive hydrodynamic equations for mackroparameters with some constitution equations relating macrovariables. In general such constitution equations have the form of integro-

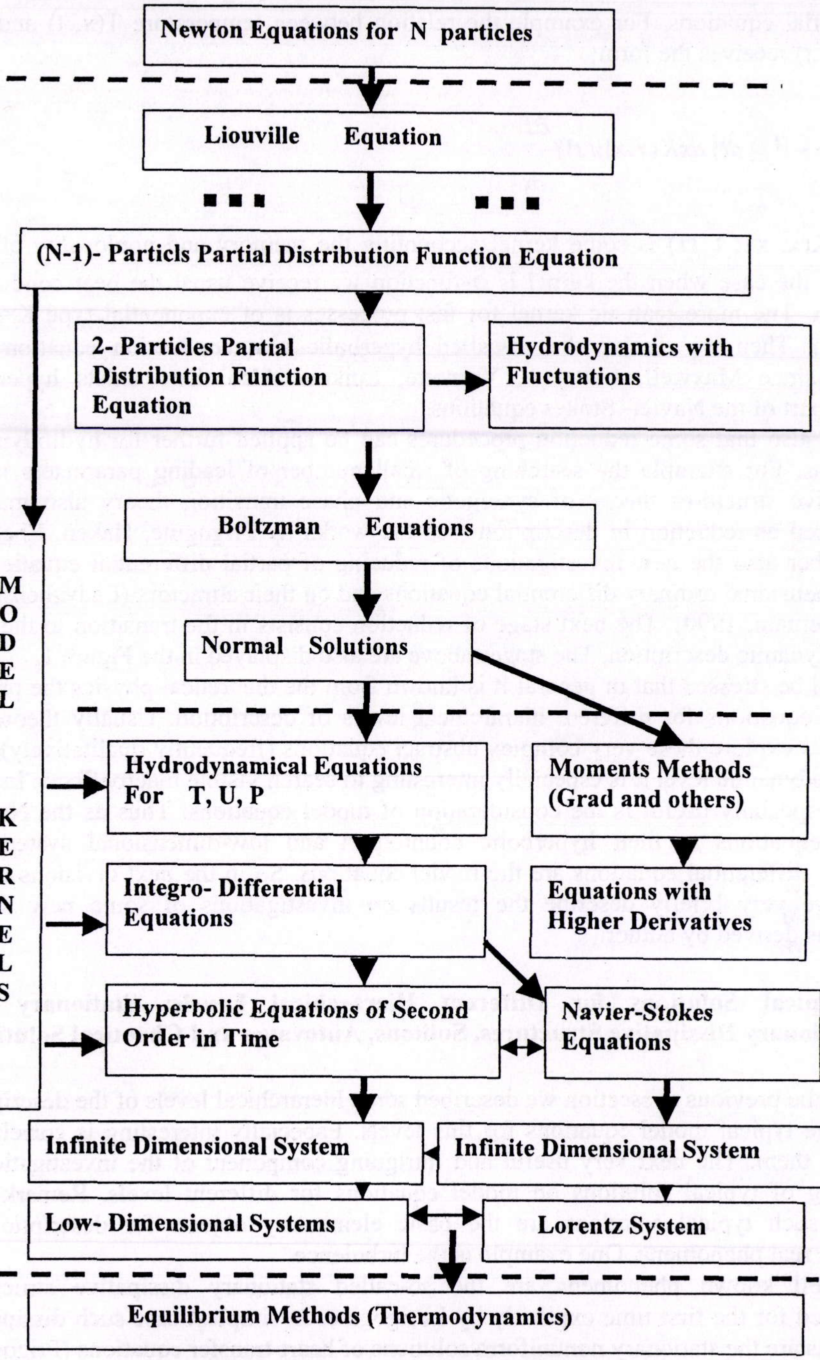

Fig.1: Hierarchical Levels of Description

differential equations. For example the relation between temperature T(x, t) and heat flux q(x,t) receives the form:

$$q(x,t) = -\int_{-\infty}^{t} dt \int dx K(x,x1;t,t1) \frac{\partial T}{\partial x}. \quad (2)$$

In (2) K(x, x1; t, t1) is some kernel accounting the memory and nonlocality effects. Only in the case when the kernel is δ-function we receive usual the heat conduction equation. The more realistic kernel for fast processes is of exponential type K=const exp(-t/τ). Then we receive the so-called hyperbolic heat conduction equation well known since Maxwell, Cattaneo, Vernotte, Luikov. Also there exists hyperbolic counterpart of the Navier- Stokes equations.

Remark also that some reduction procedures can be applied further for hydrodynamic equations. For example the searching of small number of leading parameters in the dissipative structures theory or synergetic and phase transition theory also may be considered as reduction in description (see the works by Prigogine, Haken, Ebeling). Remember also the new investigations of reducing of partial differential equations to low-dimensional ordinary differential equations and on their attractors (Ladyzhenskaya, 1969; Temam, 1990). The next stage of reduction consists in the transition to the pure thermodynamic description. The stages above are also displayed in the Figure 1.

It should be stressed that in general it is known from the theoretical physics the precise abstract equations for different hierarchical levels of description. Usually theoretical physicists explore these very complex abstract equations (frequently qualitatively). But on hydrodynamic level it is especially interesting to search visible macroeffects. In such a case especially useful is the consideration of model equations. Thus as the Navier-Stokes equations as their hyperbolic counterpart and low-dimensional systems of ordinary differential equations are the model equations. So in the next divisions of the paper we very briefly describe the results on investigations of some new model equations derived by author.

2.3 Typical Solutions for Different Hierarchical Levels: Stationary and Nonstationary Dissipative Structures, Solitons, Autovaves and Chaotical Solutions.

Thus in the previous subsection we described some hierarchical levels of the description and some typical model equations for this levels. Especially interesting is correlation between them. The next very useful and intriguing component of the investigation is searching of typical solutions on model equations for different levels. Remark that usually such typical solutions are the basic elementary objects for description of complex real phenomena. One example is the turbulence.

Next well known phenomena are the so-called stationary dissipative structures introduced for the first time explicitly by I.Prigogine. In simplest case such dissipative structures are the stationary nonuniform solutions of heart-transfer equations (Prigogine, 1980; Danilenko et al, 1992; Loskutov, Michailov, 1997). They originate in open thermodynamic systems with heart, mass and entropy fluxes through the boundaries far from equilibrium.

Second close examples constitute nonstationary dissipative structures named blow-up solutions, collapses or peaking regimes (Samarsky et al, 1987; Danilenko et al, 1992). Mathematically such solutions receive infinite value of variables in a finite time (more details see in Danilenko et al, 1992; Makarenko, 1996a, b, c).

Remember that the dissipative structures above were investigated mainly on the base of the usual parabolic heat- mass transfer equations. We investigated them on the basis of more correct hyperbolic heat and mass conduction equations with the memory and nonlocality accounting. We received as theoretical results as interesting applications to combustion (Kudinov et al, 1983). In sections below we describe some our recent investigations on the model hydrodynamic equations with the memory effects.

The next class of typical solutions constitutes solitons and autowaves. Till now such solutions was investigated in fully integrable equations such as Korteveg- de Vries, Burgers, Benny-Stewartson equations. Accounting the memory effects leads to the new equations and new possible solutions. The simplest new equation is the one-dimensional hyperbolic modification of Burgers equation. The two and three-dimensional hyperbolic systems improving the Navier-Stokes equations was introduced by author. When the relaxation time is small remarked equations is the singular perturbation of the parabolic counterpart. It is important that new equations have more solution types then usual. One distinctive feature is that in a case of the large flow velocities the equations with memory accounting allow collapses (blow-up) in solutions. Remember that an existing of blow–up solutions in the Navier- Stokes systems is still under the question now.

Finally there exists another entirely separate class of solution – the so-called chaotical solutions. Earlier the author pushed idea that each hierarchical level of the description has its own type of turbulence (Makarenko et al, 1993). Moreover different modal equations have the different types of chaotic solutions. Till now dynamical chaos for hydrodynamics was investigated mainly on the base of the parabolic type equations or Lorenz system of o.d.e. The author derived new low-dimensional systems of o.d.e. which generalized the Lorenz system. It is found that such more correct equations have some presumably new types of chaotic behaviour (Makarenko, 1998). We exposed the results of investigations very briefly in the next divisions.

3 Hyperbolic Hydrodynamics with Memory Effects.

As was mentioned more correct then the Navier-Stokes equations should be applied in such a case. Cattaneo, Vernotte, Joseph and many others had investigated more correct hyperbolic equations for heat conduction and for the thermoelasticity (see also (Danilenko et al, 1992; Makarenko, 1996; Makarenko et al, 1997)). The accounting of the memory effects also follows the origin of new hydrodynamic models. The simplest modification of the Navier-Stokes equations is the Maxwell media equations

$$\frac{\partial \vec{V}}{\partial t}+V_k\frac{\partial \vec{V}}{\partial x_k}+\tau\left(\frac{\partial^2 \vec{V}}{\partial t^2}+\frac{\partial V_k}{\partial t}\frac{\partial \vec{V}}{\partial x_k}+V_k\frac{\partial^2 \vec{V}}{\partial t \partial x}-\right)-\nu\Delta\vec{V}=-(1+\tau\frac{\partial}{\partial t})grad P+F; \quad (3a)$$

$$div\vec{V} = 0, \qquad (3b)$$

where τ is the relaxation time and ν is the viscosity, $\vec{V}$ is velocity vector, F(x, T) is external force and P – pressure.

In the second part of proposed report it is considered the result of mathematical investigations for the model hydrodynamic equations derived from system above. Than the one-dimensional model equation is the so-called hyperbolic modification of Burgers equation

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \tau\left(\frac{\partial^2 V}{\partial t^2}\right) - \nu\frac{\partial^2 V}{\partial x^2} = 0 \qquad (4)$$

The new simplest two and three-dimensional model equations have the form (k=2,3)

$$\frac{\partial \vec{V}}{\partial t} + V_k\frac{\partial \vec{V}}{\partial x_k} + \tau\left(\frac{\partial^2 \vec{V}}{\partial t^2}\right) - \nu\Delta\vec{V} = 0; \qquad (5)$$

Remember that early we had investigated another model equation with the memory effect - the so-called hyperbolic heat conduction equation. We had investigate the problems of combustion theory and had found conditions for blow-up solutions and interesting dependence the processes current from a relaxation time (Kudinov et al, 1993; Makarenko, 1996).

The hydrodynamic equations with the memory effects are more interesting. First peculiarity is the possibility of collapses existing without external forces. This effect was investigated (analytically and numerically) in the one-dimensional modification of Burgers equation (equation (4)) and results were published earlier (Danilenko et al, 1992; Makarenko, 1996; Makarenko et al, 1997). It is found that in dependence of initial velocity U0 (x) in case $|M| > 1$ we can receive blow-up for velocity u and in case $|M| < 1$ we receive decreasing solution. The quantity

$$M = \{\max U0(x)\}/C; C = \sqrt{\nu/\tau} \qquad (6)$$

represents the ratio of the flow velocity to the velocity of small disturbances (harmonics) in equation. M plays the role of well-known Mach number in gasdynamics. The behaviour of solutions of (4) in case $|M| > 1$ is entirely different from them in usual Burgers equation case.

We also had investigated the two- and three- dimensional model systems (5) with the memory effect accounting. In two-dimensional case we found numerically similar dependence of the value of M as above. For case $|M|>1$ in (Makarenko, Moskalkov, 1992; Makarenko, 1996) we described the development of vorticity collapses in a flow and an increasing of the vortices number.

Our *initial investigations* of the three-dimensional system (5) are very interesting and prospective not only for the model system (5) but also for the hydrodynamic and turbulence theory. Here we described some preliminary numerical results. In case $|M|>1$ we found the reducing in characteristic scales of vortices and increasing of vortices number in a fixed volume of space. On Figures 2,3 we display the typical behavior in a case of simplest initial and boundary condition. We used initial condition in the form

$$u(x,y,z) = By, |y| \le R;$$

$$v(x,y,z) = Bx, |x| \le R; \qquad (7)$$

$$w(x,y,z) = 0, \forall x,y,z$$

Such conditions corresponds to the rotation of the fluid ball with the radius R in the stable flow (the picture of initial conditions is similar to the "real earth in universe"). Initially the fluid in the ball rotates around the axis Z and moves in the stream in the direction of this axis. Figure 2 represents the section of the flow in the plane (X, Y) with Z=0.

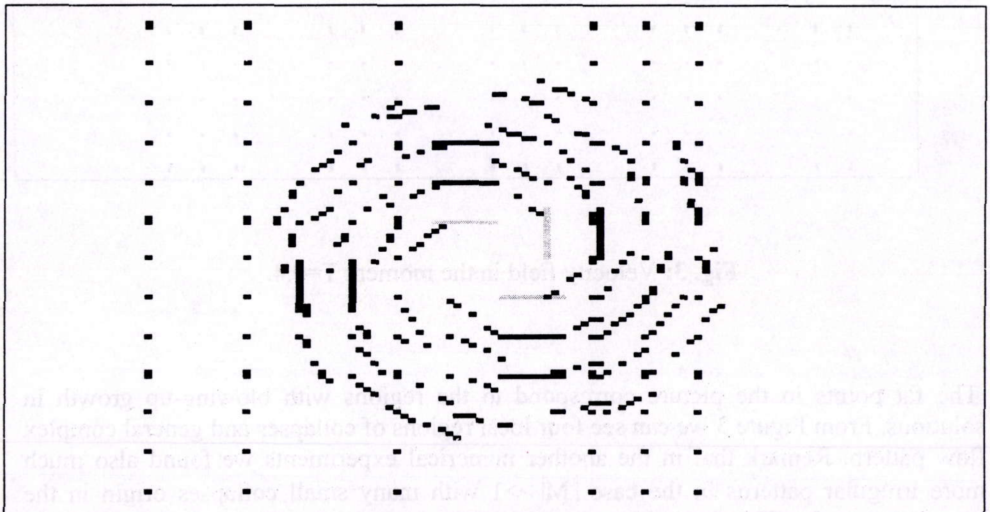

Fig. 2: Initial velocity field (equation (7)).

Short arrows display the direction of the flow in the grid points (small points in picture represent the grids). The initial nonzero velocity is localized within the circle of radius R. The flow field at the moment of time T=0.4 is displayed in the Figure 3.

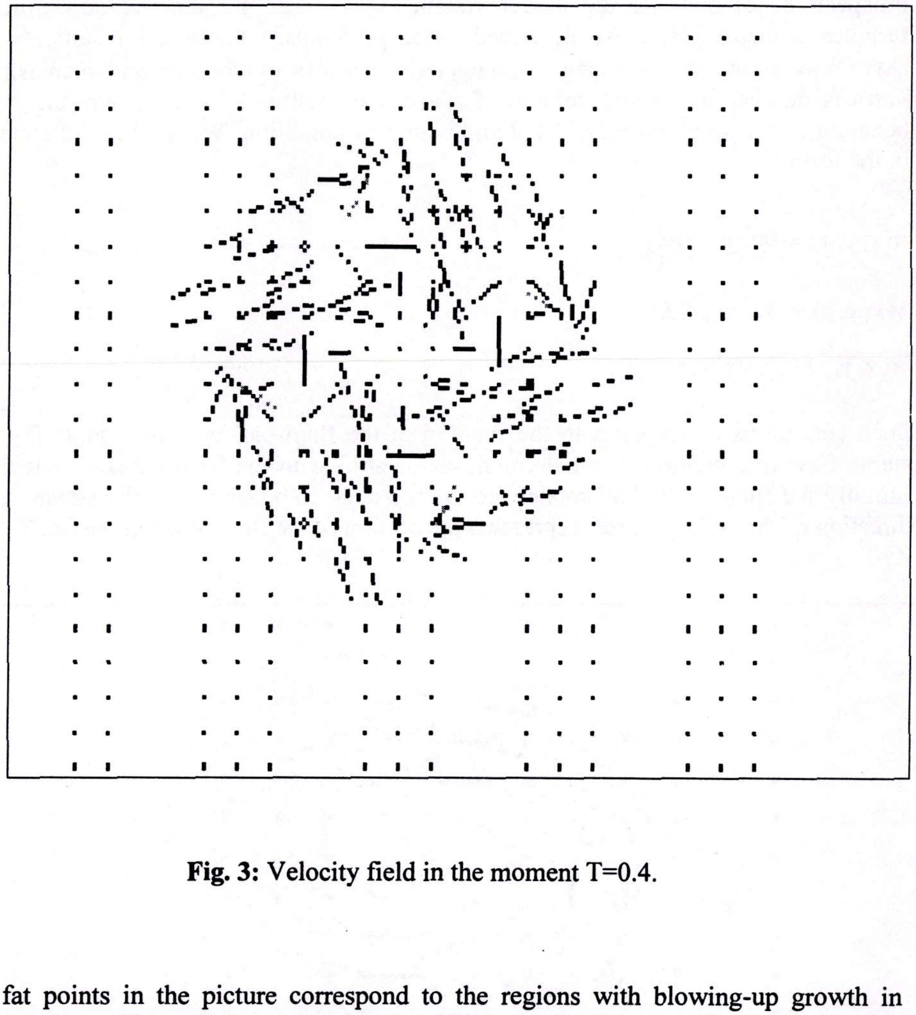

Fig. 3: Velocity field in the moment T=0.4.

The fat points in the picture correspond to the regions with blowing-up growth in solutions. From Figure 3 we can see four local regions of collapses and general complex flow pattern. Remark that in the another numerical experiments we found also much more irregular patterns in the case $|M|>>1$ with many small collapses origin in the bounded domain. This behavior remembers very closely the "dry turbulence" found earlier (Sharkovsky, 1984; 1995). Such properties of solutions of the model equation (3)-(5) open new ways for description of the real turbulence.

3. New Model Equations with Memory Effects for Chaos Investigations

It is well known that Galerkin methods are one of the approaches to the investigation of hydrodynamic equations. It is easy to construct low-dimensional dynamical systems using this method. In particular, for the Navier-Stokes equations the classical example is the well-known Lorenz system. Below we expose some results on the construction of low-dimensional analogs for the generalized hydrodynamics with memory effects. Some comparison with common equations also proposed.

3.1 Galerkin Systems for Three- Dimensional Flows

The first class of dynamical models was derived for the three-dimensional flows with the stick boundary conditions from the system (3a)-(3b). In Galerkin methods solution looks for as the series (Ladyzhenskaya, 1969)

$$v(x,t) = \sum_{k=1}^{\infty} zk(t)\psi_k(x), \tag{8}$$

where z_k, k=1,2,..., are unknown coefficients; $\{\psi_k\}$, k=1,2,... is full system of orthogonal eigenfunction of some linear eigenvalues problem. In the case F(x, t)=F(x) after application of projection procedure we receive the infinite system of o.d.e.

$$\tau \frac{d^2 zl}{dt^2} + \frac{dzl}{dt} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} cklm\, zk\, zm + \nu \mu_l zl +$$

$$+ \tau \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{dzk}{dt} zm + zk\frac{dzm}{dt}\right) cklm = f^l \tag{9}$$

l=1,2,..., where f^l some coefficients received from the right side of equations (3). The simple low-dimensional system of o.d.e. is received from (7) by reduction of system (9) to three variables and after introducing derivatives as complement variables takes the form:

$$\tau \frac{dx1}{dt} = (-x1 - x5x6 - \nu x4 + f_1) - \tau(x3x5 + x6x2); \tag{10}$$

$$\tau \frac{dx2}{dt} = (-x2 + 2x4x6 - \nu x5 + f_2) + 2\tau(x1x6 + x3x4); \tag{11}$$

$$\tau\frac{dx3}{dt}=(-x3+x5x4-\nu x6+f3)-\tau(x2x4+x1x5)\cdot \qquad (12)$$

$$\frac{dx4}{dt}=x1; \qquad \frac{dx5}{dt}=x2; \qquad \frac{dx6}{dt}=x3; \qquad (13)-(15)$$

Without memory effect the system (10) - (15) coincides with the system received in (Brushlinskaya, 1968) from the Navier-Stokes equations.

3.2 Dynamical Systems for 2- Dimensional Flows on Torus with Periodicity Condition

In 1979 Boldrighini and Franceschini investigated the five-dimensional system for flows on the torus. For the sake of comparison we derive the analogous systems for the generalized hydrodynamics with memory. When $\tau=0$ (no memory effects) this system coincides with the system from (Boldrighini, Franceschini 1979). And in the case $0<\tau<<1$ our system is the singular perturbation of Boldrighini- Franceschini system. It was considered the flows of a fluids with memory on the torus $T_2=[0,2\pi]*[0,2\pi]$ just in Boldrigini- Franchesini work. The solution was searched as the series of harmonics exp(ikx), where x- coordinate and k- wave vector. It was received infinite-dimensional system of o.d.e. Truncated system with five harmonics has the form:

$$\tau\frac{dx1}{dt}=(-x1-2x6+4x7x8+4x9x10)+4\tau(x2x9+x7x3)+4\tau(x4x10+x9x5) \qquad (16)$$

$$\tau\frac{dx2}{dt}=(-x2-9x7+3x6x8)+3\tau(x1x8+x6x3) \qquad (17)$$

$$\tau\frac{dx_3}{dt}=(-x_3-5x_6-7x_7x_6)-7\tau(x_1x_7+x_6x_2)+R \qquad (18)$$

$$\tau\frac{dx4}{dt}=(-x4-5x9-x6x10)-\tau(x1x10+x6x3) \qquad (19)$$

$$\tau\frac{dx5}{dt}=(-x5-x10-3x6x9)-3\tau(x1x9+x6x4) \qquad (20)$$

$$\frac{dx6}{dt}=x1; \qquad \frac{dx7}{dt}=x2; \qquad \frac{dx8}{dt}=x3; \qquad \frac{dx9}{dt}=x4; \qquad \frac{dx10}{dt}=x5; \qquad (21)-(25)$$

3.3 Some Properties of Dynamical Systems with Memory Effect Account

In past publications we had described some properties of the received systems of o.d.e. (Makarenko, 1996, 1998). Now for information we display some results on numerical solutions of the o.d.e. systems above.
We present some properties of such systems and a little results of computer modeling. On the Figure 4 we show the two-dimensional projection of ten-dimensional phase space. We also represent the complex behaviour of the system (16)-(25) with the parameter value R=6.0 in the vicinity of the stationary point of system. This stationary point has next coordinates: x1=x2=x3=x4=x5=x9=x10=0,

$$x6 = \sqrt{6}(-5 + \sqrt{25 + 28R/\sqrt{6}})/14; \; x7 = x6/\sqrt{6}; \quad x8 = \sqrt{3/2}.$$

Remark that the values of last five parameters (x1-x10) coincide with the stationary point of the system in (Boldrighini, Franceschini, 1979) without memory effects.
On the horizontal axis we represent the values of variable x6 and on vertical - the values of variable x2.

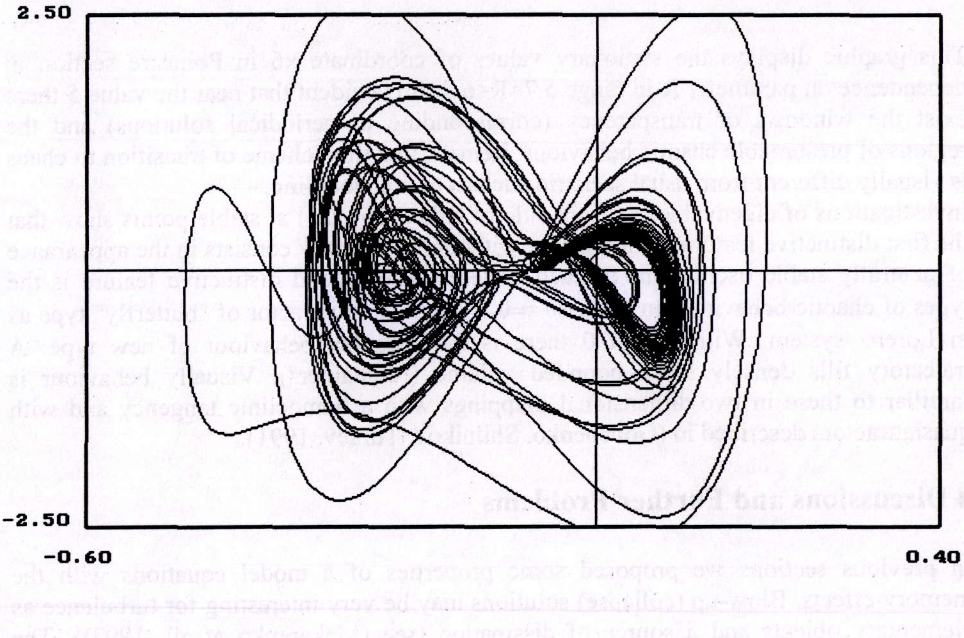

Fig. 4: The projection on the axis x2, x6 the phase portrait of the 10-dimensional system (16)- (25).

The Figure 5 corresponds to the so-called one-dimensional bifurcation diagram of the considered system of o.d.e.

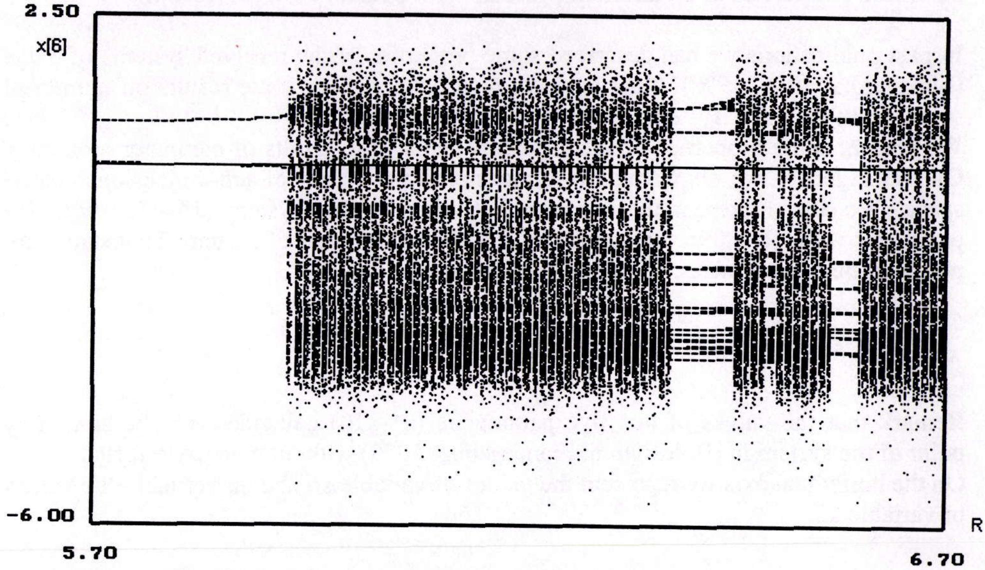

Fig. 5: One-dimensional bifurcation diagram.

This graphic displays the stationary values of coordinate x6 in Poincare section in dependence on parameter R in range 5.7<R<6.7. It is evident that near the value 5 there exist the windows of transparency (corresponding to periodical solutions) and the regions of presumable chaotic behaviour. Remark that this scheme of transition to chaos is visually different from usual scenario such as period doubling.
Investigations of eigenvalues of right side of o.d.e. (16)-(25) at stable points show that the first distinctive feature from the case without the memory consists in the appearance of neutrally stable oscillations in such systems. The second distinctive feature is the types of chaotic behaviour. In the case $\tau=0$ typical is an attractor of "butterfly" type as in Lorenz system. With τ not 0 there is the complex behaviour of new type. A trajectory fills densely some bounded volume ("container"). Visually behaviour is familiar to these in two-dimensional mappings with a homoclinic tangency and with quasiattractors described in (Gonchenko, Shilnikov, Turaev, 1991).

4 Discussions and Further Problems

In previous sections we proposed some properties of a model equations with the memory effects. Blow-up (collapse) solutions may be very interesting for turbulence as elementary objects and a source of dissipation (see (Makarenko at all, 1997)). The processes of vortex pipe scale decreasing with increasing of vortex number under condition $|M|>1$ can imitate the vortex filament creation introduced by Moffat, Kida

and Ohkitani (see Frisch, 1996, pp.187-188). It can see also the deep analogy with the Abricosov's vortex in superfluidity. Blow-up in solutions is analogs (in some sense) to negative energy waves in hydrodynamics and plasma physics (Makarenko et al, 1993). Remark also that the limit case when M tends to infinity may lead to solutions with the infinite number of small vortex. In such a case the limit velocity flow may receives probabilistic and multyvaluedness interpretation.
The o.d.e. systems above are interesting for chaotic behaviour investigations. First of all when $\tau \to 0$ these systems became singularly perturbed. So we may have relaxation oscillations and a specific chaos. At second if $\tau \to 0$ than the dimension of the attractors tends to infinity. In such a case we can have a chaotic behaviour intrinsic not to a dissipative system but rather to a conservative system. The infinite system of o.d.e. behaves just like the collection of oscillators. So the distributed systems above may manifest the properties of media constituted from oscillators. The explanation lies in fact that the leading operator in equations for description is wave operator. Presumably the structure of the chaotic behaviour in such media is simple than in dissipative media and is based on the properties of infinite systems of elementary disturbances. Earlier we proposed the notion of dispersion turbulence for conservative systems with infinite set of harmonics (Kuleshov, Makarenko, 1983). Moreover we suppose that familiar behaviour are intrinsic for systems with broken symmetry. Next we suppose also that the systems above can have large number of attractors when $N \to \infty$ (N- number of equations).

Thus we can see that the accounting the memory effect on hydrodynamical level leads to the existence of new interesting features of solution, new elementary objects of flows and new types of transition to chaos. So the investigations of such equations are very actual and perspective. Remark that nonlocality effect can be considered at the same way as memory effect. Some results on nonlocality were described in (Makarenko et al, 1997).

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