Stability in Delayed and Anticipatory Systems of Applied Mechanics

Peter B. Béda
HAS-BUTE Research Group on Dynamics of Machines and Vehicles,
Budapest University of Technology, Műegyetem rkp.1-3.
H-1111 Budapest, Hungary
+36 1 463 1728, bedap@kme.bme.hu

Abstract

In applied mechanics several papers concentrate on the comparison of delayed and non-delayed approaches of controlled machines. We may study both continuous and discrete time systems, by using both numeric and analytic methods. These analytic methods are from the qualitative theory of differential equations like Lyapunov's indirect method, or the use of monodromy operator of discrete mappings and the basic bifurcation theory. The principal points of interest in the following work are how continuous time system differs from its representation as some discrete time system in stability and robustness and how the discretisation of a continuous time subsystem acts on the stability properties of the coupled system.
Keywords : discretisation, delayed differential equations, simulation

1 Introduction

The stability of controlled mechanical systems is a key aspect. In numerous problems of mechanical engineering a machine is controlled by a digital device to perform some task [7,8,9,10,11,12,13]. Such system has essentially two different parts. The one is the machine in the sense of mechanical engineering. It is usually described as a continuous time system by using one of the traditional methods of applied mechanics. The other subsystem is the discrete controller. Generally we have a complex nonlinear system of a continuous time and a discrete time subsystem. Instability may arise from either the continuous or the discrete time parts.

In the problem of balancing the source of instability is mechanics, an unstable state of a mechanical system should be stabilised by using some sort of control. Such case was investigated in our previous paper [1]. Then the equation of motion was derived for a simple controlled inverted pendulum with length l and mass m (see Fig.1). The pendulum was attached to a cart with a hinge and its stability was achieved by applying force F to the cart

By using Lagrangian formalism the equations of motion are

$$\frac{1}{3}ml^2\ddot{\vartheta} + \frac{1}{2}ml\ddot{x}\cos\vartheta - \frac{1}{2}mgl\sin\vartheta = 0 \tag{1}$$

$$m\ddot{x} + \frac{1}{2}ml\ddot{\vartheta}\cos\vartheta - \frac{1}{2}ml\dot{\vartheta}^2\sin\vartheta = Q(t) \tag{2}$$

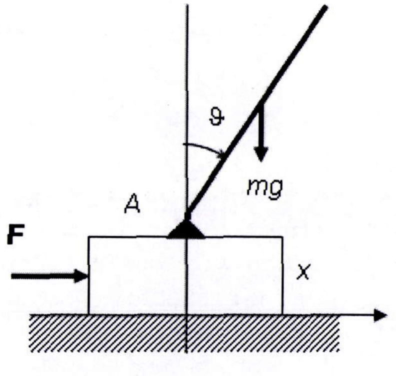


Fig. 1: Simplified model of an inverted pendulum on a cart

where the two generalised coordinates are the position x of the cart and the angular position ϑ of the pendulum measured from the upwards vertical. On the right hand side of (2) the control function Q is

$$Q\left(\equiv F\left(t-\tau\right)\right) = c_1\dot{\vartheta}\left(t-\tau\right) + c_0\vartheta\left(t-\tau\right). \tag{3}$$

In (3) the control function

$$Q = F\left(t-\tau\right) \tag{4}$$

is based on delayed values of variables. By some techniques, it is possible to compute an anticipatory control function

$$Q_A = F\left(t\right) \tag{5}$$

based on the present values of variables (see for example, Dubois [14]). In this paper, we do not give the techniques to obtain Q_A, and we just consider the two cases: Q and Q_A.

There are two different ways of thinking in investigating the dynamics of the controlled system (as it is described in our previous paper [1]). In analytical mechanics the simplest possible system of equations is used. While the mechanical system has a single degree-of-freedom we may express $\ddot{x}$ from (2) and substitute into (1) and then the equtions of motion are

$$\ddot{\vartheta} = -\frac{3\sin\vartheta\cos\vartheta}{4-3\cos^2\vartheta}\dot{\vartheta}^2 + \frac{6g\sin\vartheta}{(4-3\cos^2\vartheta)\,l} - \frac{6\cos\vartheta}{(4-3\cos^2\vartheta)\,ml}F(t-\tau) \tag{6}$$

$$F(t-\tau) = c_1\dot{\vartheta}(t-\tau) + c_0\vartheta(t-\tau)\,. \tag{7}$$

When from (7) F is substituted into (6), a single equation

$$\ddot{\vartheta} = -\frac{3\sin\vartheta\cos\vartheta}{4-3\cos^2\vartheta}\dot{\vartheta}^2 + \frac{6g\sin\vartheta}{(4-3\cos^2\vartheta)\,l}$$
$$-\frac{6\cos\vartheta}{(4-3\cos^2\vartheta)\,ml}\Big(c_1\dot{\vartheta}(t-\tau) + c_0\vartheta(t-\tau)\Big)\,. \tag{8}$$

is obtained for the only variable ϑ.

The other possibility is to keep the two generalised coordinates and force F as unknown functions from (3) and then we have a system of three equation

$$\begin{aligned}
\ddot{\vartheta} &= \frac{3l\dot{\vartheta}^2\sin\vartheta\cos\vartheta - 6g\sin\vartheta}{(-4+3\cos^2\vartheta)\,l} + \frac{6F\cos\vartheta}{(-4+3\cos^2\vartheta)\,ml} \\
\ddot{x} &= \frac{3g\sin\vartheta\cos\vartheta - 2l\dot{\vartheta}^2\sin\vartheta}{(-4+3\cos^2\vartheta)} - \frac{4F}{(-4+3\cos^2\vartheta)\,m} \\
F(t-\tau) &= c_1\dot{\vartheta}(t-\tau) + c_0\vartheta(t-\tau)\,.
\end{aligned} \tag{9}$$

A detailed derivation of the dynamical systems (6), (7) and (9) can be found in [1] and it is followed by a linear stability investigation of the upright position. Then the behaviour of the systems with delayed (4) and anticipatory (5) control was compared by numerical analysis, which requires discretisation. Such discrete system are widely studied in the literature of computing anticipatory systems [3,6]. Now instead of numerical simulation we construct a discrete mapping and its stability will be studied analytically.

2 Construction of Systems with Anticipatory and Feedback Controls

2.1 Reduced Order System with a Hidden Anticipatory Control

As we have already seen two ways are possible to get the equtions of motion, but in mechanics the sets of equations (6), (7) and (9) are equivalent with equation (10).

While (10) is a functional differential equation, in mechanical engineering we often neglect delay effect to get an appoximate equation

$$\ddot{\vartheta} = \frac{-3\sin\vartheta\cos\vartheta}{4-3\cos^2\vartheta}\dot{\vartheta}^2 + \frac{6g\sin\vartheta}{(4-3\cos^2\vartheta)\,l} - \frac{6\cos\vartheta\left(c_1\dot{\vartheta}(t) + c_0\vartheta(t)\right)}{(4-3\cos^2\vartheta)\,ml}, \tag{10}$$

which is suitable for further analytical studies. As we have seen in the previous part this step means to use an anticipatory control function Q_A from (5) instead of feedback Q of (4). Now the question is, how the stability properties of the original and the "simplified" dynamical systems relate to each other.

In case of the anticipatory control function (5) an incursive feed-in-time system [14] is obtained

$$F(t) = c_1\dot{\vartheta}(t) + c_0\vartheta(t). \tag{11}$$

Let us introduce new variables:

$$y_1 = \vartheta,\ y_2 = \dot{\vartheta},$$

then from (10)

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \frac{-3\sin y_1\cos y_1}{4-3\cos^2 y_1}{y_2}^2 + \frac{6g\sin y_1}{(4-3\cos^2 y_1)\,l} - \frac{6\cos y_1\left(c_1y_2 + c_0y_1\right)}{(4-3\cos^2 y_1)\,ml} \end{aligned} \tag{12}$$

is obtained. To find the solutions of (12) a discrete time $t \in [t_0,\ t_1, \ldots, t_i, \ldots]$ system will be introduced with constant time steps

$$t_i = t_0 + i\Delta t, \qquad \text{where} \qquad i = 1, 2, \ldots,$$

and simplifying notations

$$y_k(i) = y_k(t_i) \qquad (k = 1, 2, \quad i = 1, 2, \ldots)$$

are used. Now for the numerical integration of (12)

$$\begin{aligned} y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\ y_2(i+1) &= y_2(i) + \left(\frac{-3\sin y_1(i)\cos y_1(i)}{4-3\cos^2 y_1(i)}\left(y_2(i)\right)^2\right. \\ &+ \frac{6g\sin y_1(i)}{(4-3\cos^2 y_1(i))\,l} - \frac{6c_1\cos y_1(i)}{(4-3\cos^2 y_1(i))\,ml}y_2(i) \\ &- \left.\frac{6c_0\cos y_1(i)}{(4-3\cos^2 y_1(i))\,ml}y_1(i)\right)\Delta t \end{aligned} \tag{13}$$

is obtained. Remark that expression (13) is a recursion, but it describes the behavior of a system with anticipatory control (11).

2.2 Multiple Variables with Feedback Control

Now we study the delayed system and use all the physical variables ϑ, x, F. Similarly to the previous case we introduce new variables:

$$y_1 = \vartheta,\ y_2 = \dot{\vartheta}, y_3 = x,\ y_4 = \dot{x},\ y_5 = F.$$

From (9) a set of equations

$$\begin{aligned}
\dot{y}_1 &= y_2(t) \\
\dot{y}_2 &= \frac{3y_2^2(t)l\sin y_1\cos y_1(t) - 6g\sin y_1(t)}{(-4+3\cos^2 y_1(t))\,l} \\
&+ \frac{6\cos y_1(t)\,y_5(t)}{(-4+3\cos^2 y_1(t))\,ml} \\
\dot{y}_3 &= y_4(t) \\
\dot{y}_4 &= \frac{3g\sin y_1(t)\cos y_1(t) - 2y_2^2(t)\,l\sin y_1(t)}{(-4+3\cos^2 y_1(t))} \\
&- \frac{4y_5(t)}{(-4+3\cos^2 y_1(t))\,m} \\
y_5 &= c_1 y_2(t-\tau) + c_0 y_1(t-\tau)
\end{aligned} \tag{14}$$

is obtained. Let us introduce discrete time system as before. Using similar simplifying notations

$$y_k(i) = y_k(t_i) \qquad (k = 1, 2, ..., 5, \quad i = 1, 2, \ldots).$$

and by assuming that $\tau = \Delta t$ the numerical integration of (14) leads to a discrete dynamical system

$$\begin{aligned}
y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\
y_2(i+1) &= y_2(i) + \left(\frac{3y_2^2(i)\,l\sin y_1(i)\cos y_1(i) - 6g\sin y_1(i)}{(-4+3\cos^2 y_1(i))\,l}\right. \\
&+ \left.\frac{6\cos y_1(i)\,y_5(i)}{(-4+3\cos^2 y_1(i))\,ml}\right)\Delta t \\
y_3(i+1) &= y_3(i) + \Delta t y_4(i) \\
y_4(i+1) &= y_4(i) + \left(\frac{3g\sin y_1(i)\cos y_1(i) - 2y_2^2(i)\,l\sin y_1(i)}{(-4+3\cos^2 y_1(i))}\right. \\
&- \left.\frac{4y_5(i)}{(-4+3\cos^2 y_1(i))\,m}\right)\Delta t \\
y_5(i+1) &= c_1 y_2(i) + c_0 y_1(i)
\end{aligned} \tag{15}$$

Now to compare the effect of controls (4) and (5) we should study systems (15) and (13). For this reason a linear operator will be constructed, which maps the i-th state vector into the $i+1$-st. This map is called the monodromy operator.

3 The use of Monodromy Operator in the Linear Stability Analysis

The linear stability of a mapping can generally be investigated by studying the eigenvalues λ_i of its monodromy operator (see for example [4, 5]).

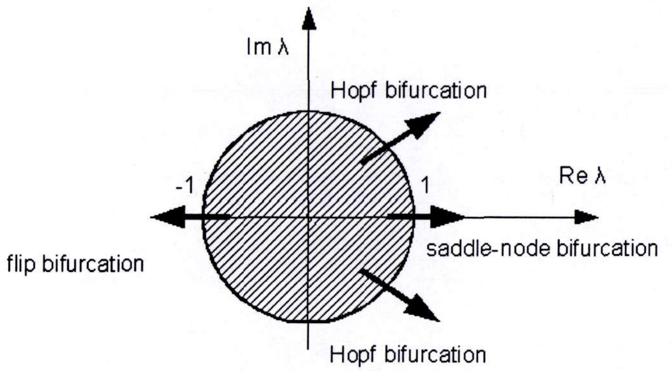


Fig. 2: Stability boundary and bifurcations in the plane of complex eigenvalues

In the complex plane of eigenvalues (see Fig. 2) the stable region is in the unit disc and loss of stability happens when one of the eigenvalues leaves the unit disc. The regions of three possible types of instabilities the flip, saddle-node and Hopf bifurcations are also shown in Fig. 2.

For anticipatory case from the linearisation of the right-hand-side of (13)

$$
\begin{aligned}
y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\
y_2(i+1) &= y_2(i) + \left(+ \frac{6 g y_1(i)}{l} \right. \\
&- \left. \frac{6c_1}{ml} y_2(i) - \frac{6c_0}{ml} y_1(i) \right) \Delta t
\end{aligned}
$$

monodromy operator reads

$$\mathbf{A} = \begin{bmatrix} 1 & \Delta t \\ 6\,\frac{\Delta t\, g}{l} - 6\,\frac{c_0\,\Delta t}{ml} & 1 - 6\,\frac{c_1\,\Delta t}{ml} \end{bmatrix}.$$

Its characteristic equation is

$$\lambda^2 + 2\,\frac{(-ml + 3\,c_1\,\Delta t)}{ml}\lambda - \frac{6\,(\Delta t)^2\,gm - 6\,(\Delta t)^2\,c_0 - ml + 6\,c_1\,(\Delta t)}{ml} = 0$$

The solutions are

$$\lambda_{1,2} = 1 - 3\,\widetilde{c}_1\,\Delta t \pm \sqrt{(3\,\widetilde{c}_1\Delta t)^2 + 6\,(\Delta t)^2\,(\alpha^2 - \widetilde{c}_0)} \tag{16}$$

where

$$\widetilde{c}_1 = \frac{c_1}{ml}, \quad \widetilde{c}_0 = \frac{c_0}{ml} \quad \text{and } \alpha^2 = \frac{g}{l}.$$

The stability condition is

$$|\lambda| < 1$$

for all solutions of (16).

In the case of feedback control the linearized system from (15) is

$$\begin{aligned}
y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\
y_2(i+1) &= 6\frac{\Delta t g}{l} y_1(i) + y_2(i) - 6\frac{\Delta t}{ml} y_5(i) \\
y_3(i+1) &= y_3(i) + \Delta t y_4(i) \\
y_4(i+1) &= -3\frac{\Delta t g}{l} y_1(i) + y_4(i) + 4\frac{\Delta t}{m} y_5(i) \\
y_5(i+1) &= c_0 y_1(i) + c_1 y_2(i)
\end{aligned}$$

and the monodromy operator is

$$\mathbf{B} = \begin{bmatrix} 1 & \Delta t & 0 & 0 & 0 \\ 6\,\frac{\Delta t\, g}{l} & 1 & 0 & 0 & -6\,\frac{\Delta t}{ml} \\ 0 & 0 & 1 & \Delta t & 0 \\ -3\,\frac{\Delta t\, g}{l} & 0 & 0 & 1 & 4\,\frac{\Delta t}{m} \\ c_0 & c_1 & 0 & 0 & 0 \end{bmatrix}.$$

Its characteristic equation is

$$\det\left(\mathbf{B} - \lambda \mathbf{I}\right) = 0 \tag{17}$$

where **I** denotes identity operator as usual.

Introducing $\varepsilon = -1 + \lambda$ the form of (17) is

$$\det \begin{bmatrix} -\varepsilon & \Delta t & 0 & 0 & 0 \\ 6\,\frac{\Delta t\, g}{l} & -\varepsilon & 0 & 0 & -6\,\frac{\Delta t}{ml} \\ 0 & 0 & -\varepsilon & \Delta t & 0 \\ -3\,\frac{\Delta t\, g}{l} & 0 & 0 & -\varepsilon & 4\,\frac{\Delta t}{m} \\ c_0 & c_1 & 0 & 0 & -\varepsilon - 1 \end{bmatrix} = 0$$

thus

$$\varepsilon^5 + \varepsilon^4 - 6\,\frac{\Delta t\,(-c_1 + \Delta t\, gm)\,\varepsilon^3}{ml} - 6\,\frac{(\Delta t)^2\,(-c_0 + gm)\,\varepsilon^2}{ml} = 0$$

$\varepsilon = 0$, that is, $\lambda = 1$ is solution of multiplicity two.

This result is implied by the physical fact that the motion of the inverted pendulum has a rigid body mode with indeterminate stability properties. Here the stability boundary appears for this reason.

To find the other roots

$$(-1+\lambda)^3 + (-1+\lambda)^2 - 6\,\frac{\Delta t\,(-c_1 + \Delta t\, gm)\,(-1+\lambda)}{ml} - 6\,\frac{(\Delta t)^2\,(-c_0 + gm)}{ml} = 0 \tag{18}$$

should be solved for λ. The stability condition is again $|\lambda| < 1$ for the solutions of (18).

The stability domains of cases with anticipatory and feedback controls can be compared, if numerical data are used. By fixing $\Delta t = 0.005$ and $\alpha = 1$ we have from (18)

$$(-1+\lambda)^3 + (-1+\lambda)^2 - 6\,(0.005 - \widetilde{c}_1)\,(-1+\lambda) + 0.000150\,\widetilde{c}_0 - 0.000150 = 0 \tag{19}$$

The solutions of (19) are plotted in Fig. 3 as functions of $ch0 = \frac{c_0}{ml}$ and $ch1 = \frac{c_1}{ml}$. The same numerical data are used in the case of the anticipatory control. They are substituted into (16) and λ_i $(i = 1, 2)$ are plotted in Fig. 4.

By comparing Figs. 3 and 4 we find that the results for eigenvalue λ_1 are visibly similar, in both cases they are larger then 1 at the origin and near to the axis $\widetilde{c}_1 = 0$. We can also observe that in anticipatory case λ_1 decreases "faster" for $\widetilde{c}_1 > 0$. The

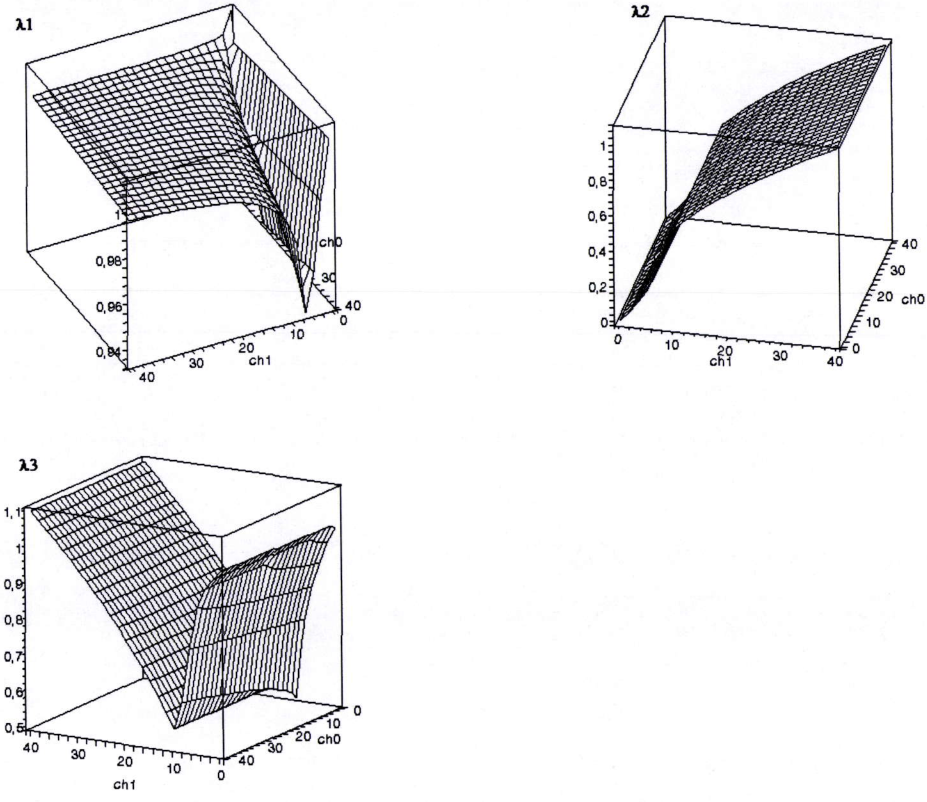


Fig. 3: Eigenvalues $\lambda_i = \lambda_i(\widetilde{c}_0, \widetilde{c}_1)$, i=1,2,3 by the feedback control case

main difference is in the form of the surfaces calculated for the other eigenvalues, because in the feedback case we see a stability limit for $\widetilde{c}_1 > 0$.

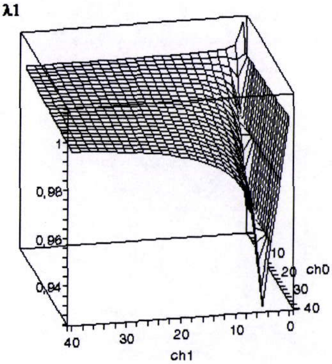
To find exact stability boundaries for (16) and (19), we should substitute

$$\lambda = \beta \pm \sqrt{1-\beta^2}, \qquad 0 \le \beta \le 1$$

into (16) and (19), and solve them to $\widetilde{c}_1, \widetilde{c}_0$. Then the stability charts presented in Fig. 5 are obtained.

Here the main difference is that for the anticipatory system there is no upper boundary for the stable region in both directions, while in case of feedback system stable region is bounded. In this sense stability properties are weaker in the feedback case.

These results show that in case of a mechanical system, which is controlled by some digital device the finite time step causes a decrease in the stable region of the control parameter plane. When delay is omitted a hidden anticipatory effect may



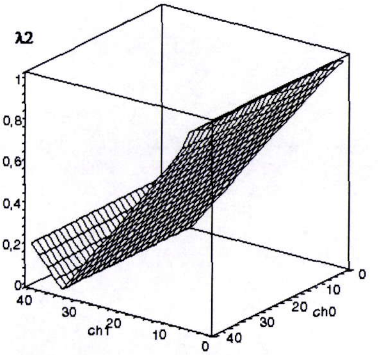
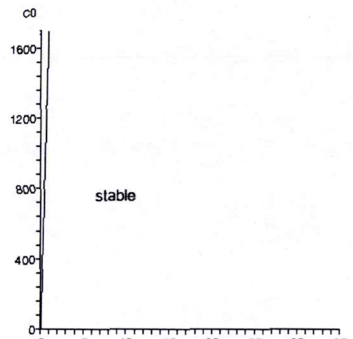
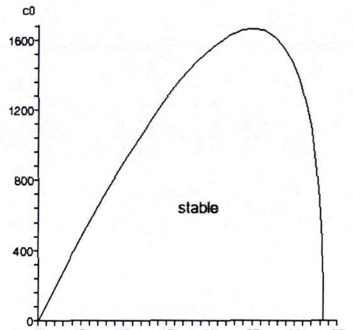


Fig. 4: The eigenvalues by the anticipatory case: $\lambda_i = \lambda_i(\widetilde{c}_0, \widetilde{c}_1)$, i=1,2



anticipatory



feedback

Fig. 5: Stability charts in plane $(\widetilde{c}_1, \widetilde{c}_0)$

result in better stability properties and larger stability region.

4 Conclusion

The result of our study show that we should be careful in modeling of digitally controlled mechanical systems. When the concepts of the classical analytical dynamics are used and we reduce the number of independent variables an equations to the minimum (the number of the mechanical degrees of freedom) a hidden anticipatory effect may appear. In this paper it can be interpreted as an anticipatory control.

At last we should add two remarks. First, remember that the recursive system has a double critical eigenvalue. It shows that the system is on stability boundary, but the physical system is a free system in coordinate x and this fact is the source of such kind of instabilty. Second, when (13) and (11) are compared we see how incursion is hidden.

The results of the study show that in modeling we should keep control force apart to see the real nature of control. It is quite different from the method used in analytical mechanics, where control is often treated as a mechanical constraint.

Acknowledgements

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ERRATUM

There are typing errors in Béda's paper [1]. The correct version of equation (21) of [1] is

$$\begin{bmatrix} \ddot{\vartheta} \\ \ddot{x} \end{bmatrix} - \begin{bmatrix} \frac{3l\dot{\vartheta}^2 \sin\vartheta \cos\vartheta - 6g \sin\vartheta}{(-4+3\cos^2\vartheta)l} \\ \frac{3g \sin\vartheta \cos\vartheta - 2l\dot{\vartheta}^2 \sin\vartheta}{(-4+3\cos^2\vartheta)} \end{bmatrix} = \begin{bmatrix} \frac{6\cos\vartheta\left(c_1\dot{\vartheta}(t-\tau)+c_0\vartheta(t-\tau)\right)}{(-4+3\cos^2\vartheta)ml} \\ -\frac{4\left(c_1\dot{\vartheta}(t-\tau)+c_0\vartheta(t-\tau)\right)}{(-4+3\cos^2\vartheta)m} \end{bmatrix}$$

and equation (22) of [1] reads

$$\begin{bmatrix} \ddot{\vartheta} \\ \ddot{x} \end{bmatrix} - \begin{bmatrix} \frac{3l\dot{\vartheta}^2 \sin\vartheta \cos\vartheta - 6g \sin\vartheta}{(-4+3\cos^2\vartheta)l} \\ \frac{3g \sin\vartheta \cos\vartheta - 2l\dot{\vartheta}^2 \sin\vartheta}{(-4+3\cos^2\vartheta)} \end{bmatrix} = \begin{bmatrix} \frac{6F\cos\vartheta}{(-4+3\cos^2\vartheta)ml} \\ -\frac{4F}{(-4+3\cos^2\vartheta)m} \end{bmatrix}.$$

All the other equations and consequently the numerical analysis presented in [1] are correct.

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