On Anticipatory Systems at Continua

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Abstract
In case of classical continuum mechanics the set of basic equations consists of the equation of motion, the kinematic equation and the constitutive equations. The paper concentrates on the stability problems and the effects of discretization on material modeling. The method of investigation is analytic, the monodromy operator of the discrete system is studied. We study how discretization, stability and anticipation act on one another. As results we show cases, when the anticipatory nature of a material model leads to instability.
Keywords : constitutive equation, discretization, monodromy operator.

1 Introduction

There are numerous papers dealing with anticipatory systems discretization and stability analysis [4, 8, 2, 1]. In this paper we study a problem, which appears at numerical analysis of a continuum. Such calculations are performed quite frequently in mechanical or structural engineering and stability problems are of great importance [3].

In case of classical continuum mechanics [6] the set of basic equations contains the equation of motion

$$\rho \dot{v} = \frac{\partial \sigma}{\partial x}, \tag{1}$$

the kinematical equation

$$\dot{\varepsilon} = \frac{\partial v}{\partial x} \tag{2}$$

and the constitutive equation.

Such group of equations has a twofold nature. On the one hand it creates a mathematical complete set of partial differential equations, which could be solved for the unknown field like the stress, strain and velocity fields. On the other hand, all material properties are included via constitutive modeling. All the data of experimental investigations and material tests appear in this part of the system of

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equations. For this reason constitutive equations are often referred as material model, or material law.

In the following the generally used notations of the continuum mechanics are used [6], where ρ is mass density and v, σ and ε denotes velocity, stress and strain fields. These are the basic unknown functions of the state vector $[v, \varepsilon, \sigma]$. Derivatives with respect to time are denoted by dots

$$\frac{d\varepsilon}{dt} = \dot{\varepsilon}.$$

In engineering the basic equations should be solved and the common way is to use numerical methods (the most popular is Finite Element Method in lots of commercial software toolkits). When some numerical analysis is performed to solve that system of basic equations a time discretization is necessary. In the next part an elastic material will be studied in the uniaxial case.

2. Elastic rate independent material

The easiest case for modeling elastic materials is, when Hooke's law is used [6]

$$\sigma = E\varepsilon, \tag{3}$$

where E denotes Young modulus. When time discretization is performed [1], such constitutive equation reads in a differential form

$$d\sigma = Ed\varepsilon,$$

while discretization results

$$\Delta\sigma = E\Delta\varepsilon,$$

where time step is Δt and for the functions of the state vector $\sigma_i = \sigma(i\Delta t)$ etc. Now

$$\Delta\sigma = \sigma_{i+1} - \sigma_i$$

then

$$\sigma_{i+1} - \sigma_i = E(\varepsilon_{i+1} - \varepsilon_i),$$

which implies

$$\sigma_i = E\varepsilon_i$$

or

$$\sigma_{i+1} = E\varepsilon_{i+1} \tag{4}$$

being anticipatory (incursive) forms [7], because the stress at step i (or $i+1$) is determined by the value of the strain at the same instant of time.

(1) and (2) are

$$\frac{dv}{dt} = \frac{1}{\varrho}\frac{\partial \sigma}{\partial x}$$

$$\frac{d\varepsilon}{dt} = \frac{\partial v}{\partial x}$$

hence in differential form we have

$$dv = \frac{1}{\varrho}\frac{\partial \sigma}{\partial x} dt$$

and

$$d\varepsilon = \frac{\partial v}{\partial x} dt$$

and the formal discretized form reads

$$v_{i+1} = v_i + \frac{1}{\varrho}\left.\frac{\partial \sigma}{\partial x}\right|_i \Delta t, \qquad \varepsilon_{i+1} = \varepsilon_i + \left.\frac{\partial v}{\partial x}\right|_i \Delta t \tag{5}$$

by adding (4) the basic set of equations forms a strong anticipatory system.

From equations (1), (2) and (4) we form

$$\rho \dot{v} = E\frac{\partial \varepsilon}{\partial x}$$

$$\dot{\varepsilon} = \frac{\partial v}{\partial x}$$

and the associated discrete dynamical system reads

$$v_{i+1} = v_i + \frac{E}{\varrho}\left.\frac{\partial \varepsilon}{\partial x}\right|_i \Delta t, \qquad \varepsilon_{i+1} = \varepsilon_i + \left.\frac{\partial v}{\partial x}\right|_i \Delta t$$

in matrix notation

$$\begin{bmatrix} v_{i+1} \\ \varepsilon_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{E}{\varrho}\Delta t\left(\frac{\partial}{\partial x}\right) \\ \Delta t\left(\frac{\partial}{\partial x}\right) & 1 \end{bmatrix} \begin{bmatrix} v_i \\ \varepsilon_i \end{bmatrix}$$

Then the eigenvalues of the monodromy operator [7]

$$M := \begin{bmatrix} 1 & \frac{E}{\varrho}\Delta t\left(\frac{\partial}{\partial x}\right) \\ \Delta t\left(\frac{\partial}{\partial x}\right) & 1 \end{bmatrix}$$

can be obtained by solving the eigenvalue equation

$$Mu - \lambda u = 0$$

for λ, where u is a proper function satisfying homogeneous boundary conditions. Then stability is present, if $|\lambda| < 1$, [9].

For the sake of simplicity we restrict the study to periodic perturbations in form

$$u = \begin{bmatrix} v_0 \\ \varepsilon_0 \end{bmatrix} \exp(i\alpha x) \tag{6}$$

then the eigenvalue equation results

$$\lambda^2 - 2\lambda + 1 + \alpha^2 \frac{E}{\varrho} (\Delta t)^2 = 0$$

thus

$$\lambda_{1,2} = 1 \pm \frac{\sqrt{4 - 4\left(1 + \alpha^2 \frac{E}{\varrho} (\Delta t)^2\right)}}{2}$$

$$\lambda_{1,2} = 1 \pm \alpha \Delta t \sqrt{-\frac{E}{\varrho}} \tag{7}$$

When $E > 0$ (7) is a complex number. Its absolute value is

$$|\lambda_{1,2}| = \sqrt{1 + \frac{E}{\varrho} \alpha^2 (\Delta t)^2} > 1 \tag{8}$$

while at $E = 0$

$$|\lambda_{1,2}| = 1$$

and when $E < 0$

$$\lambda_1 = 1 - \alpha \Delta t \sqrt{-\frac{E}{\varrho}} < 1, \quad \lambda_2 = 1 + \alpha \Delta t \sqrt{-\frac{E}{\varrho}} > 1 \tag{9}$$

By comparing $(\lambda_{1,2})^2$ (8) and λ_2^2 from (9) we have

$$r_1^2 := 1 + \frac{E}{\varrho} \alpha^2 (\Delta t)^2 \qquad \text{and} \quad \mathrm{r}_2^2 := 1 + 2\alpha \Delta \mathrm{t} \sqrt{-\frac{\mathrm{E}}{\varrho}} + \frac{\mathrm{E}}{\varrho} \alpha^2 (\Delta \mathrm{t})^2$$

thus as $\Delta t \to 0$ $\lim r_1^2 < \lim r_2^2$. Thus in both case instability is found, but case $E < 0$ is "more" unstable. At $E = 0$ the static bifurcation condition is satisfied.

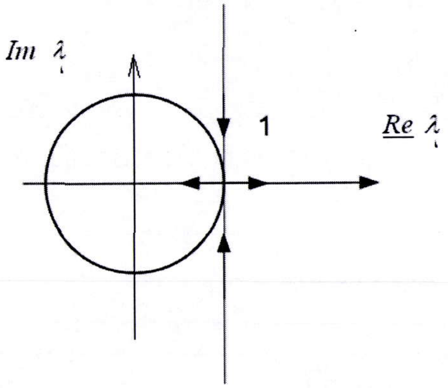


Fig. 1: The change of location of roots at the unit circle as E decreases

3. Damping effect

Let us change the constitutive equation to a rate dependent one by adding term $D\dot{\sigma}$. Then the constitutive equation reads

$$\sigma + D\dot{\sigma} = E\varepsilon, \tag{10}$$

which happens in case of visco-elastic materials. From (10)

$$\dot{\sigma} = \frac{E}{D}\varepsilon - \frac{1}{D}\sigma$$

or in differential form we have

$$d\sigma = \frac{E}{D}\varepsilon dt - \frac{1}{D}\sigma dt.$$

After discretization

$$\Delta\sigma = \frac{E}{D}\varepsilon_i \Delta t - \frac{1}{D}\sigma_i \Delta t$$

or

$$\sigma_{i+1} = \sigma_i - \frac{1}{D}\sigma_i \Delta t + \frac{E}{D}\varepsilon_i \Delta t$$

and by adding equations (5) a recursive system

$$\begin{bmatrix} v_{i+1} \\ \varepsilon_{i+1} \\ \sigma_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{\varrho}\Delta t \left(\frac{\partial}{\partial x}\right) \\ \Delta t \left(\frac{\partial}{\partial x}\right) & 1 & 0 \\ 0 & \frac{E}{D}\Delta t & \left(1 - \frac{1}{D}\Delta t\right) \end{bmatrix} \begin{bmatrix} v_i \\ \varepsilon_i \\ \sigma_i \end{bmatrix}$$

is obtained. Now the monodromy operator is

$$Mu := \begin{bmatrix} 1 & 0 & \frac{1}{\varrho}\Delta t\left(\frac{\partial}{\partial x}\right) \\ \Delta t\left(\frac{\partial}{\partial x}\right) & 1 & 0 \\ 0 & \frac{E}{D}\Delta t & \left(1-\frac{1}{D}\Delta t\right) \end{bmatrix} u$$

and the eigenvalue equation for similar periodic perturbations as in (6) is

$$\exp\left(i\alpha x\right) \begin{bmatrix} (1-\lambda) & 0 & \frac{1}{\varrho}\Delta t\,(i\alpha) \\ \Delta t\,(i\alpha) & (1-\lambda) & 0 \\ 0 & \frac{E}{D}\Delta t & \left(1-\frac{1}{D}\Delta t-\lambda\right) \end{bmatrix} \begin{bmatrix} v_0 \\ \varepsilon_0 \\ \sigma_0 \end{bmatrix} = 0$$

The characteristic equation is

$$(1-\lambda)^2\left((1-\lambda)\,D-\Delta t\right)-\alpha^2\frac{E}{\varrho}\left(\Delta t\right)^3=0 \tag{11}$$

While the square of the elastic wavespeed is

$$c^2=\frac{E}{\varrho}$$

from (11)

$$-D\lambda^3+\lambda^2\left(D-\Delta t\right)+\lambda\left(2\Delta t-3D\right)+D-\alpha^2c^2\left(\Delta t\right)^3=0 \tag{12}$$

which is a third order algebraic equation for the eigenvalues of the monodromy operator. Its solutions can be obtained by using for example Cardano's formula. Then the absolute value of the roots decides stability.

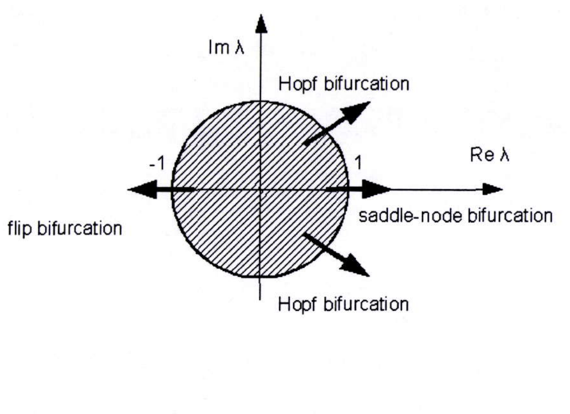
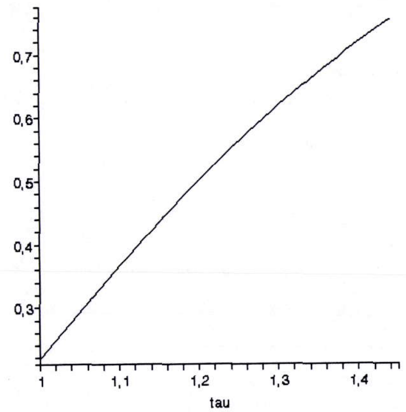


Fig. 2: Stability boundary in the plane of complex eigenvalues



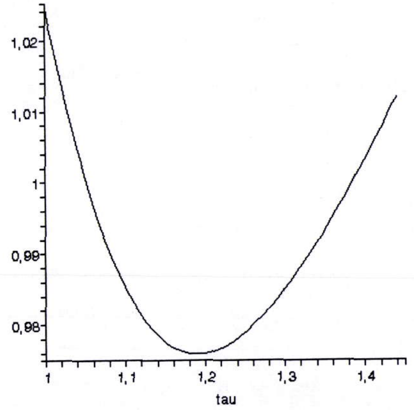


Fig. 3: Absolute values of the eigenvalues of the monodromy operator: $|\lambda_1|$ and $|\lambda_2| = |\lambda_3|$

In the complex plane of eigenvalues (see Fig. 2) the stable region is in the unit circle and loss of stability happens when one of the eigenvalues leaves the unit circle. The regions of three possible types of instabilities the flip, saddle-node and Hopf bifurcations are also shown in Fig. 2.

In Fig. 3 solutions $|\lambda_i| = |\lambda_i(\tau)|$ of (11) were calculated at $\Delta t = 0.00037s$ and $\alpha = 1$. It shows that there exists a region of stability in contrast to the anticipatory (rate independent) constitutive equation. The stability boundary can also be calculated by substituting the critical unit eigenvalue

$$\lambda = \pm\sqrt{1-\beta^2} \pm i\beta, \qquad 0 \leqq \beta \leqq 1$$

into (12), (see Fig. 4)

In Fig. 4 the stability region is under the curve, where c denotes wavespeed, which has the value 3000 m/s for elastic case and less for non-elastic case. Parameter $\tau = \frac{\Delta t}{D}$ denotes the ratio of time step and the rate dependent parameter.

By studying the way of loss of stability on the stability boundary [9], Hopf bifurcation happens except at flip ($\lambda = -1$) and saddle-node ($\lambda = 1$). If $\lambda = 1$ from (12),

$$\alpha^2 c^2 D (\Delta t)^2 = 0,$$

thus $c = 0$ or $D = 0$. The first case is the so-called divergence type of material instability or strain localization, while the second leads to Hooke's elastic rate independent material (part 2). When $\lambda = -1$, from (12)

$$D = \frac{4\Delta t - c^2 (\Delta t)^3}{8}$$

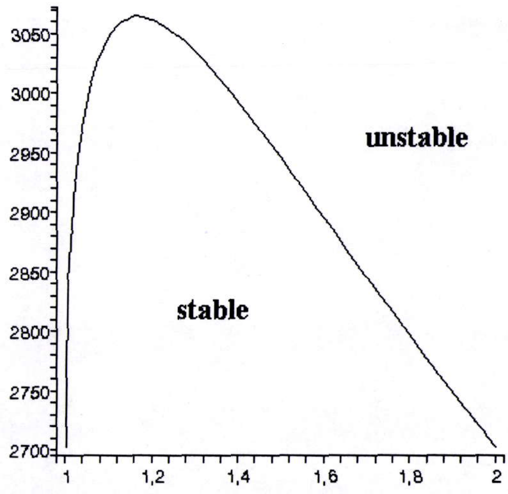


Fig. 4: Stability chart in parameters τ, c

is obtained. Thus the condition of flip bifurcation contains time step and material constants.

4 Conclusion

Numerical solution of all dynamic problems of mechanics of continua require discrete time systems. When it is constructed from the classical basic set of equations, we may obtain incursive systems. In rate independent elasticity such system leads to unstable behaviour. When rate dependence is added, the resulting discrete time system remains recursive and stable.

In studying the case of a continuum satisfying Hook's law with and without rate dependence we find that simply time discretization results instability by causing anticipatory nature.

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