Laws of Form and Discrete Physics

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Abstract
This essay is a discussion of the concept of reflexivity and its relationships with Laws of Form, self-reference, re-entry , eigenform and the foundations of physics.

Keywords: marked state unmarked state, eigenform, token, reflexive domain, eigenvector, nexus, iterant, discrete derivative, recursion, process, commutator, position, momentum, quantum process, diffusion

1. INTRODUCTION

Reflexive is a term that refers to the presence of a relationship between an entity and itself. Nowhere is there a way to effectively cut an individual participant from the world and make him into a purely objective observer. His action in the world is concomitant to being reflexively linked with that world. Just so for theorists of the world (cyberneticists) for their theories, if communicated, become part of the action and decision-making of that world.

How then, shall we describe a reflexive domain? We shall give an abstract definition that captures, what I believe to be the main conceptual feature of reflexivity. We then immediately prove that eigenforms, fixed points of transformations, are present for all transformations of the reflexive domain.

Eigenforms are the natural emergence of signs and tokens through recursion. The existence of fixed points for arbitrary transformations shows us that the domain we have postulated is very wide. It is not an objectively existing domain. It is a clearing in which structures can arise and new structures can arise. A reflexive domain is not an already-existing structure. To be what it claims to be, a reflexive domain must be a combination of existing structure and an invitation to create new structure and new concepts.

We are particularly interested in the way these concepts of reflexivity affect fundamentals of topology and fundamentals of physics. The last part of this essay is a formulation of elementary mathematics of matrices, complex numbers and exponentials in terms of process, reflexivity and eigenform.

We then show how quantum mechanics and discrete physics can be seen in the light of these interpretations.

Our essay begins with explication of Spencer-Brown's Laws of Form [3] and of the notion of eigenform as pioneered by Heinz von Foerster in his papers [4, 5,6,7] and explored in papers of the author [11, 12, 22, 23]. In [5] The familiar objects of our existence can be seen as tokens for the behaviors of the organism, creating apparently stable forms.

An object , in itself , is a symbolic entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions. We ourselves are

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such objects, we as human beings are "signs for ourselves", a concept originally due to the American philosopher C. S. Peirce [10]. Eigenforms are mathematical companions to Peirce's work.

In an observing system, what is observed is not distinct from the system itself, nor can one make a separation between the observer and the observed. The observer and the observed stand together in a coalescence of perception. From the stance of the observing system all objects are non-local, depending upon the presence of the system as a whole. It is within that paradigm that these models begin to live, act and enter into conversation with us.

We take a wider stance and consider the structure of spaces and domains that partake of the reflexivity of object and process. We make a definition of a reflexive domain (compare [1] and [18]). Our definition populates a space (domain) with entities that can be construed as objects. We assume that each object acts as a transformation on the space. This means that given entities A and B, then there is a new entity C that is the result of A and B acting together in the order AB (so that one can say that "A acts on B" for AB and one can say "B acts on A" for BA). This means that the reflexive space is endowed with a non-commutative and non-associative algebraic structure. The reflexive space is expandable in the sense that whenever we define a process, using entities that have already been constructed or defined, then that process can take a name, becoming a new entity/transformation of a space that is expanded to include itself. Reflexive spaces are open to evolution in time, as new processes are invented and new forms emerge from their interaction.

Reflexive spaces always have eigenforms for every element/transformation/entity in the space. The proof is simple:

Given F in a reflexive domain, define G by Gx = F(xx).
Then GG = F(GG) and so GG is an eigenform for F.

Just as promised, in a reflexive domain, every entity has an eigenform. From this standpoint, one should start with the concept of reflexivity and see that from it emerge eigenforms. Are we satisfied with this approach? We are not satisfied. For in order to start with reflexivity, we need to posit objects and processes. As we have already argued in this essay, objects are tokens for eigenbehaviours. And a correct or natural beginning is a field of processes where objects are seen as tokens of these processes.
The story is circular. Objects beget processes and processes beget objects. And with this we can be satisfied. We weave a tale the goes back and forth between recursion and eigenforms. We also relate these ideas of reflexivity and fixed points to left distributive non-associative algebras. We relate this with approaches to wholeness in physics and philosophy such as the work of Barbara Piechosinska [16]. A magma is a non-associative algebra with a single binary operation that is left-associative:

$$\mathbf{a*(b*c) = (a*b)*(a*c).}$$

This axiom says that every element A of the magma is a structure preserving mapping of the magma to itself with

$$\mathbf{A[x] = A*x,}$$
$$\mathbf{A[x*y] = A[x]*A[y].}$$

The notion of a magma is another view of self-reflexivity. We raise questions about the relationship of magmas and reflexive domains.

This paper explores the analogies of fixed points, observations and observables, eigenvectors and recursive processes in relation to foundations of physics.

For the complex numbers, think of the oscillatory process generated by **R(x) = -1/x.** The fixed point is **i** with $\mathbf{i^2 = -1}$, but the processes generated over the real numbers must be directly related to the idealized **i**. We shall let **I{+1,-1}** stand for an undisclosed alternation or ambiguity between **+1** and **-1** and call **I{+1,-1}** an *iterant*. There are two *iterant views*: **[+1,-1]** and **[-1,+1]**. These, seen as points of view of alternating process will become the square roots of negative unity. We introduce a temporal shift operator η such that

$$[a,b]\eta = \eta\ [b,a] \text{ and } \eta\ \eta\ = 1$$

so that contcatenated observations can include a time step of one-half period of the process **...abababab...** . We combine iterant views term-by-term as in **[a,b][c,d] = [ac,bd]**. Then we have, with i = $[1,-1]\eta$ (**i** is view/operator),

$$ii = [1,-1]\eta\ [1,-1]\eta\ = [1,-1][-1,1]\eta\ \eta\ = [-1,-1] = -1.$$

This gives rise to a new process-oriented construction of the complex numbers. The key to rethinking the complex numbers in these terms is to understand that **i** represents a discrete oscillating temporal process. We take as a matter of principle that the usual real variable **t** for time is better represented as **it** so that time is seen to be a process, an observation and a magnitude all at once. This principle of the temporal nexus is supported by our analysis of i as an eigenform. The central metaphor of this paper is the temporal nexus where time is implicit, and time is explicit and keeping time. In the nexus there is neither form nor sign, motion nor time. Time, the measurement of time and time's indication all emerge at once from the nexus in the form of action that is embodied in **it**. The metaphor suggests that it is no accident that deeper physical reality is revealed when mere numerical time **t** is replaced by the time of the nexus **it**. The time of the nexus is at once flowing, beyond motion, an eigenform, a geometric operator and a discrete dynamics counting below where counting cannot go.

2. OBJECTS AS TOKENS FOR EIGENBEHAVIOURS

In his paper "Objects as Tokens for Eigenbehaviours" [5] von Foerster suggests that we think seriously about the mathematical structure behind the constructivist doctrine that *perceived worlds are worlds created by the observer.* At first glance such a statement appears to be nothing more than solipsism. At second glance, the statement appears to be a tautology, for who else can create the rich subjectivity of the immediate impression of the senses? In that paper he suggests that the familiar objects of our experience are the fixed points of operators. These operators *are* the structure of our perception. To the extent that the operators are shared, there is no solipsism in this point of view. It is the beginning of a mathematics of second order cybernetics.

Consider the relationship between an observer O and an "object" A. "The object remains in constant form with respect to the observer". This constancy of form does not preclude motion or change of shape. Form is more malleable than the geometry of

Euclid. In fact, ultimately the form of an "object" is the form of the distinction that "it" makes in the space of our perception. In any attempt to speak absolutely about the nature of form we take the form of distinction for the form. (paraphrasing Spencer-Brown [3]). It is the form of distinction that remains constant and produces an apparent object for the observer. How can you write an equation for this? We write

$$\mathbf{O(A) = A.}$$

The object **A** is a fixed point for the observer **O**. The object is an eigenform. We must emphasize that this is a most schematic description of the condition of the observer in relation to an object **A.** We record only that the observer as an actor (operator) manages to leave the (form of) the object unchanged. This can be a recognition of symmetry, but it also can be a description of how the observer, searching for an object, makes that object up (like a good fairy tale) from the very ingredients that are the observer herself.

And what about this matter of the object as a token for eigenbehaviour? This is the crucial step. We forget about the object and focus on the observer. We atttempt to "solve" the equation **O(A) = A** with **A** as the unknown. Not only do we admit that the "inner" structure of the object is unknown, we adhere to whatever knowledge we have.

We can start anew from the dictum that the perceiver and the perceived arise together in the condition of observation. This is mutuality. Neither perceiver nor the perceived have priority over the other. A distinction has emerged and with it a world with an observer and an observed. The distinction is itself an eigenform.

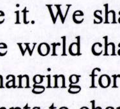
3. COMPRESENCE AND COALESCENCE

We identify the world in terms of how we shape it. We shape the world in response to how it changes us. We change the world and the world changes us. Objects arise as tokens of a behaviour that leads to seemingly unchanging forms. Forms are seen to be unchanging through their invariance under our attempts to change, to shape them.

For an observer there are two primary modes of perception -- *compresence* and *coalesence.* Compresence connotes the coexistence of separate entities together in one including space. Coalesence connotes the one space holding, in perception, the observer and the observed, inseparable in an unbroken wholeness. Coalesence is the constant condition of our awareness. Coalesence is the world taken in simplicity. Compresence is the world taken in apparent multiplicity. This distinction of compresence and coalesence, drawn by Henri Bortoft [2], can act as a compass in traversing the domains of object and reference. *Eigenform is a first step towards a mathematical description of coalesence.* In the world of eigenform the observer and the observed are one in a process that recursively gives rise to them both.

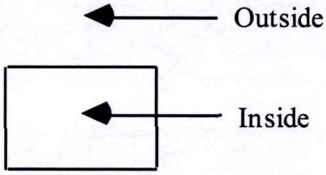
4. LAWS OF FORM

Laws of Form [LOF] is a lucid book with a topological notation based on one symbol, the mark:



The Mark

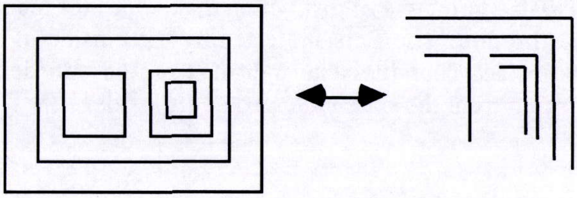
This single symbol is figured to represent a distinction between its inside and its outside:



Inside and Outside

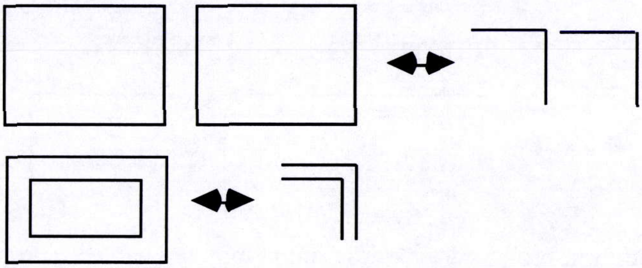
As is evident from the figure above, the mark is to be regarded as a shorthand for a rectangle drawn in the plane and dividing the plane into the regions inside and outside the rectangle. Spencer-Brown's mathematical system made just this beginning.

In this notation the idea of a distinction is instantiated in the distinction that the mark is seen to make in the plane. Patterns of non-intersecting marks (that is non-intersecting rectangles) are called expressions. For example,



Expressions

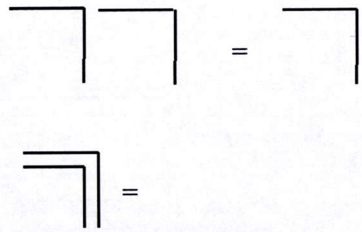
In this example, I have illustrated both the rectangle and the marks version of the expression. In an expression you can say definitively of any two marks whether one is or is not inside the other. The relationship between two marks is either that one is inside the other, or that neither is inside the other. These two conditions correspond to the two elementary expressions shown below.



Adjacent and Nested

The mathematics in Laws of Form begins with two laws of transformation about these two basic expressions.

Symbolically, these laws are:

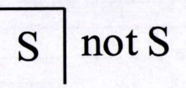


Calling and Crossing

In the first of these equations, the law of calling, two adjacent marks (neither is inside the other) condense to a single mark, or a single mark expands to form two adjacent marks. In the second equation, the law of crossing, two marks, one inside the other, disappear to form the unmarked state indicated by nothing at all. Alternatively, the unmarked state can give birth to two nested marks. A calculus is born of these equations, and the mathematics can begin. But first some epistemology:

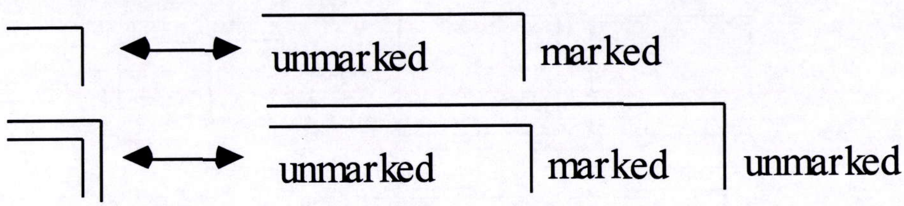
First we elucidate a principle of distinction that delineates the use of the mark.

Principle of Distinction: The state indicated by the outside of a mark is *not* the state indicated by its inside. Thus the state indicated on the outside of a mark is the state obtained by crossing from the state indicated on its inside.



Dichotomy

It follows from the principle of distinction, that the outside of an empty mark indicates the marked state (since its inside is unmarked). It also follows from the principle of distinction that the outside of a mark having another mark inscribed within it indicates the unmarked state.



Space and Value

Notice that the form produced by a description may not have the properties of the form being described. For example, the inner space of an empty mark is empty, but we describe it by putting the word unmarked there, and in the description that space is no longer empty. Thus do words obscure the form and at the same time clarify its representations.

Spencer-Brown begins his book, before introducing this notation, with a chapter on the concept of a distinction.

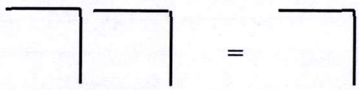
"We take as given the idea of a distinction and the idea of an indication, and that it is not possible to make an indication without drawing a distinction. We take therefore the form of distinction for the form."

From here he elucidates two laws:

1. The value of a call made again is the value of the call.
2. The value of a crossing made again is not the value of the crossing.

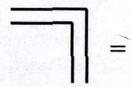
The two symbolic equations above correspond to these laws. The way in which they correspond is worth discussion.

First look at the law of calling. It says that the value of a repeated name is the value of the name. In the equation



Calling

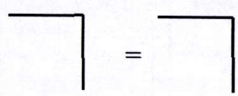
one can view either mark as the name of the state indicated by the outside of the other mark. In the other equation



Crossing

the state indicated by the outside of a mark is the state obtained by crossing from the state indicated on the inside of the mark. Since the marked state is indicated on the inside, the outside must indicate the unmarked state. The Law of Crossing indicates how opposite forms can fit into one another and vanish into the Void, or how the Void can produce opposite and distinct forms that fit one another, hand in glove.

There is an interpretation of the Law of Crossing in terms of movement across a boundary. In this story, a mark placed over a form connotes the crossing of the boundary from the Domain indicated by that form to the Domain that is opposite to it. Thus in the double mark above, the connotation is a crossing *from* the single mark on the inside. The single mark on the inside stands for the marked state. Thus by placing a cross over it, we transit to the unmarked state. Hence the disappearance to Void on the right-hand side of the equation. The value of a crossing made again is not the value of the crossing. The same interpretation yields the equation



Identity of Process and Name

where the left-hand side is seen as an instruction to cross from the unmarked state, and the right hand side is seen as an indicator of the marked state. The mark has a double carry of meaning. It can be seen as an operator, transforming the state on its inside to a different state on its outside, and it can be seen as the name of the marked state. That combination of meanings is compatible in this interpretation.

In this calculus of indications we see a precise elucidation of the way in which markedness and unmarkedness are used in language. In language we say that if you cross from the marked state then you are unmarked. This distinction is unambiguous in the realm of words. Not marked is unmarked. In this calculus of the mark these patterns are captured in a simple and non-trivial mathematics, the mathematics of the laws of form. Spencer-Brown makes the point that one can follow the analogy of introducing imaginary numbers in ordinary algebra to introduce *imaginary boolean values* in the arithmetic of logic. An apparently paradoxical equation such as

$$J = \overline{J \quad}\rceil$$

The Reentering Mark

can be regarded as an analog of the quadratic $\mathbf{x = -1/x}$, and its solutions will be values that go beyond marked and unmarked, beyond true and false. The reentering mark is the first representative of an object that is seen to arise as the fixed point of a process.

5. THE EIGENFORM MODEL

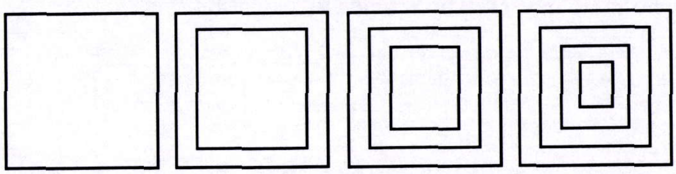
We have seen how the concept of an object has evolved. The notion of a fixed object has become the notion of a process that produces the apparent stability of an object. This process can be simplified in modelling to become a recursive process where a rule or rules are applied time and time again. The resulting object is the fixed poiint or *eigenform* of the process, and the process itself is the *eigenbehaviour.*

In this way we have a model for thinking about object as token for eigenbehaviour. This model examines the result of a simple recursive process carried to its limit.
For example, suppose that

$$F(X) = \boxed{\quad X \quad}$$

Figure 1

Each step in the process encloses the results of the previous step within a box. Here is an illustration of the first few steps of the process applied to an empty box X:



X F(X) F(F(X)) F(F(F(X)))

Figure 2

If we continue this process, then successive nests of boxes resemble one another, and in the limit of infinitely many boxes, we find that

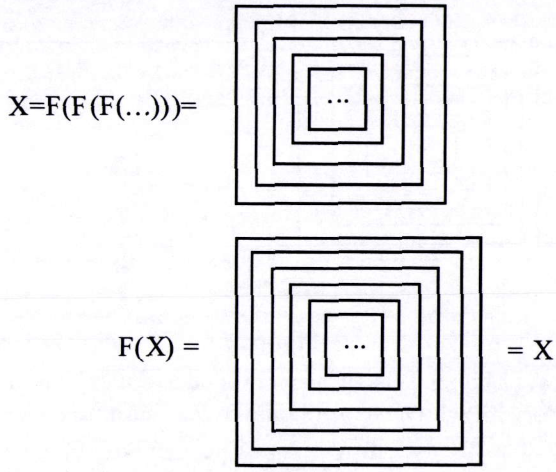


Figure 3

the infinite nest of boxes is invariant under the addition of one more surrounding box. Hence this infinite nest of boxes is a fixed point for the recursion. In other words, if X denotes the infinite nest of boxes, then

$$\mathbf{X = F(X).}$$

This equation is a description of a state of affairs. The form of an infinite nest of boxes is invariant under the operation of adding one more surrounding box.

In the process of observation, we interact with ourselves and with the world to produce stabilities that become the objects of our perception. These objects, like the infinite nest of boxes, often go beyond the specific properties of the world in which we operate. We make an imaginative leap to complete such objects to become tokens for eigenbehaviours. It is impossible to make an infinite nest of boxes. We do not make it. We *imagine* it. And in imagining that infinite nest of boxes, we arrive at the eigenform.

The leap of imagination to the infinite eigenform is a model of the human ability to create signs and symbols. In the case of the eigenform **X** with **X = F(X), X** can be regarded as the name of the process itself or as the name of the limit process. Note that if you are told that

$$\mathbf{X = F(X),}$$

then substituting **F(X)** for **X**, you can write

$$\mathbf{X = F(F(X)).}$$

Substituting again and again, you have

$$\mathbf{X = F(F(F(X))) = F(F(F(F(X)))) = F(F(F(F(F(X))))) = ...}$$

The process arises from the symbolic expression of its eigenform. In this view the eigenform is an implicate order for the process that generates it.

Sometimes one stylizes the structure by indicating where the eigenform **X** reenters its own indicational space by an arrow or other graphical device. See the picture below for the case of the nested boxes.

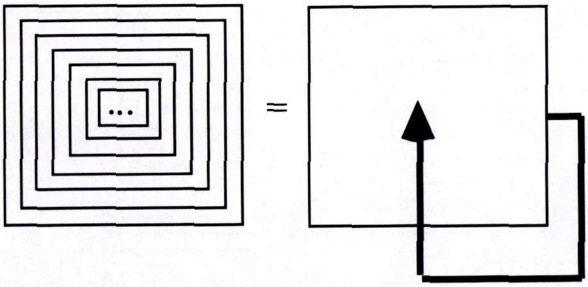


Figure 4

Does the infinite nest of boxes exist? Certainly it does not exist in this page or anywhere in the physical world with which we are familiar. The infinite nest of boxes exists in the imagination. It is a symbolic entity. Eigenform is the imagined boundary in the reciprocal relationship of the object (the "It") and the process leading to the object (the process leading to "It"). In the diagram below we have indicated these relationships with respect to the eigenform of nested boxes. Note that the "It" is illustrated as a finite approximation (to the infinite limit) that is sufficient to allow an observer to infer/perceive the generating process that underlies it.

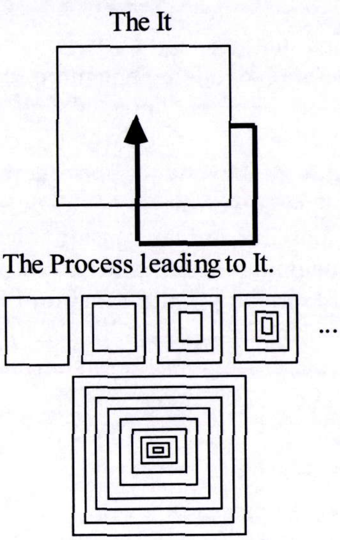


Figure 5

An object in the world (cognitive, physical, ideal,...) provides a conceptual center for the exploration of relationships related to its context and to the processes that generate it. If we take the suggestion to heart that objects are tokens for eigenbehaviors, then an object in itself is an entity, participating in a network of interactions, taking on its apparent solidity and stability from these interactions.

An object is an amphibian between the symbolic and imaginary world of the mind and the complex world of personal experience. The object, when viewed as process, is a dialogue between these worlds. The object when seen as a sign for itself, or in and of itself, is imaginary. The eigenform model can be expressed in abstract and general terms. Suppose that we are given a recursion (not necessarily numerical) with the equation

$$\mathbf{X(t+1) = F(X(t))}.$$

Here **X(t)** denotes the condition of observation at time **t.** Then **F(X(t))** denotes the result of applying the operations symbolized by **F** to the condition at time **t**. You could, for simplicity, assume that **F** is independent of time. Time independence of the recursion **F** will give us simple answers and we can later discuss what will happen if the actions depend upon the time. In the time independent case we can write

$$\mathbf{J = F(F(F(...)))}$$

the infinite concatenation of **F** upon itself. Then

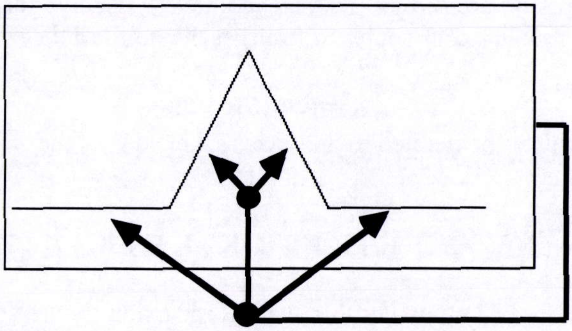
$$\mathbf{F(J) = J}$$

since adding one more **F** to the concatenation changes nothing.
Thus **J,** the infinite concatenation of the operation upon itself leads to a fixed point for **F. J** is said to be the eigenform for the recursion **F**. We see that every recursion has an eigenform. Every recursion has an (imaginary) fixed point.

We end this section with one more example. This is the eigenform of the Koch fractal [14]. In this case one can write
symbolically the eigenform equation

$$\mathbf{K = K\{\,K\;\;K\,\}\,K}$$

to indicate that the Koch Fractal reenters its own indicational space four times (that is, it is made up of four copies of itself, each one-third the size of the original. The curly brackets in the center of this equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.



K = K { K K } K

Figure 6

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above.
The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.

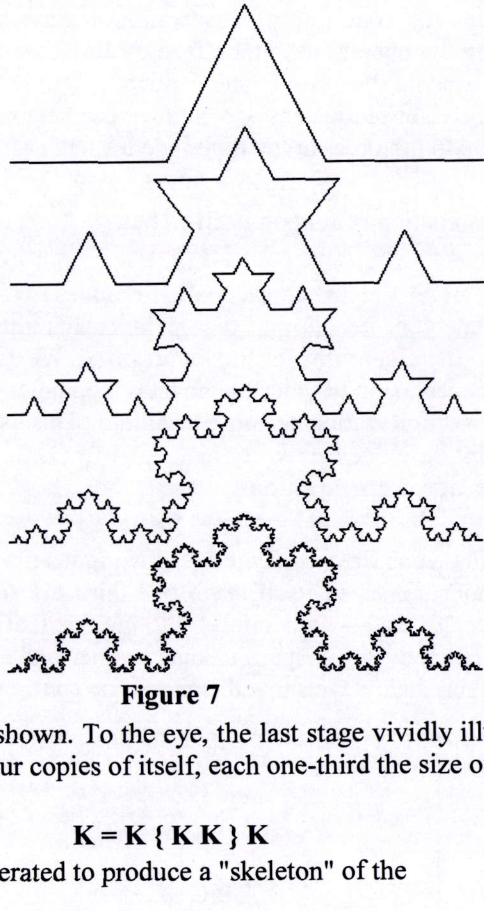


Figure 7

Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

$$\mathbf{K = K\,\{\,K\,K\,\}\,K}$$

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

$$\mathbf{K = K\,\{\,K\,K\,\}\,K}$$
$$\mathbf{= K\,\{\,K\,K\,\}\,K\ \{\ K\,\{\,K\,K\,\}\,K\ \ K\,\{\,K\,K\,\}\,K\ \}\,K\,\{\,K\,K\,\}\,K}$$
$$\mathbf{= \ldots}$$

We have only performed one line of this skeletal recursion. There are sixteen **K**'s in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison with this symbolic recursion shows how geometry aids the intuition. The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of exploration.

It is usually thought that the recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. This is a fine tuning to the point where the action of the perceiver and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in the mathematical levels, such as number or fractal pattern, part of the process is articulated. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind. The perceiving mind is itself an eigenform of its own perception.

6. THE SQUARE ROOT OF MINUS ONE

The purpose of this section is to place **i**, square root of minus one, and its algebra in the context of eigenform and reflexivity. We then see that the square root of minus one is intimately related to time and the mathematical expression of time. We then return to this theme in later sections, showing how this point of view works in relation to quantum mechanics. The main point here is that **i** can be interpreted as a primitive dynamical system that undergoes oscillation and nevertheless has a significant eigenform associated with this iteration.

Traditionally **i**, the square root of minus one, arises from the fact that it is suggested by the solutions to quadratic equations. The simplest instance is the equation $\mathbf{x^2 + 1 = 0}$. A solution to this equation must have square equal to **-1**, but there are no real numbers whose square is **-1**. The square of a negative number is positive and the square of a postive number is positive and the square of zero is zero. Thus mathematicians (reluctantly at first) introduced an ideal number **i** such that $\mathbf{i^2 = -1}$. This number can be used to find solutions to other quadratic equations. For example **1+i** and **1-i** are the roots of $\mathbf{x^2 - 2x + 2 = 0}$.

The use of such complex numbers became indispensible when it was realized that they could be used in the solution of cubic equations in such a way that the real solution of a cubic was expressed naturally as a combination of two complex numbers such that the imaginary parts canceled out and all that was left was the real solution.

Here is an example showing how complex numbers can combine in non-trivial ways to form real numbers.

We write **x = Sqrt(1 + i) + Sqrt(1-i)**

and then we see that $\mathbf{x^2}$ **= 1+i + 1-i + 2Sqrt((1+i)(1-i)) = 2 + 2Sqrt(2).**

Thus we can conclude that **x = Sqrt(2 + 2 Sqrt(2))**

and so **Sqrt(1 + i) + Sqrt(1-i) = Sqrt(2 + 2 Sqrt(2)),**

showing that certain real numbers can be obtained as combinations of complex numbers. This theme of finding real solutions by entering and leaving the complex domain is fundamental to the applications of complex numbers in mathematics and natural science.

Mathematicians began to seriously use complex numbers in the 1500's. It was not until around 1800 that Gauss and Argand discovered a geometric interpretation that opened up the subject to genuinely deep explorations.

The geometric interpretation of the complex numbers is well-known and we shall not repeat it here except to remark that **i** seen as a vector of unit length in the plane, and perpendicular to the horizontal axis. The horizontal axis is taken to be the real numbers. Multiplication by **i** is interpreted geometrically as rotation by **90** degrees. Thus **ii** is rotation by **180** degrees. Indeed the **180** degree rotate of **+1** is **-1,** and so **ii = -1**.

We now describe a process point of view for complex numbers.
Think of the oscillatory process generated by

$$R(x) = -1/x.$$

The fixed point would satisfy

$$i = -1/i$$

and multiplying, we get that

$$ii = -1.$$

On the other hand the interation of **R** on **1** yields

$$+1, R(1) = -1, R(R(1)) = +1, -1, +1, ...$$

Thus there must be a linkage between this ideal number **i** whose square is **-1** and the recursion that leads to an oscillation. The square root of minus one is a perfect example of an eigenform that occurs in a new and wider domain than the original context in which its recursive process arose. The process has no fixed point in the original domain.

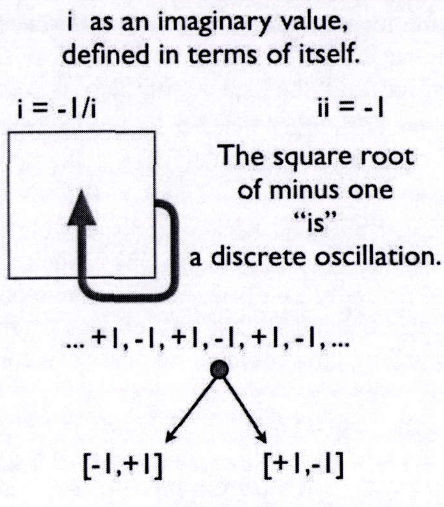


Figure 8

Looking at the oscillation between **+1** and **-1**, we see that there are naturally two viewpoints that we denote by **[+1,-1]** and **[-1,+1]**. These viewpoints corrsepond to whether one regrads the oscillation at time zero as starting with **+1** or with **-1**.

We shall let **I{+1,-1}** stand for an undisclosed alternation or ambiguity between **+1** and **-1** and call **I{+1,-1}** an *iterant*. There are two *iterant views*: **[+1,-1]** and **[-1,+1]**.

Given an iterant **[a,b]**, we can think of **[b,a]** as the same process with a shift of one time step. These two iterant views, seen as points of view of an alternating process, will become the square roots of negative unity, **i** and **-i**.

We introduce a *temporal shift operator* η such that

$$[a,b]\eta = \eta\,[b,a] \text{ and } \eta\,\eta = 1$$

for any iterant **[a,b]**, so that contcatenated observations can include a time step of one-half period of the process

$$...abababab....$$

We combine iterant views term-by-term as in

$$[a,b][c,d] = [ac,bd].$$

We now define i by the equation

$$i = [1,-1]\eta\,.$$

This makes **i** both a view of the iterant process and an operator that takes into account a step in time. We calculate

$$ii = [1,-1]\eta\,[1,-1]\eta = [1,-1][-1,1]\eta\,\eta = [-1,-1] = -1.$$

Thus we have constructed the square root of minus one by using an iterant viewpoint. The key to rethinking the complex numbers in these terms is to understand that **I** represents a discrete oscillating temporal process and it is an eigenform participating in the algebraic structure of the complex numbers.

The Temporal Nexus

We take as a matter of principle that the usual real variable **t** for time is better represented as **it** so that time is seen to be a process, an observation and a magnitude all at once. This principle of "imaginary time" is justified by the eigenform approach to the structure of time and the structure of the square root of minus one. An example of the use of the Temporal Nexus, consider the expression $x^2 + y^2 + z^2 + t^2$, the square of the Euclidean distance of a point **(x,y,z,t)** from the origin in Euclidean four-dimensional space. Now replace **t** by **it**, and find

$$x^2 + y^2 + z^2 + (it)^2 = x^2 + y^2 + z^2 - t^2,$$

the squared distance in hyperbolic metric for special relativity. By replacing **t** by its process operator value **it** we make the transition to the physical mathematics of special relativity. We will discuss this theme further in later sections. There, we relate the Temporal Nexus, the role of complex numbers in quantum mechanics and the role of temporal shift operators in discrete physics.

7. REFLEXIVE DOMAINS

A reflexive domain **D** is an arena where actions and processes that transform the domain can also be seen as the elements that compose the domain. Every element of the domain can be seen as a transformation of the domain to itself. In actual practice an element of a domain may be a person or company (collective of persons) or a physical object or mechanism that is seen to be in action. In actual practice we must note that what are regarded as objects or entities depends upon the way in which observers inside or outside the domain divide their worlds. It is very difficult to make a detailed mathematical model of such situations. Each actor is an actor in more than one play. His actions undergo separate but related interpretations, depending upon the others with

whom he interacts. Mutual feedback of a multiplicity of ongoing processes is not easily described in the Platonic terms of pure mathematics.
Nevertheless, we take as a general principle for a mathematical model that **D** is a certain set (possibly evolving in time), and we let **[D,D]** denote a selected collection of mappings from **D** to **D**. An element **F** of **[D,D]** is a mapping **F:D ----->D**.
We shall assume that there is a 1-1 correspondence **I:D -----> [D,D]**. This is the assumption of reflexivity. Every element of the reflexive domain is a transformation of that domain.
Each denizen of the reflexive domain has a dual role of actor and actant.

Given an element **g** in **D**, **I(g):D ----> D** is a mapping from **D** to **D**, and for every mapping **F:D ----> D**, there is an element **g** in **D** such that **I(g) = F**. The reflexive domain embodies a perfect correspondence between actions, and entities that are the recipients of these actions. We now prove a fundamental theorem about reflexive domains. We show that every mapping **F:D -----> D** has a fixed point **p**, an element **p** in **D** such that **F(p) = p**. This means that there is another way, in a reflexive domain, to associate a point to a transformation. The point can be seen as the fixed point of a transformation and in that way, the points of the domain disappear into the self-referential nature of the transformations.

Fixed Point Theorem. Let **D** be a reflexive domain with actor/actant correspondence **F:D -----> [D,D]**. Then every **F** in **[D,D]** has fixed point. That is, there exists a **p** in **D** such that **F(p) = p**.

Proof. Define **G:D ----> D** by the equation **Gx = F(I(x)x)** for each **x** in **D**.
Since **I:D -----> [D,D]** is a 1-1 correspondence, we know that **G = I(g)** for some **g** in **D**.
Hence **Gx = I(g)x = F(I(x)x)** for all **x** in **D**.
Therefore, letting **x = g**, **I(g)g = F(I(g)g)** and so **p = I(g)g** is a fixed point for **F**.
Q.E.D.

We shall discuss this proof and its meaning right now in a series of remarks, and later in the paper in regard to examples that will be constructed.

Remark1.

Suppose that we reduce the notational complexity of our description of the reflexive domain by simply saying that for any two entities **g** and **x** in the domain there is a new entity **gx** that is the result of the interaction of **g** and **x**. (We think of **gx** as **I(g)x.)**
In mathematical terms, we define **gx = I(g)x**. Then the proof of the fixed point theorem appears in a simpler form: We define **Gx = F(xx)** and note that **GG = F(GG)**.
Thus **GG** is the fixed point for **F!** I like to call **G** "**F**'s Gremlin". This is an apt description of our **G**. At first **G** looks quite harmless. Applying **G** to any **A** we just apply **A** to itself and apply **F** to the result. **GA = F(AA)**. The dangerous mixture comes when it is possible to apply **G** to itself! Then **GG = F((GG)** and **GG** is sitting right in there surrounded by **F** and you cannot stop the action. Off goes the recursion

GG = F(GG)
= F(F(GG))
= F(F(F(F(GG))))
= F(F(F(F(F(F(F(F(GG))))))))

The diabolical nature of the Gremlin is that he represents a process that once started, is hard to stop. Such are the processes by which we make the world into a field of tokens and symbols and forget the behaviours and processes and reflexive spaces from which they came. Fixed points and self-references are the unavoidable fruits of reflexivity, and reflexivity is the natural condition in a universe where there is no complete separation of part from the whole.

Remark 2.

A reflexive domain is a place where actions and events coincide. An action as a mapping of the whole space, because there is no intrinsic separation of the local and the global. Feedback is an attempt to handle the lack of separation of part and whole by describing their mutual influence.

When we define a new element g of **D** via **gx = F(x)** for any mapping **F:D---->D**, and we have a notion of combination of elements of **D**: **a,b -----> ab**, then we can define **gx = F(xx)** and so get **gg = F(gg)**. Here we have not made a big separation between the elements of **D** and the mappings, since each element **g** of **D** gives the mapping **I(g)x = gx**. But in fact, we could define **ab = I(a)b** in a reflexive domain.

Whenever there is a transformation **F**, we make that transformation into an element of the domain by the definition **gx = F(x)**. We transmute verbs to nouns. The reflexive domain evolves. The space is not given a priori. The space evolves in relation to actions and definitions. The road unfolds before us as we travel.

Remark 3.

We create languages for evolving concepts. The outer reaches of set theory (and category theory) lead to clear concepts, but these concepts are not themselves sets or categories. A good example is the famous Russellian concept of sets that are not members of themselves. Russell's concept is not a set. Another example is the concept of set itself. There is no set that is all sets.

This very limitation on the notion of set is its opening. It shows us that set theory is an evolving language. Language and concepts expand in time.

Here is a transformation on sets: **F(X) = { X }**. The transform of a set **X** is the singleton set whose member is **X**. If **X** is not a member of itself, then **F(X)** is also not a member of itself. But a fixed point of the transformation **F** is an entity **U** such that **{U} = U**, and this would be a set that is a member of itself.

Left Distributivity and Magma

In a *left distributive* formalism we have a binary operation **a*b** on a domain **Q** such that for all **A**, **b** and **c** the following equation holds.

$$A*(b*c) = (A*b)*(A*c).$$

This corresponds exactly to the interpretation that each element **A** in **Q** is a mapping of **Q** to **Q** where the mapping **A[x] = A*x** is a structure preserving mapping from **Q** to **Q**.

$$A[b*c] = A[b]*A[c].$$

We can ask of a domain that every element of the domain is itself *a structure preserving mapping of that domain.* This is very similar to the requirement of reflexivity.

We call a domain M with an operation * that is left distributive a *magma*. A magma with no other relations than left-distributivity is called a *free magma.* For interpretations in terms of the theory of knots and links, see [9], [22] and [23].

The simplest example of a Magma is an algebra on three elements **{a,b,c}** where **ab = c = ba, bc = a = cb, ca = b = ac** and **aa = a, bb=b, cc = c**. Here we have three distinct entities, each pair interacts to produce the third entity and each entity interacts with itself to produce itself. Note that this system is not associative. For example, **(aa)b = ab = c**, while **a(ab) = ac = b**. It is easy to check the left distributive law. For example **a(bc) = aa = a** while **(ab)(ac) = cb=a** so that **a(bc) = (ac)(bc)**.

The search for structure preserving mappings can occur in rarefied contexts. See for example the work of Laver and Dehornoy [21, 9] who studied mappings of set theory to itself that would preserve all definable structure in the theory. Dehornoy realized that many of the problems he studied in relation to set theory were accessible in more concrete ways via the use of knots and braids. Thus the knots and braids become a language for understanding formal properties of self-embedded structure.

Structure preserving mappings of set theory must begin as the identity mapping since the relations of sets are quite rigid at the beginning. (You would not be able to map an empty set to a set that was not empty for example, and so the empty set would have to go to itself.) The existence of non-trivial structure preserving mappings of set theory questions the boundaries of definablity and involves the postulation of sets of very large size. See [16] for a good exposition of the philosophical issues about such embeddings and for an approach to wholeness in physics that is based on these ideas.

I shall call a magma M *reflexive* if it has the property that every structure preserving mapping of the algebra is realized by an element of the algebra *and* **(x*x)*z = x*z** for all **x** and **z** in M.

A special case of this last property would be where **x*x = x** for all **x** in M.

Fixed Point Theorem for Reflexive Magmas.

Let **M** be a reflexive magma.

Let **F:M ----> M** be a structure preserving mapping of **M** to itself.

Then there exists an element **p** in **M** such that **F(p) = p**.

Proof. Let **F:M -----> M** be any structure preserving mapping of the magma M to iteself. This means that we assume that **F(x*y) = F(x)*F(y)** for all **x** and **y** in **M**.

Define **G(x) = F(x*x)** and regard **G:M ----> M**. Is **G** structure preserving?

We must compare **G(x*y) = F((x*y)*(x*y)) = F(x*(y*y))**

with **G(x)*G(y) = F(x*x)*F(y*y) = F((x*x)*(y*y))**.

Since **(x*x)*z = x*z** for all **x** and **z** in **M**, we conclude that **G(x*y) = G(x)*G(y)** for all **x** and **y** in M. Thus **G** is structure preserving and hence there is an element **g** of **M** such that **G(x) = g*x** for all **x** in **M**.

Therefore we have **g*x = F(x*x)**, whence **g*g = F(g*g)**.

For **p = g*g**, we have **p = F(p)**.

This completes the proof. //

This analysis shows that the concept of a magma is very close to our notion of reflexive domain.

8. THE SECRET

What is the simplest language that is capable of self-reference?
We are all familiar with the abilities of natural language to refer to itself. Self reference is a most accessible example of eigenform. Why this very sentence is an example of self-referentiality. The American dollar bill declares "This bill is legal tender.". The sentence that you are now reading declares that you, the reader, are complicit in its own act of reference. But what is the simplest language that can refer to itself?

Such anguage would have a simple alphabet. Let us say it has only the letter **R**. The words in this language will be all strings of **R**'s. Call the language **LS**. The words in **LS** are the following:

R,
RR,
RRR,
RRRR,
and so on.

Two words are equal if they have the same number of letter **R**'s.
We now create a rule of reference of this language.

Each word makes a meaningful statement of reference via the rule:
If **X** is a word in **LS**, then **RX** refers to **XX**. **RX** refers to **XX**, the repetition of **X.**

Thus **RRR** refers to **RRRR** (not to itself), and **R** refers to the empty word.

There is a word in LS that refers to itself. Can you find it? Lets see.
RX refers to **XX**. So we need **XX** = **RX** if **RX** would refer to **RX**.
If XX = RX, then X = R. So we need **X = R**. And **RR** refers to itself.

The little language **LS** looks like a pedantic triviality, but it is actually at the root of reflexivity, Godel's incompleteness Theorem, recursion theory, Russell's paradox and the notion of self-observing and self-referring systems. It seems paradoxical that what looks like a trick of repeating a symbol can be so important. The trick is more than just a trick.

I would like to think that when we eventually discover the true secret of the universe it will turn out to be this simple. The snake bites its tail. The Universe is constructed in such a way that it can refer to itself. In so doing, the Universe must divide itself into a part that refers and part to which it refers, a part that sees and a part that is seen.

Let us say that **R** is the part that refers and **U** is the referent. The divided universe is **RX** and **RX = U** and **RX** refers to **U** (itself).

Our solution suggests that the Universe divides itself into two identical parts each of which refers to the universe as a whole. This is **RR**.

In other words, the universe can pretend that it is two and then let itself refer to the two, and find that it has in the process referred only to the one, that is itself.

The Universe plays hide and seek with herself, pretending to divide herself into two when she is really only one. And that is the secret of the Universe.

And Physics. From the point of view of physics, the universe should have a universal equation of the form

U |phi> =0

Where **|phi>** is the state of the universe and **U** is the universe as an operator on its own state. We follow our own principles and identify the universe as state |phi> the universe as process U. Then the equation of the universe becomes

$$\mathbf{U\,U = 0}.$$

This is the fundamental nilpotent form for a universal equation of physics. See [24] for a detailed evocation of this philosophy of nilpotence and its relationship with the Dirac equation. In a sequence to the present paper, the physical considerations in the next section will be brought into line with this nilpotent viewpoint of Peter Rowlands.

9. QUANTUM PHYSICS

Our pimary reason for introducing quantum physics into this essay is that it is inextricably related to the complex numbers and in particular to the square root of minus one. We wish to show that the eigenform view of the square root of minus one and the Temporal Nexus inform the epistemology of quantum theory. The reader should recall the Temporal Nexus from the section before. We take as a matter of principle that the usual real variable **t** for time is better represented as **it** so that time is seen to be a process, an observation and a magnitude all at once. This principle of "imaginary time" is justified by the eigenform approach to the structure of time and the structure of the square root of minus one. Quantum mechanics has been a powerful force in asking us to rethink our notions of objects and causality. Von Foerster's notion of eigenform was an outgrowth of his background as a quantum physicist. We should ask what eigenforms might have to do with quantum theory and with the quantum world.

In this section we meet the concurrence of the view of object as token for eigenbehavior and the observation postulate of quantum mechanics. In quantum mechanics observation is modeled not by eigenform but by its mathematical relative the eigenvector. The reader should recall that a *vector* is a quantity with magnitude and direction, often pictured as an arrow in the plane or in three dimensional space.

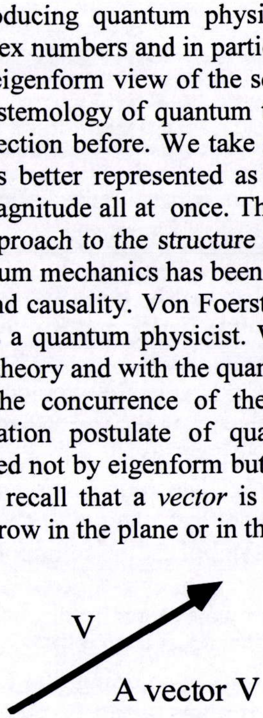


Figure 9

In quantum physics [11], the state of a physical system is modeled by a vector in a high-dimensional space, called a Hilbert space. As time goes on the vector rotates in this high dimensional space. Observable quantities correspond to (linear) operators **H** on these vectors **v** that have the property that the application of **H** to **v** results in a new vector that is a multiple of **v** by a real factor **λ.**

(An operator is said to be linear if **H(av +w) = aH(v) + H(w)** for vectors v and w, and any number a. Linearity is usually a simplifying assumption in mathematical models, but it is an essential feature of quantum mechanics.)

In symbols this has the form

$$\mathbf{Hv = \lambda v.}$$

One says that **v** is an *eigenvector* for the operator **H**, and that **λ** is the *eigenvalue.* The constant **λ** is the quantity that is observed (for example the energy of an electron). These are particular properties of the mathematical context of quantum mechanics. The **λ** can be eliminated by replacing **H** by $\mathbf{G = H/\lambda}$ (when **λ** is non zero) so that

$$\mathbf{Gv = (H/\lambda)v = (Hv)/\lambda = \lambda v/k = v.}$$

Thus

$$\mathbf{Gv = v.}$$

In quantum mechanics observation is founded on the production of eigenvectors **v** with **Gv=v** where **v** is a vector in a Hilbert space and **G** is a linear operator on that space.

Many of the strange and fascinating properties of quantum mechanics emanate directly from this model of observation. In order to observe a quantum state, its vector is projected into an eigenvector for that particular mode of observation. By projecting the vector into that mode and not another, one manages to make the observation, but at the cost of losing information about the other possibilities inherent in the vector. This is the source, in the mathematical model, of the complementarities that allow exact determination of the position of a particle at the expense of nearly complete uncertainty about its momentum (or vice versa the determination of momentum at the expense of knowledge of the position).

Observation and quantum evolution (the determinate rotation of the state vector in the high dimensional Hilbert space) are interlocked.

Each observation discontinuously projects the state vector to an eigenvector. The intervals between observations allow the continuous evolution of the state vector. This tapestry of interaction of the continuous and the discrete is the basis for the quantum mechanical description of the world.

The theory of eigenforms is a sweeping generalization of quantum mechanics that creates a context for understanding the remarkable effectiveness of that theory. If indeed the world of objects is a world of tokens for eigenbehaviors, and if physics demands forms of observations that give numerical results, then a simplest example of such observation is the observable in the quantum mechanical model.

Is the quantum model, in its details, a consequence of general principles about systems? This is an exploration that needs to be made. We can only ask the question here. But the mysteries of the interpretation of quantum mechanics all hinge on an assumption of a world external to the quantum language. Thinking in terms of eigenform we can begin to look at how the physics of objects emerges from the model itself. Where are the eigenforms in quantum physics? They are in the mathematics itself. For example, we have the simplest wave-function

$$\boldsymbol{\varphi(x,t) = e^{i(kx - \omega t)}}.$$

Since we know that the function $\mathbf{E(x) = e^{x}}$ is an eigenform for orperation of differentiation with respect to **x,** $\boldsymbol{\varphi(x,t)}$ is a special multiple eigenform from which the energy can be extracted by temporal differentiation, and the momentum can be extracted by spatial differentiation. We see in $\boldsymbol{\varphi(x,t)}$ the complexity of an individual who presents many possible sides to the world.

$\varphi(x,t)$ is an eigenform for more than one operator and it is a composition of e^x and **i**, the square root on minus one, a primordial eigenvalue related to time. It is this internal complexity that is mirrored in the uncertainty relations of Heisenberg and the complementarity of Bohr. The eigenforms themselves, as wave-functions, are *inside* the mathematical model, on the *other side* of that which can be observed by the physicist.

We have seen eigenforms as the constructs of the observer, and in that sense they are on the side of the observer, even if the process that generates them is outside the realm of his perception. This suggests that we think again about the nature of the wave function in quantum mechanics. Is it also a construct of the observer?

To see quantum mechanics and the world in terms of eigenforms requires a turning around, a shift of perception where indeed we shall find that the distinction betweem model and reality has disappeared into the world of appearance.

This is a reversal of epistemology, a complete turning of the world upside down. Eigenform has tricked us into considering the world of our experience and we find that it is our world, generated by our actions. The world has become objective through the self-generated stabilities of those actions.

A Quick Review of Quantum Mechanics

DeBroglie hypothesized two fundamental relationships: between energy and frequency, and between momentum and wave number. These relationships are summarized in the equations

$$E = \hbar w,$$
$$p=\hbar k,$$

where **E** denotes the energy associated with a wave and **p** denotes the momentum associated with the wave. Here $\hbar = h/2\pi$ where **h** is Planck's constant.

Schrödinger answered the question: Where is the wave equation for DeBroglie's waves? Writing an elementary wave in complex form

$$\psi = \psi(x,t) = \exp(i(kx - wt)),$$

we see that we can extract DeBroglie's energy and momentum by differentiating:

$$i\hbar\partial\Psi/\partial t = E\psi \quad \text{and} \quad -i\hbar\partial\Psi/\partial x = p\psi.$$

This led Schrödinger to postulate *the identification of dynamical variables with operators* so that the first equation ,

$$i\hbar\partial\Psi/\partial t = E\Psi,$$

is promoted to the status of an equation of motion while the second equation becomes the definition of momentum as an operator:

$$p = -ih\partial/\partial x\ .$$

Once **p** is identified as an operator, the numerical value of momentum is associated with an eigenvalue of this operator, just as in the example above. In our example $p\psi = \hbar k\psi$.

In this formulation, the position operator is just multiplication by **x** itself. Once we have fixed specific operators for position and momentum, the operators for other physical quantities can be expressed in terms of them.

We obtain the energy operator by substitution of the momentum operator in the classical formula for the energy:

$$\mathbf{E = (1/2)mv^2 + V}$$
$$\mathbf{E = p^2/2m + V}$$
$$\mathbf{E = -(\hbar^2/2m)\partial^2/\partial x^2 + V.}$$

Here **V** is the potential energy, and its corresponding operator depends upon the details of the application. With this operator identification for **E**, Schrödinger's equation

$$\mathbf{i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2 + V\psi}$$

or equivalently,

$$\mathbf{\partial\psi/\partial t = i(\hbar/2m)\partial^2\psi/\partial x^2 - (i/\hbar)V\psi}$$

is an equation in the first derivatives of time and in second derivatives of space. In this form of the theory one considers general solutions to the differential equation and this in turn leads to excellent results in a myriad of applications. It is useful to point out that Schrödinger's equation without the potential term has the form

$$\mathbf{\partial\psi/\partial t = i(\hbar/2m)\partial^2\psi/\partial x^2.}$$

It is in this form that we shall compare it with discrete processes in the next section. There we shall use the Temporal Nexus to obtain new insight into the role of **i** in this equation.

In quantum theory, observation is modelled by the concept of eigenvalues for corresponding operators. *The quantum model of an observation is a projection of the wave function into an eigenstate.*

An energy spectrum **{Ek}** corresponds to wave functions $\boldsymbol{\psi}$ satisfying the Schrödinger equation, such that there are constants Ek with $\mathbf{E\psi = Ek\psi}$. An *observable* (such as energy) **E** is a Hermitian operator on a Hilbert space of wavefunctions. Since Hermitian operators have real eigenvalues, this provides the link with measurement for the quantum theory.

It is important to notice that there is no mechanism postulated in this theory for how a wave function is "sent" into an eigenstate by an observable. Just as mathematical logic need not demand causality behind an implication between propositions, the logic of quantum mechanics does not demand a specified cause behind an observation. This absence of an assumption of causality in logic does not obviate the possibility of causality in the world. Similarly, the absence of causality in quantum observation does not obviate causality in the physical world. Nevertheless, the debate over the interpretation of quantum theory has often led its participants into asserting that causality has been demolished in physics.

Note that the operators for position and momentum satisfy the equation $\mathbf{xp - px = \hbar i.}$ This corresponds directly to the equation obtained by Heisenberg, on other grounds, that dynamical variables can no longer necessarily commute with one another. In this way, the points of view of DeBroglie, Schrödinger and Heisenberg came together, and quantum mechanics was born. In the course of this development, interpretations varied widely. Eventually, physicists came to regard the wave function not as a generalized wave packet, but as a carrier of information about possible observations. In this way of thinking $\psi^*\psi$ (ψ^* denotes the complex conjugate of ψ) represents the probability of finding the "particle" (A particle is an observable with local spatial characteristics.) at a *given point in* spacetime. Strictly speaking, it is the spatial integral of $\psi^*\psi$ that is interpreted as a total probability with $\psi^*\psi$ the probablility density. This way of thinking is supported by the fact that the total spatial integral is time-invariant as a consequence of Schrodinger's equation!

10. ITERANTS, COMPLEX NUMBERS AND QUANTUM MECHANICS

We have seen that there are indeed eigenforms in quantum mechanics. The eigenforms in quantum mechanics are the mathematical functions such as

$$e^x$$

that are invariant under operators such as $\mathbf{D = d/dx}$.

But we wish to examine the relationship between recursion, reflexive spaces and the properties of the quantum world.

The hint we have received from quantum theory is that we should begin with mathematics which is replete with eigenforms.

In fact, this hint seems very rich when we consider that **i**, the square root of minus one, *is a key eigenform in* our panoply of eigenforms and it is a key ingredient in quantum mechanics intimately related to the role of time.

Revisiting the Temporal Nexus

If we define

$$\mathbf{R(A) = -1/A}$$

then $\mathbf{R(i) = i}$ since $i^2 = \mathbf{-1}$ is equivalent to $\mathbf{i = -1/i}$.

Using the infinite recursion we would then write

$$\mathbf{i = -1/-1/-1/-1/-1/...},$$

making **i** an infinite reentry form for the operator **R.**

We will write

$$\mathbf{i = [-1/*]}$$

where * denotes the reentry of the whole form into that place in the right-hand part of the expression.

Similarly, if $\mathbf{F(x) = 1 + 1/x}$, then the eigenform would be **[1 + 1/*]** and we could write $\mathbf{(1+\sqrt{5})/2 = [1 + 1/*]}$.

With this in place we can now consider wave functions in quantum mechanics such as

$$\psi(x,t) = \exp(i(kx - wt)) = \exp([-1/*]\,(kx - wt))$$

and we can consider classical formulas in mathematics such as *Euler's formula*

$$\exp([-1/*]\varphi) = \cos(\varphi) + [-1/*]\sin(\varphi)$$

in this light. We start here with Euler's formula, for this formula is the key relation between complex numbers, **i**, waves and periodicity.

We return to the finite nature of **[-1/*]**. This eigenform is an oscillator between **-1** and **+1**. It is only **i** in its idealization.

In its appropriate synchronization it has the property that **i = -1/i**. As a real oscillator, the equation **R(i) = -1/i** tells us that when **i** is **1**, then i is transformed to **-1** and when **i** is **-1** then **i** is transformed to **+1**. There is no fixed point for **R** in the real domain. The eigenform is achieved by leaving the real domain for a new and larger domain.

We know that this larger domain can be conceptualized as the plane with Euclidean rotational geometry, but here we explore the larger domain in terms of eigenforms.

We find that **i** itself is a fundamental discrete process, and it is in the "microworld" of such discrete physical processes that not only quantum mechanics, but also classical mechanics is born.

Iterants and Iterant Views

In order to think about i, consider an infinite oscillation between **+1** and **-1**:

... -1,+1,-1,+1,-1,+1,-1,+1,...

This oscillation can be seen in two distinct ways. It can we seen as **[-1,+1]** (a repetition in this order) or as **[+1,-1]** (a repetition in the opposite order). This suggests regarding an infinite alternation such as

... a,b,a,b,a,b,a,b,a,b,a,b,a,b,...

as an entity that can be seen in two possible ways, indicated by the ordered pairs **[a,b]** and **[c,d]**. We shall call the infinite alternation of **a** and **b** the *iterant* of **a** and **b** and denote it by **I{a,b}**. Just as with a set **{a,b}**, the iterant is independent of the order of **a** and **b**. We have **I{a,b} = I{b,a}**, but there are two distinct views of any iterant and these are denoted by **[a,b]** and **[b,a]**. The key to iterants is that two representatives of an iterant can *by themsleves* appear identical, but *taken together* are seen to be different. For example, consider

... a,b,a,b,a,b,a,b,a,b,a,b,a,b,...

and also consider

...b,a,b,a,b,a,b,a,b,a,b,a,b,a,...

There is no way to tell the difference between these two iterants except by a direct comparision as shown below

... a,b,a,b,a,b,a,b,a,b,a,b,a,b,...
... b,a,b,a,b,a,b,a,b,a,b,a,b,a,...

In the direct comparison we see that if one of them is **[a,b]**, then the other one should be **[b,a]**. Still, there is no reason to assign one of them to be **[a,b]** and the other **[b,a]**. It is a strictly relative matter. The two iterants are entangled (to borrow a term from quantum mechanics) and if one of them is observed to be **[a,b]**, then the other is necessarily observed to be **[b,a]**.

Lets go back to the square root of minus one as an oscillatory eigenform.

... -1,+1,-1,+1,-1,+1,-1,+1,...

What is the operation **R(x) = -1/x** in this case? We usually think of a starting value and then the new operation shifts everything by one value with **R(+1) = -1** and **R(-1) = +1**. Thus would suggest that

R(... -1,+1,-1,+1,-1,+1,-1,...) = ... +1,-1,+1,-1,+1,-1,+1,...

and these sequences will be different when we compare, them even though they are identical as individual iterants.

... -1,+1,-1,+1,-1,+1,-1,+1,...
... +1,-1,+1,-1,+1,-1,+1,-1,...

However, we would like to take the eigenform/iterant concept and make a more finite algebraic model by using the iterant views

[-1,+1] and [+1,-1].

Certainly we should consider the transform **P[a,b] = [b,a]** and we take

-[a,b]=[-a, -b],

so that

-P[a,b] = [-b,-a].

Then

-P[1,-1] = [1,-1].

In this sense the operation **-P** has eigenforms **[1,-1]** and **[-1,1]**.

You can think of **P** as the shift by one-half of a period in the process

...abababababab.... .

Then **[-1,1]** is an eigenform for the operator that combines negation and shift.

We will take a shorthand for the operator **P** via

P[a,b] = [a,b]' = [b,a].

If **x=[a,b]** then **x' = [b,a]**. We can add and multiply iterant views by the combinations

[a,b][c,d] = [ac,bd],
[a,b] + [c,d] = [a+c, b+ d],
k[a,b] = [ka,k,b] when k is a number.

We take **1 = [1,1]** and **-1 = [-1,-1]**. This is a natural algebra of iterant views, but note that **[-1,+1][-1,+1] = [1,1] = 1**, so we do not yet have the square root of minus one.

Consider **[a,b]** as representative of a process of observation of the iterant **I{a,b}. [a,b]** is an *iterant view.* We wish to combine **[a,b]** and **[c,d]** as *processes of observation.* Suppose that observing **I{a,b}** requires a step in time. That being the case, **[a,b]** will have shifted to **[b,a]** in the course of the single time step. We need an algebraic structure to handle the temporality. To this end, we introduce an operator η with the property that

$$[a,b]\eta = \eta[b,a] \text{ with } \eta^2 = \eta\eta = 1$$

where **1** means the identity operator. You can think of η as a *temporal shift operator* that can act on a sequence of individual observations. The algebra generated by iterant views and the operator η is taken to be associative.

Here the interpretation is that **XY** denotes "first observe **X**, then observe **Y**". Thus $X\eta Y\eta = XY'\eta\eta = XY'$ and we see that **Y** has been shifted by the presence of the operator η, just in accord with our temporal interpretation above.

We can now have a theory where **i** and its conjugate **-i** correspond to the two views of the iterant **I{-1,+1}.**

Let $\mathbf{i} = [1,-1]\eta$ **and** $-\mathbf{i} = [-1,1]\eta$. We get a square root of minus one:

$$\mathbf{ii} = [1,-1]\eta[1,-1]\eta = [1,-1][-1,1]\eta\eta = [-1,-1] = -[1,1] = -1.$$

The square roots of minus one are iterant views coupled with temporal shift operators. Not so simple, but not so complex either!

If $\mathbf{e} = [1,-1]$ then $\mathbf{e}' = [-1,1] = -\mathbf{e}$ and $\mathbf{ee} = [1,1] = 1$ with $\mathbf{ee}' = -1$.

$$\mathbf{i} = \mathbf{e}\eta$$

$$\mathbf{ii} = \mathbf{e}\eta\mathbf{e}\eta = \mathbf{ee}'\eta\eta = \mathbf{ee}' = -1$$

With this defintion of **i**, we have an algebraic interpretation of complex numbers that allows one to think of them as observations of discrete processes.

This algebra contains more than just the complex numbers.

With $\mathbf{x} = [a,b]$ and $\mathbf{y} = [c,d]$, consider the products $(\mathbf{x}\eta)(\mathbf{y}\eta)$ and $(\mathbf{y}\eta)(\mathbf{x}\eta)$:

$$(\mathbf{x}\eta)(\mathbf{y}\eta) = [a,b]\eta[c,d]\eta = [a,b][d,c] = [ad,bc]$$

$$(\mathbf{y}\eta)(\mathbf{x}\eta) = [c,d]\eta[a,b]\eta = [c,d][b,a] = [cb,da].$$

Thus

$$(\mathbf{x}\eta)(\mathbf{y}\eta) - (\mathbf{y}\eta)(\mathbf{x}\eta) = [ad-bc, -(ad-bc)] = (ad-bc)[1,-1].$$

Thus

$$\mathbf{x}\eta\mathbf{y}\eta - \mathbf{y}\eta\mathbf{x}\eta = (ad-bc)\mathbf{i}\,\eta.$$

We see that, with temporal shifts, the algebra of observations is non-commutative.

Note that for these processes, represented by vectors **[a,b]**, the commutator $\mathbf{x}\eta\mathbf{y}\eta - \mathbf{y}\eta\mathbf{x}\eta = (ad-bc)\mathbf{i}\eta$ is given by the determinant of the matrix corresponding to two process vectors, and hence will be non-zero whenever the two process vectors are non-zero and represent different spatial rays in the plane. There is more. The full algebra of iterant views can be taken to be generated by elements of the form

$$[a,b] + [c,d]\eta$$

and it is not hard to see that this is isomorphic with 2 x 2 matrix algebra with the correspondence given by the diagram below.

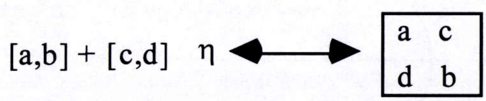


Figure 10

We see from this excursion that there is a full interpretation for the complex numbers (and indeed matrix algebra) as an observational system taking into account time shifts for underlying iterant processes.

Let $\mathbf{A} = [a,b]$ and $\mathbf{B} = [c,d]$ and let $\mathbf{C} = [r,s]$, $\mathbf{D} = [t,u]$. With $\mathbf{A}' = [b,a]$, we have

$$(\mathbf{A} + \mathbf{B}\eta)(\mathbf{C}+\mathbf{D}\eta) = (\mathbf{AC} + \mathbf{BD}') + (\mathbf{AD} + \mathbf{BC}')\eta.$$

This writes 2 x 2 matrix algebra in the form of a hypercomplex number system. From the point of view of iterants, the sum **[a,b]** + **[b,c]η** can be regarded as a superposition of two types of observation of the iterants **I{a,b}** and **I{c,d}.** The operator-view **[c,d]η** includes the shift that will move the viewpoint from **[c,d]** to **[d,c]**, while **[a,b]** does not contain this shift. Thus a shift of viewpoint on **[c,d]** in this superposition does not affect the values of **[a,b].** One can think of the corresponding process as having the form shown below.

... a a a a a a a a a a a a a a a ...
... c d c d c d c d c d c d c d ...
... b b b b b b b b b b b b b b b ...

The snapshot **[c,d]** changes to **[d,c]** in the horizontal time-shift while the vertical snapshot **[a,b]** remains invariant under the shift.
It is interesting to note that in the spatial explication of the process we can imagine the horizontal oscillation corresponding to **[c,d]η** as making a boundary (like a frieze pattern), while the vertical iterant parts a and b mark the two sides of that boundary.

Returning to Quantum Mechanics

You can regard **ψ(x,t) =exp(i(kx - wt))** as containing a micro-oscillatory system with the special synchronizations of the iterant view **i = [+1,-1]η**. It is these synchronizations that make the big eigenform of the exponential **ψ(x,t)** work correctly with respect to differentiation, allowing it to create the appearance of rotational behaviour, wave behaviour and the semblance of the continuum. Note that **exp(iφ) = cos(φ) + i sin(φ)** in this way of thinking is an infinite series involving powers of **i**. The exponentional is synchronized via **i** to separate out its classical trigonometric parts. In the parts we have **cos(φ) + i sin(φ) = [cos(φ), cos(φ)] + [sin(φ), -sin(φ)]J**, a superposition of the constant cosine iterant and the oscillating sine iterant. Euler's formula is the result of a synchronization of iterant processes.

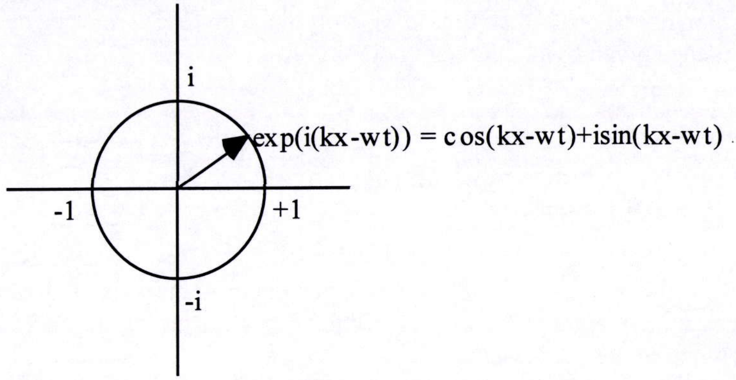


Figure 11

One can blend the classical geometrical view of the complex numbers with the iterant view by thinking of a point that orbits the origin of the complex plane, intersecting the real axis periodically and producing, in the real axis, a periodic oscillation in relation to its orbital movement in the higher dimensional space. The diagram above is the familiar depiction of a vector in the complex plane that represents the phase of a wave-function. I hope that the reader can now look at this picture in a new way, seeing $\mathbf{i = [+1,-1]\eta}$ as a discrete oscillation with built-in time shift and the exponential as a process oscillating between **cos(kx-wt) + sin(kx-wt)** and **cos(kx-wt)-sin(kx-wt)**. The exponential function takes the simple oscillation between **+(kx-wt)** and **-(kx-wt)** and converts it by a complex of observations of this discrete process to the trigonometric wave-forms. All this goes on beneath the surface of the Schrodinger equation. This is the production of the eigenforms from which may be extracted the energy, position and momentum.

Higher Orders of Iterant Structure. What works for 2 x 2 matrices generalizes to n x n matrix algebra, but then the operations on a vector $[x_1,x_2,...,x_n]$ consitute all permutations of n objects. A generating element of iterant algebra is now of the form

$$x\,\sigma = [x_1,x_2,...,x_n]\sigma$$

where σ is an element of the symmetric group S_n. The iterant algebra is the linear span of all elements $x\,\sigma$, and we take the rule of multiplication as

$$x\,\sigma\, y\,\tau = xy^{\sigma}\,\sigma\tau$$

where y^{σ} denotes the vector obtained from **y** by permuting its coordinates via σ ; **xy** is the vector whose k-th coordinate is the product of the k-th coordinate of **x** and the k-th coordinate of **y** ; $\sigma\tau$ is the composition of the two permutations σ and τ .

Hamilton's Quaternions

Here is an example. Hamilton's Quaternions are generated by the iterant views

$$I = [+1,-1,-1,+1]\sigma,\ J= [+1,+1,-1,-1]\lambda,\ K= [+1,-1,+1,-1]\tau$$

where

$$\sigma =(12)(34),\ \lambda= (13)(24),\ \tau =(14)(23).$$

Here we represent the permutations as products of transpositions **(ij)**. The transposition (ij) interchanges i and j, leaving all other elements of **{1,2,...,n}** fixed.

One can verify that

$$I^2 = J^2 = K^2 = IJK = -1.$$

For example,

$$I^2 = [+1,-1,-1,+1]\sigma\ [+1,-1,-1,+1]\sigma = [+1,-1,-1,+1][-1,+1,+1,-1]\sigma\,\sigma$$
$$= [-1,-1,-1,-1] = -1.$$

and

$$IJ = [+1,-1,-1,+1]\sigma\ [+1,+1,-1,-1]\lambda = [+1,-1,-1,+1][+1,+1,-1,-1]\ \sigma\,\lambda$$
$$= [+1,-1,+1,-1]\ (12)(34)(13)(24) = [+1,-1,+1,-1]\ (14)(23) = [+1,-1,+1,-1]\ \tau.$$

In a sequel to this paper, we will investigate this iterant approach to the quaternions and other algebras related to fundmental physics. For now it suffices to point out that the quaternions of the form $a + bI + cJ + dK$ with $a^2 + b^2 + c^2 + d^2 = 1$ (a,b,c,d real numbers) consitutute the group **SU(2)**, ubiquitous in physics and fundamental to quantum theory. Thus the formal structure of all processes in quantum mechanics can be represented as actions of iterant viewpoints. Nevertheless, we must note that making an iterant interpretation of an entity like **I = [+1,-1,-1,+1]σ** is a conceptually natural departure from our original period two iterant notion. Now we are considering iterants such as **I{+1,-1,-1,+1}** where the iterant is a multi-set and the permutation group acts to produce all possible orderings of that multi-set. The iterant itself is not an oscillation. It represents an implicate form that can be seen in any of its possible orders. Once seen, these orders are subject to permutations that produce the possible views of the iterant. Algebraic structures such as the quaternions appear in the explication of such implicate forms.

The reader will also note that we have moved into a different conceptual domain from the original emphasis in this paper on eigenform in relation to to recursion. Indeed, each generating quaternion is an eigenform for the transformation **R(x) = -1/x**.

The richness of the quaternions arises from the closed algebra that arises with its infinity of eigenforms that satisfy this equation, all of the form **U = aI + bJ + cK** where $a^2 + b^2 + c^2 = 1$.

This kind of significant extra structure in the eigenforms comes from paying attention to specific aspects of implicate and explicate structure, relationships with geometry and ideas and inputs from the perceptual, conceptual and physical worlds. Just as with our earlier examples (with cellular automata) of phenomena arising in the course of the recursion, we see the same phenomena here in the evolution of matheamatical and theoretical physical structures in the course of the recursion that constitutes scientific conversation.

Quaternions and SU(2) Using Complex Number Iterants

Since complex numbers commute with one another, we could consider iterants whose values are in the complex numbers. This is just like considering matrices whose entries are complex numbers.

For this purpose we shall allow given a version of **i** that commutes with the iterant shift operator η. Let this commuting **i** be denoted by ι (iota). Then we are assuming that

$$\iota 2 = -1, \eta \iota = \iota \eta, \eta 2 = +1.$$

We then consider iterant views of the form $[a + b\iota, c + d\iota]$ and $[a + b\iota , c + d\iota]\eta = \eta\ [c + d\iota , a + b\iota]$. In particular, we have **e = [1,-1]**, and $i = e\eta$ is quite distinct from ι . Note, as before, that $e\eta = -\eta e$ and that $e^2 = 1$. Now let

$$I = \iota e , J = e\eta , K = \iota\eta .$$

We have used the commuting version of the square root of minus one in these definitions, and indeed we find the quaternions once more.

$$I^2 = \iota e\, \iota e = \iota\, \iota\, e\, e = (-1)(+1) = -1, J^2 = e\eta\, e\eta = e\, (-e)\, \eta\, \eta = -1,$$
$$K^2 = \iota\eta\, \iota\eta = \iota\, \iota\, \eta\, \eta = -1, IJK = \iota e\, e\eta\, \iota\eta = \iota\, 1\, \iota\, \eta\, \eta = \iota\, \iota = -1.$$

Thus

$$I^2 = J^2 = K^2 = IJK = -1.$$

This must look a bit cryptic at first glance, but the construction shows how the structure of the quaternions comes directly from the non-commutative structure of our period two iterants. In other, words, quaternions can be represented by 2 x 2 matrices. This is way it has been presented in standard language. The group **SU(2)** of 2 x2 unitary matrices of determinant one is isomorphic to the quaternions of length one.

$$\begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} = [\, z \, , \overline{z}\,] + [\, w \, , -\overline{w}\,]\,\eta$$

Figure 12

In the equation above, we indicate the matrix form of an element of **SU(2)** and its corresponding complex valued iterant. You can easily verify that

1: z=1, w=0, I: z=ι, w =0, J: z=0, w = 1, K: z=0, w = ι.

This gives the generators of the quaternions as we have indicated them above and also as generators of **SU(2)**.

Similarly, **H = [a,b] + [c + di, c-di]η** represents a Hermitian 2 x 2 matrix and hence an observable for quantum processes mediated by **SU(2)**. Hermitian matrices have real eigenvalues. It is curious how certain key iterant combinations turn out to be essential for the relations with quantum observation.

11. TIME SERIES AND DISCRETE PHYSICS

In this section we shall use the convention (outside of iterants) that successive observations, first A and then B will be denoted BA rather than AB. This is to follow previous conventions that we have used. We continue to interpret iterant observation sequences in the opposite order as in the previous section. This section is based on our work in [20] but takes a different interpretation of the meaning of the diffusion equation in relation to quantum mechanics.

We have just reformulated the complex numbers and expanded the context of matrix algebra to an intepretation of **i** as an oscillatory process and matrix elements as combined spatial and temporal oscillatory processes

(in the sense that **[a,b]** is not affected in its order by a time step, while **[a,b]η** includes the time dynamic in its interactive capability, and 2 x 2 matrix algebra is the algebra of iterant views **[a,b] + [c,d]η**).

We now consider elementary discrete physics in one dimension. Consider a time series of positions **x(t), t = 0, Δt, 2Δt, 3Δt, ...** . We can define the velocity **v(t)** by the formula **v(t) = (v(t+ Δ) - v(t))/Δt = Dx(t)** where **D** denotes this discrete derivative. In order to obtain **v(t)** we need at least one tick **Δt** of the discrete clock. Just as in the iterant algebra, we need a time-shift operator to handle the fact that once we have observed **v(t),** the time has moved up by one tick.

Thus we shall add an operator **J** that in this context accomplishes the time shift:

$$x(t)J = Jx(t+\Delta t).$$

We then *redefine* the derivative to include this shift:

$$\mathbf{Dx(t) = J(x(t+ \Delta) - x(t))/\Delta t} .$$

The result of this definition is that a successive observation of the form **x(Dx)** is distinct from an observation of the form **(Dx)x**. In the first case, we observe the velocity and then **x** is measured at **t + Δt** . In the second case, we measure **x** at **t** and then measure the velocity. Here are the two calculations:

$$\mathbf{x(Dx) = x(t)\,(J(x(t+ \Delta) - x(t))/\Delta t\,)}$$
$$\mathbf{= (J/\Delta)(x(t+ \Delta t))(x(t+ \Delta t) - x(t))}$$
$$\mathbf{= (J/\Delta t)(x(t+ \Delta t)^2\ - x(t+ \Delta t)x(t)).}$$
$$\mathbf{(Dx)x = (J(x(t+ \Delta t) - x(t))/\Delta t\,)x(t)}$$
$$\mathbf{= (J/\Delta t)(x(t+ \Delta t)x(t) - x(t)^2).}$$

We measure the difference between these two results by taking a commutator
[A,B] = AB - BA and we get the following formula where we write **Δx = x(t+ Δt) - x(t).**

$$\mathbf{[x,(Dx)] = x(Dx) - (Dx)x = (J/\Delta t)(x(t+ \Delta t) - x(t))^2 = J\ (\Delta x)^2/\Delta t}$$

This final result is worth marking:

$$\mathbf{[x,(Dx)] = J\ (\Delta x)^2/\Delta t.}$$

From this result we see that the commutator of **x** and **Dx** will be constant if
$(\Delta x)^2/\Delta t = K$ is a constant.
For a given time-step, this means that $(\Delta x)^2 = K\,\Delta t$
so that $\Delta x = +\sqrt{(K\,\Delta t\,)}$ or $-\sqrt{(K\,\Delta t\,)}$.
In other words,
$x(t + \Delta t\,) = x(t) + \sqrt{(K\,\Delta t\,)}$ or $x(t) - \sqrt{(K\,\Delta t\,)}$.
This is a Brownian process with diffusion constant equal to **K.**

Digression on Browian Processes and the Diffusion Equation

Assume, for the purpose of discussion that in the above process, at each next time, it is equally likely to have + or - in the formulas
$x(t + \Delta t\,) = x(t) + \sqrt{(K\,\Delta t\,)}$ **or** $x(t) - \sqrt{(K\,\Delta t\,)}$.
Let **P(x,t)** denote the probability of the particle being at the location x at time t in this process. Then we have

$$\mathbf{P(x, t + \Delta t\,) = (1/2)(P(x - \Delta x) + P(x + \Delta x)).}$$

Hence

$$\mathbf{(P(x, t + \Delta t\,) - P(x,t))/\Delta t)}$$
$$\mathbf{= ((\Delta x)^2/2\Delta t)(P(x - \Delta x) - 2P(x,t) + P(x + \Delta x,t))/(\Delta x)^2)}$$
$$\mathbf{= (K/2)(P(x - \Delta x) - 2P(x,t) + P(x + \Delta x,t))/(\Delta x)^2).}$$

Thus we see that **P(x,t)** satisfies the a discretization of the diffusion equation

$$\mathbf{\partial\, P/\partial t = (K/2)\partial^2\, P/\partial x^2.}$$

Compare the diffusion equation with the Schrodinger equation iwith zero potential shown below.

$$\mathbf{i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2}$$

In the Schrodinger equation we see that we can rewrite it in the form

$$\partial\psi/\partial t = i(\hbar/2m)\partial^2\psi/\partial x^2$$

Thus, if we were to make a literal comparison with the difusion equation we would take $K = i(\hbar/m)$ and we would identify

$$(\Delta x)^2/\Delta t = i(\hbar/m).$$

Whence

$$\Delta x = ((1+i)/\sqrt{2}) \sqrt{[(\hbar/m)\Delta t]}$$

and the corresponding Brownian process is

$$x(t + \Delta t) = x + \Delta x \text{ or } x - \Delta x.$$

The process is a step-process along a diagonal line in the complex plane. We are looking at a Brownian process with complex values! What can this possibly mean? Note that if we take this point of view, then **x** is a complex variable and the partial derivative with respect to **x** is taken with respect to this complex variable. In this view of a complexified version of the Schrodinger equation, the solutions for Δx as above are real probabilities.

We shall have to move the **x** variation to real **x** to get the usual Schrodinger equation, and this will result in complex valued wave functions in its solutions.

In our context, the complex numbers are themselves oscillating and synchronized processes. We have $i = [1,-1]\eta$ where η is a shifter satisfying the rules of the last section, and **[1,-1]** is a view of the iterant that oscillates between plus and minus one. Thus we are now observing that solutions to the Schrodinger equation can be construed as Brownian paths in a more complicated discrete space that is populated by both probabilistic and synchronized oscillations. This demands further discussion, which we now undertake.

The first comment that needs to be made is that since in the iterant context Δx is an oscillatory quantity it does make sense to calculate the partial derivatives using the limits as Δx and Δt approach zero, but this means that the interpretation of the Schrodinger equation as a diffusion equation and the wave function as a probability is dependent on this generalization of the derivative. *If we take Δx to be real, then we will get complex solutions to Schrodinger's equation.* In fact we can write

$$\psi(x, t + \Delta t) = (1-i)\psi(x, t) + (i/2)\psi(x - \Delta x) + (i/2)\psi(x + \Delta x)$$

and then we will have, in the limit,

$$\partial\psi/\partial t = i(\hbar/2m)\partial^2\psi/\partial x^2$$

if we take $(\Delta x)^2/\Delta t = (\hbar/m)$.It is interesting to compare these two choices. In one case we took

$$(\Delta x)^2/\Delta t = i(\hbar/m)$$

and obtained a Brownian process with imaginary steps. In the other case we took

$$(\Delta x)^2/\Delta t = (\hbar/m)$$

and obtained a real valued process with imaginary *probability weights.* These are complementary points of view about the same structure.

With $(\Delta x)^2/\Delta t = (\hbar/m)$, $\psi(x, t)$ is no longer the classical probability for a simple Brownian process. We can imagine that the coefficients **(1-i)** and **(i/2)** in the expansion of $\psi(x, t + \Delta t)$ are somehow analogous to probability weights, and that these weights would correspond to the generalized Brownian process where the real-valued particle can move left or right by Δx or just stay put.

Note that we have **(1-i) + (i/2) + (i/2) = 1**, signalling a direct analogy with probability where the probability values are imaginary. But this must be explored in the iterant epistemology! Note that **1-i = [1,1] -[1,-1]**η and so at any given time represents either **[1,1] - [1,-1] = [0,2]** or **[1,1] - [-1,1] = [2,0]**.

It is very peculiar to try to conceptualize this in terms of probability or amplitudes.

Yet we know that in the standard interpretations of quantum mechanics one derives probability from the products of complex numbers and their conjugates.

To this end it is worth seeing how the product of **a+bi** and **a-bi** works out:

$$(a + bi)(a-bi) = aa + bia + a(-bi) + (bi)(-bi) = aa + abi - abi - bbii$$
$$= aa - bb(-1) = aa + bb.$$

It is really the rotational nature of **exp(it)** that comes in and makes this work.

$$\exp(it)\exp(-it) = \exp(it - it) = \exp(0) = 1$$

The structure is in the exponent. The additive combinatory properties of the complex numbers are all under the wing of the rotation group.

A fundamental symmetry is at work, and that symmetry is a property of the synchronization of the periodicities of underlying process. The fundamental iterant process of **i** disappears in the multiplication of a complex number by its conjugate. In its place is a pattern of apparent acuality. It is actual just to the extent that one regards **i** as only possibility. On making a reality of **i** itself we have removed the boundary between mathematics and the reality that "it" is supposed to describe. There is no such boundary.

12. EPILOGUE AND SIMPLICITY

Finally, we arrive at the simplest place. Time and the square root of minus one are separable in the temporal nexus. The square root of minus one is a symbol and algebraic operator for the simplest oscillatory process.

As a symbolic form, **i** is an eigenform satisfying the equation

$$i = -1/i.$$

One does not have an increcemt of time all alone as in classical Δt. One has $i\Delta t$, a combination of an interval and the elemental dynamic that is time. With this understanding, we can return to the commutator for a discrete process and use $i\Delta t$ for the temporal incerment.

We found that discrete observation led to the commutator equation

$$[x,Dx] = J\,(\Delta x)^2/\Delta t$$

which we will simplify to

$$[q, p/m] = (\Delta x)^2/\Delta t.$$

taking **q** for the position **x** and **p/m** for velocity, the time derivative of position.

Understanding that Δt should be replaced by $i\Delta t$, and that, by comparison with the diffusion equation,

$$(\Delta x)^2/\Delta t = \hbar/m,$$

we have

$$[q, p/m] = (\Delta x)^2/i\,\Delta t = -i\,\hbar/m,$$

whence

$$[p,q] = i\hbar,$$

and we have arrived at Heisenberg's fundamental relatiionship between position and momentum. This mode of arrival is predicated on the recognition that only $i\,\Delta t$ represents a true interval of time.

In the notion of time there is an inherent clock or an inherent shift of phase that is making a sychrony in our ability to observe, a precise dynamic beneath the apparent dynamic of the observed process. Once this substitution is made, *once the correct imaginary value is placed in the temporal circuit, the patterns of quantum mechanics appear.*

The problem that we have examined in this paper is the problem to understand the nature of quantum mechanics. In fact, we hope that the problem is seen to disappear the more we enter into the present viewpoint. A viewpoint is only on the periphery. The iterant from which the viewpoint emerges is in a superposition of indistinguishables, and can only be approached by varying the viewpoint until one is released from the particularities that a point of view contains.

It is not just the eigenvalues of Hermitian operators that are the structures of the observation, but rather a multiplicity of eigenforms that populate mathematics at all levels. These forms are the indicators of process. Mathematics comes alive as an interrelated orchestration of processes. It is these processes that become the exemplary operators and elements of the mathematics that are put together to form the physical theory. We hope that the reader will be unable, ever again, to look at Schrodinger's equation or Heisenberg's commutator the same way, after reading this argument.

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This paper is a modified treatment of the same ideas as we have written about them in [22] and [23].

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