Determination of optimal control strategy in strict hierarchical manpower system

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Abstract
Present paper addresses an approach to determine optimal recruitment and transition strategies in strict hierarchical manpower system by the application of simulation modeling and optimization methods. The transition model is represented in the form of discrete state space. The target values for each particular rank are determined by the user defined trajectory function. Optimal recruitment and transition dynamics is determined by the minimization of the differences between desired and actual state values. Analytical approach to the optimization is considered in order to provide proper control strategy.
Keywords : state space, modeling, manpower, optimization, Potryagin

1 Introduction

Strict hierarchical model of human resources transition addresses the organizational structure, where only sequential transitions between different ranks are possible i.e. jumps in the promotion are not allowed. Such systems could be found e.g. in military, where only the sequential, highest, ranks are considered [1; 2]. Long-term manpower planning in an army is a strategic and important task involving enormous amounts resources therefore the anticipative value [3; 4] of developed model should be applied. An army is a part of a complex social system, in which dynamics over a longer time frame must be considered due to large time constants and delays across feedback loops. Non-sequential transitions are not permissible due to the mandatory training. The system is described by the principle of system dynamics [5; 6; 7] in discrete form as:

$$x(k) = x(k_0) + \sum_{i=k_0}^{k-1} (R_{in}(i) - R_{out}(i))\ \Delta t \quad (1)$$

$$\frac{\Delta x(i)}{\Delta t} = R_{in}(i) - R_{out}(i) \quad (2)$$

In continuous form the system could be represented as:

$$x(t) = \int_{t_0}^{t} [R_{in}(t) - R_{out}(t)]\mathrm{d}t + x(t_0) \quad (3)$$

$$\frac{\mathrm{d}(x)}{\mathrm{d}t} = \text{net change x} = R_{in}(t) - R_{out}(t) \quad (4)$$

where Eq. 2 represents the *net change* of the state x. The state variables $x_1, x_2, \ldots, x_n$ describe the system state which are in our case the member numbers of particular rank $x_1, x_2, \ldots, x_n$. The rate elements R and F represent the intensity of the state change. Here the transitions between particular ranks and wastages are considered. The meaning of the symbols is the following:

- R_0 rate element which represents the input to the system determined by the value u.
- R_1 rate element which represents the transitions from rank x_1 into rank x_2. R_1 is determined by the value of x_1 and the coefficient r_1.
- R_2 rate element which represents the transitions from rank x_2 into rank x_3. R_2 is determined by the value of x_2 and the coefficient r_2, etc.
- F_1 rate element which represents the wastages from the rank x_1. F_1 is determined by the values of x_1 and by coefficient f_1.
- F_2 rate element which represents the wastages from the rank x_2. F_2 is determined by the values of x_2 and by coefficient f_2, etc.

Writen in the discrete state space matrix form where $\Delta t = 1$, the considered system has the following representation:

$$\left\{ \begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \right. \quad (5)$$

here the matrix **A** from the Eq. 5 is determined as:

$$\mathbf{A} = \begin{bmatrix} 1-r_1(k)-f_1(k) & 0 & 0 & \dots \\ r_1(k) & 1-r_2(k)-f_2(k) & 0 & \dots \\ 0 & r_2(k) & 1-r_3(k)-f_3(k) & \dots \\ 0 & 0 & r_3(k) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6}$$

The input of the considered system $u(k)$ is determined by the matrix $\mathbf{B}$ into the state x_1 by stating:

$$x_1(k+1) = [1 - r_1(k) - f_1(k)]\; x_1(k) + u(k) \tag{7}$$

According to the stated Eq. 7 $u(k)$ represents the number of new recruits which enter the rank determined by the state variable x_1. $u(k)$ represents the single input to the considered system. The system is therefore dependant on the parameters of input, promotion and wastages.

2 Optimization Approach

One would strive for optimal control according to the initially given trajectories which should be achieved by the values of state elements. In order to reach this goal, the Pontryagin maximum prinicple will be applied [8]. Let us state the Lagrangian multiplicator for the boudary $\dot{x} = f(x, u)$:

$$L = J + \int_0^T \lambda\,[f(x,u) - \dot{x}]\,dt \tag{8}$$

Stated in different form:

$$L = \int_0^T V(x,u)\,dt + \int_0^T \lambda\,[f(x,u) - \dot{x}]\,dt \tag{9}$$

$$= \int_0^T [V(x,u)\,dt + \lambda\, f(x,u) - \lambda\,\dot{x}]\,dt \tag{10}$$

Hamiltonian function is stated as:

$$H(x,u) = V(x,u) + \lambda\, f(x,u) \tag{11}$$

Therefore one gets:

$$L = \int_0^T [H(x,u,t) - \lambda\,\dot{x}]\,dt \tag{12}$$

The change in Lagrangian ΔL is stated as:

$$\Delta L = \int_0^T \left[\frac{\partial H}{\partial u} du + \left(\frac{\partial H}{\partial u} + \dot{\lambda}\right) dx\right] dt - \lambda (T)\, dx^T \tag{13}$$

If one considers that the $\Delta L = 0$ the following conditions are obtained:

$$\frac{\partial H}{\partial u} = 0 \qquad 0 \leq t \leq T \tag{14}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \qquad 0 \leq t \leq T \tag{15}$$

$$\lambda(T) = 0 \vee x(T) = x^T \tag{16}$$

The condition stated by the Eq. 14 declares, that the Hamiltonian function is maximized by the selection of the control variable on the path of the stated trajectory. Here we anticipate, that there is no limitations at the control variable. The condition stated by the Eq. 15 addresses the intensity of the change of the costate variable λ. The intensity of the change of the costate variable is equal to the negative value of the Hamiltonian function with respect to the proper costate variable. In the case, that more variables would be considered, e.g. $x_1, x_2, ...$ one would get $\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1}, \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2}, ...$ The condition stated by the Eq. 16 determines the end value of the costate variable. The end value could be 0 or, if the final value is stated as $x(T) = x^T$; then the $dx^T = 0$.

According to the stated Hamiltonian function the state variable could be determined as:

$$\dot{x} = f(x, u) = \frac{\partial H}{\partial \lambda} \tag{17}$$

At the determination of the optimal strategy one should first add the costate variable λ to the initial equations and define the Hamiltonian function $H(x, u) = V(x, u) + \lambda f(x, u)$ and solve by the trajectories $\{u(t)\}$, $\{\lambda(t)\}$ and $\{x(t)\}$, here the following conditions should be fullfiled:

$$\frac{\partial H}{\partial u} = 0 \qquad 0 \leq t \leq T \tag{18}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \qquad 0 \leq t \leq T \tag{19}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f(x, u) \tag{20}$$

$$\lambda(T) = 0 \vee x(T) = x^T \tag{21}$$

$$x(0) = x^0 \tag{22}$$

Hamiltonian function could be expanded for the arbitrary number of state variables $x_1, x_2, ...$, here the proper costate variables λ should be added as well as control variables $u(t)$.

3 Application Example

Let us consider, as the example, the system with the one state element. The criteria function is stated as:

$$J = \max - \int_{0}^{T} (4\,t + 10 - x)^2 \tag{23}$$

In the Eq. 23 one considers, that the trajectory is determined by the linear equation; here we want that the response of the system $x(t)$is as close as possible to the stated trajectory. The criteria function is in the boundaries of 0 to $T = 10$. In the Eq. 23, before the integral sign, the $-$ has been put, since the deviation should be as small as possible. Boundaries in the sense of Pontryagin actually represent the state space with the input:

$$\dot{x} = r_0 - \frac{1}{10}\,x \tag{24}$$

Boundary conditions for our case are defined as $x(0) = 10, x(10) = 50$. Hamiltonon-ian function for our case is stated as:

$$H(x,u) = V(x,u) + \lambda\,f(x,u) = -(4\,t + 10 - x)^2 + \lambda\,(r_0 - \frac{1}{10}\,x) \tag{25}$$

Partial derivatiove of the Hamiltonian function according to our input $u = r_0$:

$$\frac{\partial H}{\partial r_o} = \lambda \tag{26}$$

By regarding the condition $\frac{\partial H}{\partial r_o} = 0$ we have $\lambda = 0$. Partial derivative of the Hamiltonian function according to the state of system x:

$$-\frac{\partial H}{\partial x} = \frac{1}{10}\lambda - 2\,(10 + 4\,t - x) \tag{27}$$

By considering that $\lambda = 0$ one gets:

$$x = 4\,t + 10 \tag{28}$$

He differential equation is deterimned as [9]:

$$\dot{x}(t) = r_0(t) - \frac{1}{10}\,(4\,t + 10) \tag{29}$$

where the function of interest is $r_o(t)$:

$$r_0(t) = 1 + \frac{2}{5}\,t + \dot{x}(t) \tag{30}$$

We consider that $x = 4\,t + 10$ and first derivative $\dot{x} = 4$ and also consider the value of the derivative in the Eq. 30; therefore one gets the optimal control:

$$r_0(t) = \frac{2}{5}\,t + 5 \tag{31}$$

Let us proceed with the solution of the second order system where we will anticipate, that the value on the state element is $x_2 = 10$ i.e. constant:

$$J = \max - \int_0^T (10 - x_2)^2 \tag{32}$$

In the Eq. 32 we will consider, that the desired trajectory is declared by the constant, besides, we want that the response of the system stated by $x_2(t)$ is as close as possible to the desired trajectory. Criteria function is stated in the boundaries of 0 to $T = 10$. The boundaries in the sense of Pontryagin i.e. the determination in the state space is:

$$\dot{x}_2 = (4\,t + 10) \times \frac{1}{10} - x_2 R_2 - \frac{1}{10}\,x_2 \tag{33}$$

Boundary values for our case are declared as $x(0) = x(10) = 10$. Further, we will consider, that $4\,t + 10)$ represents value $x(t)$, if we also consider the coefficient $\frac{1}{10}$ we get the inflow to the state element x_2, which is $R_1 = \frac{1}{10}(4\,t + 10)$. Hamiltonian function for the state x_2 is:

$$H(x_2, u) = V(x_2, u) + \lambda\, f(x_2, u) = -(10 - x_2)^2 + \lambda \left((4\,t + 10) \times \frac{1}{10} - x_2 R_2 - \frac{1}{10}\,x_2 \right)$$

Partial derivative of Hamiltonian function with regard to the input $u = r_2$:

$$\frac{\partial H}{\partial r_2} = -\lambda\, x \tag{34}$$

At the condition that $\frac{\partial H}{\partial r_2} = 0$ we have $\lambda = 0$. Partial derivative of the Hamiltonian function with the regard of the system state x_2:

$$-\frac{\partial H}{\partial x_2} = \lambda\, R_2 - 2\,(10 - x) \tag{35}$$

The differential equation is state as:

$$\dot{x}_2(t) = (4\,t + 10) \times \frac{1}{10} - x_2(t)\, R_2(t) \tag{36}$$

where the variable of interset is $R_2(t)$:

$$R_2(t) = \frac{5 + 2\,t - 5\,\dot{x}_2(t)}{5\,x(t)} \tag{37}$$

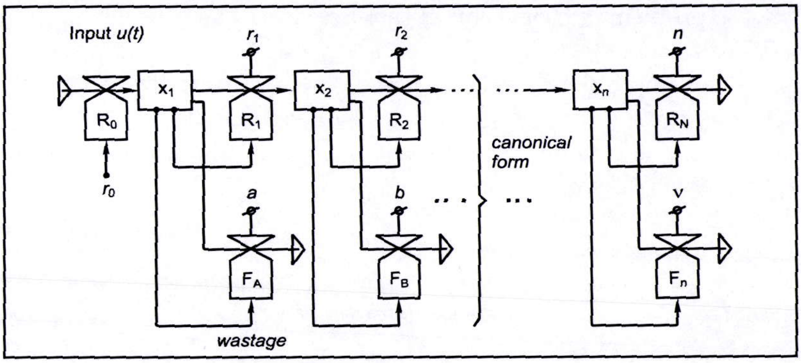

Fig. 1: Structure of the System Dynamics model of the transitions in the cannonical form. x_n represents the state element.

We consider that the $x_2 = 10$ and the first derivative $\dot{x}_2 = 0$ as well as the value of the derivative defined by Eq. 37; therefore we get the optimal control:

$$R_2(t) = \frac{1}{25}t + \frac{1}{10} \tag{38}$$

The system could be represented by the elements of System Dynamics as shown in the Fig. 1, where the part of the system structure is shown. Here the transitions between particular ranks are represented as well as wastages and recruitment. In this case the part of the structure is considered which takes into account the number of entities in rank x_1 which represent the state element. Here the canonical form is anticipated.

4 Conclusion

Determination of the optimal strategy in the strict hierarchical system is demanding task. The importance of the optimal strategy is determined by the ubiquitous presence of such systems e.g. supply chains, manpower systems etc. The SD methodology proved to be a proper tool to present the users with the hard methodological concepts such as state space. The cascading exponential delay structure represents an abstract form of the considered system whose main characteristics could be easily presented to the users. If the user does not fully understand the concepts applied in the developed system, the acceptance of the developed strategies cannot be reached and therefore tha anticipative advantage of applying the developed model is not taken. The described methodology of the system optimization by the Pontryagin maximum principle for the case of first and second order offers the possible alternative of optimization for the systems of higher order. The optimization is due to the large number of parameters demanding. Besides the problem of equal strategies occurs for which the Hamiltonian function could not be determined. In the further development the problem of oscillation in the system will be considered. In our case the oscillation of system states as well as strategy functions should not exercise the oscillations. For example the optimal strategy could be stated which considers the oscillation in the parameter values which, for the real case, would not be appropriate. In this case the additional criterion should be introduced in order to provide the proper, optimal strategy.

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