# Backstepping-based Control Design for the Stabilization of Nonholonomic Chained Systems: Applications to Mobile Robots*

Abdelhamid Tayebi and Ahmed Rachid

*Laboratoire des Systèmes Automatiques*
*Université de Picardie-Jules Verne*
*7, Rue du Moulin Neuf, 80000 AMIENS, FRANCE.*
*E-mail : tayebi@u-picardie.fr*

***Abstract.*** *This paper presents a backstepping procedure for the design of discontinuous time-invariant state feedback controllers for the stabilization of nonholonomic systems in chained form. To highlight its effectiveness, our procedure is applied to two nonholonomic physical systems, namely, a unicycle and a car-like mobile robots. It is worth noticing that this approach may be considered as new systematic way to design time-invariant discontinuous controllers for nonholonomic systems.*

***Keywords.*** *Nonholonomic systems, Mobile robots, Stabilization problem, Backstepping approach, Lyapunov method.*

## *1. Introduction*

The problem we are concerned with is the stabilization of the class of nonholonomic systems in chained form described by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{aligned} \tag{1}$$

where $(x_1, x_2, \cdots, x_n) \in D_1$ denote the state variables and $(u_1, u_2) \in D_2$ denote the input variables, $D_1$ and $D_2$ are, respectively, open subsets of $\Re^n$ and $\Re^2$, both containing the origin.

Such a class of nonholonomic systems was treated for the first time in (Murray et al. 1991), and sufficient conditions under which any mechanical system, with two inputs (e.g. wheeled mobile robots), can be transformed via coordinate and feedback transformations into the chained form (1) are given in ( Murray et al. 1993). One of the reasons for the interest in

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such systems is that they fall into the class of systems which cannot be stabilized via any smooth time-invariant feedback, as pointed out in (Brocket 1983). This fact motivated the search for stabilizers of another type. Among all, recall smooth time-varying feedback resulting in oscillating trajectories (see Pomet 1992, Samson 1995, Tayebi et al. 1996A and references therein), and discontinuous feedback resulting in exponential stability and yielding to nonoscillating trajectories (see, for example, (Astolfi 1996), (Bloch et al 1994), (Canudas et al 1992), (Canudas et al 1995), (Reyhanoglu 1995) and Tayebi et al. 1996B and 1997). The backstepping approach introduced in (Krstic et al 1995), has also been used to derive time-varying controllers as shown in (Jiang et al 1996) and (Jiang et al 1995). However, in this paper, using the elegant backstepping technique, we propose a systematic way to design the discontinuous time-invariant controllers for system (1).

For the sake of simplicity, two nonholonomic physical systems are studied instead of system (1):

*(i)* the unicycle-like mobile robot described in the Cartesian space by

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{cases} \tag{2}$$

which can be transformed into the following third order chained form

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \end{cases} \tag{3}$$

using the following transformations $\{x_1 = \theta,\ x_2 = x\cos\theta + y\sin\theta,\ x_3 = x\sin\theta - y\cos\theta\}$ and $\{u_1 = \omega,\ u_2 = v - x_3\omega\}$.

*(ii)* and the car-like mobile described, in the Cartesian space, by

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \dfrac{v}{L}\tan\phi \\ \dot{\phi} = \omega \end{cases} \tag{4}$$

which can be transformed into a forth order chained system

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \\ \dot{x}_4 = x_3 u_1 \end{cases} \tag{5}$$

using the following local transformations $\left\{ x_1 = x, x_2 = \frac{tan\phi}{Lcos^3\theta}, x_3 = tan\theta, x_4 = y \right\}$ and $\left\{ v = \frac{u_1}{\cos\theta}, \omega = -\frac{3\sin^2\phi\sin\theta}{L\cos^2\theta}u_1 + L\cos^2\phi\cos^3\theta\ u_2 \right\}$ defined over the subset $\Gamma = \left\{ (x, y, \theta, \phi) \in \Re^4 / \theta \neq \frac{\pi}{2}\text{mod}\,\pi, \phi \neq \frac{\pi}{2}\text{mod}\,\pi \right\}$ of $\Re^4$.

In section 2, we describe the procedure yielding in a systematic way to discontinuous time-invariant controllers for nonholonomic systems in chained form and we suggest a solution for the stabilization problem of a car-like vehicle (4) and a unicycle-like vehicle (2). Section 3 contains some simulation results. Section 4 concludes this paper.

## *2. Design Procedure*

In order to apply the backstepping procedure, let us consider the following change of coordinates
$y_i = x_{n-i+1}$ for $1 \le i \le n$. System (1), then becomes

$$\begin{aligned} \dot{y}_1 &= y_2 u_1 \\ \dot{y}_2 &= y_3 u_1 \\ \dot{y}_3 &= y_4 u_1 \\ &\vdots \\ \dot{y}_{n-1} &= u_2 \\ \dot{y}_n &= u_1 \end{aligned} \tag{6}$$

To render the last coordinate $y_n$ exponentially stable, let us take a linear state feedback $u_1 = -k_1 y_n$, where $k_1$ is a strictly positive parameter.
System (6) then becomes

$$\begin{aligned} \dot{y}_1 &= -k_1 y_2 y_n \\ \dot{y}_2 &= -k_1 y_3 y_n \\ \dot{y}_3 &= -k_1 y_4 y_n \\ &\vdots \\ \dot{y}_{n-1} &= u_2 \\ \dot{y}_n &= -k_1 y_n \end{aligned} \tag{7}$$

Now, the problem consists in finding the control law $u_2$ for the stabilization of (7), using the backstepping approach (Krstic 1995). The first step consists in finding an adequate control Lyapunov function (clf) $V_1$ for the first equation of (7) and a virtual control $y_2 = \psi_1(y_1, y_n)$ which stabilizes $y_1$. In the next step, the previous clf is augmented to obtain an adequate one $V_2$ which leads to the virtual control $y_3 = \psi_2(y_1, y_2, y_n)$ that stabilizes both $y_1$ and $(y_2 - \psi_1(y_1, y_n))$. Progressing in this way, we arrive to the $(n-2)$-th step, where we obtain the clf $V_{n-2}$ which leads to the virtual control $y_{n-1} = \psi_{n-2}(y_1, y_2, y_3, \cdots, y_{n-2}, y_n)$ that stabilizes all of $y_i$ $1 \le i \le n-2$. Finally, we augment $V_{n-2}$ to obtain the final clf $V_{n-1}$ yielding to the stabilizing control $u_2 = \psi_{n-1}(y_1, y_2, y_3, \cdots, y_{n-2}, y_{n-1}, y_n)$ for the system (7).

In order to guarantee that the whole state is bounded and tends to zero when $t$ goes to infinity, one must ensure that all of $\psi_i$ are bounded and vanishes when $t$ tends to infinity. As it will be shown later, the $\psi_i$ functions are not defined for $y_n = 0$. Therefore, the control law $u_2$ is defined over the domain $\Omega_n = \{y = (y_1, y_2, \cdots, y_n) \in \Re^n / y_n \neq 0\}$. Due to the exponential convergence of $y_n$,, the discontinuity manifold is not very restrictive since we have just to avoid $y_n(0) = 0$.

### 2.1. The Car-like Mobile Robot Case

Let us consider system (5) written in the form (6) with $u_1 = -k_1 y_4$, we obtain

$$\begin{cases} \dot{y}_1 = -k_1 y_2 y_4 \\ \dot{y}_2 = -k_1 y_3 y_4 \\ \dot{y}_3 = u_2 \\ \dot{y}_4 = -k_1 y_4 \end{cases} \tag{8}$$

***Step 1.*** Let us take the following Lyapunov function candidate for the first equation of (8)

$$V_1(y_1) = \frac{1}{2} y_1^2 \tag{9}$$

Under the following virtual control, defined over $\Omega_4$,

$$y_2 = \psi_1(y_1, y_4) = \frac{k_2}{k_1} \frac{y_1}{y_4}, \tag{10}$$

the time derivative of (9) becomes

$$\dot{V}_1(y_1) = -k_2 y_1^2, \tag{11}$$

where $k_2$ is a positive parameter.

***Step 2.*** Now, let us introduce a new variable $z_2 = y_2 - \psi_1(y_1, y_4)$ which represents the deviation between $y_2$ and the virtual control $\psi_1$ and consider the first two equations of (8) where $y_2$ is substituted by $z_2 + \psi_1(y_1, y_4)$

$$\begin{cases} \dot{y}_1 = -k_1 y_4 (z_2 + \psi_1) \\ \dot{z}_2 = -k_1 y_3 y_4 + k_2 z_2 + (k_2 - k_1) \psi_1 \end{cases} \tag{12}$$

Using the following Lyapunov function candidate

$$V_2(y_1,z_2)=V_1(y_1)+\frac{1}{2}z_2^2\,, \tag{13}$$

and the following virtual control defined over $\Omega_4$

$$y_3=\psi_2(y_1,y_2,y_4)=-y_1+\frac{1}{k_1y_4}\big((k_2+k_3)z_2+(k_2-k_1)\psi_1\big), \tag{14}$$

leads to

$$\dot{V}_2(y_1,z_2)=-k_2y_1^2-k_3z_2^2\,, \tag{15}$$

where $k_3$ is a positive parameter.

***Step 3.*** As in step 2, let us introduce the new variable $z_3=y_3-\psi_2(y_1,y_2,y_4)$ and consider system (12) augmented by $\dot{z}_3=\dot{y}_3-\dot{\psi}_2$

$$\begin{cases}\dot{y}_1=-k_1y_4(z_2+\psi_1)\\ \dot{z}_2=-k_1(z_3+\psi_2)y_4+k_2z_2+(k_2-k_1)\psi_1\\ \dot{z}_3=u_2-\dot{\psi}_2\end{cases} \tag{16}$$

where

$$\dot{\psi}_2=k_1y_2y_4-(k_2+k_3)y_3+(2k_2+k_3+\frac{k_2k_3}{k_1})\frac{y_2}{y_4}-2\frac{k_2}{k_1}(k_1+k_3)\frac{y_1}{y_4^2}$$

(17)

Consider the following Lyapunov function candidate

$$V_3(y_1,z_2,z_3)=V_2(y_1,z_2)+\frac{1}{2}z_3^2 \tag{18}$$

Differentiating (18) with respect to time and using the following control law defined over $\Omega_4$

$$u_2=k_1y_2y_4-(k_2+k_3)y_3+(2k_2+k_3+\frac{k_2k_3}{k_1})\frac{y_2}{y_4}-2\frac{k_2}{k_1}(k_1+k_3)\frac{y_1}{y_4^2}-k_4z_3+k_1y_4z_2$$

(19)

yields

$$\dot{V}_3(y_1,z_2,z_3)=-k_2y_1^2-k_3z_2^2-k_4z_3^2 \tag{20}$$

where $k_4$ is a positive parameter.

Now, one can easily conclude that $y_1,z_2$ and $z_3$ are bounded and tend to zero when $t$ tends to infinity. Therefore,

$$y_2\to\frac{k_2}{k_1}\frac{y_1}{y_4}\text{ and }y_3\to-y_1+\frac{(k_2+k_3)}{k_1}\frac{y_2}{y_4}+\frac{k_2(k_2-k_1)}{k_1}\frac{y_1}{y_4}-\frac{k_2(k_2+k_3)}{k_1^2}\frac{y_1}{y_4^2}.$$

To guarantee the boundedness and the convergence to zero of $y_2$ and $y_3$, one must ensure the boundedness and the convergence to zero of $\frac{y_1}{y_4}$, $\frac{y_1}{y_4^2}$ and $\frac{y_2}{y_4}$. So, when $t \to \infty$, from (9) and (11) one can conclude that $y_1$ decays to zero as $exp(-k_2 t)$. Therefore, if we take $k_2 > 2k_1$ the boundedness and the convergence to zero of $\frac{y_1}{y_4}$ and $\frac{y_1}{y_4^2}$ becomes obvious whenever $y_4(0) \neq 0$ , since $y_4$ decays to zero as $exp(-k_1 t)$.

Now, to seek for the boundedness and the convergence to zero of $\frac{y_2}{y_4}$, one must do some manipulations. So, from equation (13), one has

$$z_2^2 = 2V_2 - y_1^2 \tag{21}$$

Substituting the latter in (15) yields

$$\dot{V}_2 = -2k_3 V_2 + (k_3 - k_2) y_1^2. \tag{22}$$

When $t$ tends to infinity $y_1 \to y_1(0)\exp(-k_2 t)$ and the latter gives

$$\dot{V}_2 = -2k_3 V_2 + (k_3 - k_2) y_1^2(0)\exp(-2k_2 t) \tag{23}$$

which leads to

$$V_2 = c_1\, exp(-2k_3 t) + c_2\, exp(-2k_2 t) \tag{24}$$

where $c_1$ and $c_2$ are two constants depending on the initial conditions.
Returning to (21) with view of (24), we obtain

$$z_2^2 = 2c_1 \exp(-2k_3 t) + (2c_2 - y_1^2(0)) \exp(-2k_2)t \tag{25}$$

Deviding (25) by $y_4^2$ and assuming that $y_4(0) \neq 0$, we obtain

$$\frac{z_2^2}{y_4^2} \equiv \left(\frac{y_2}{y_4} - \frac{k_2}{k_1}\frac{y_1}{y_4^2}\right)^2 = \alpha_1 \exp\left(-2(k_3 - k_1)t\right) + \alpha_2 \exp\left(-2(k_2 - k_1)t\right) \tag{26}$$

where $\alpha_1$ and $\alpha_2$ are constants depending on the initial conditions.
As we have seen previously, the choice $k_2 > 2k_1$ ensures the convergence to zero of $\frac{y_1}{y_4^2}$, hence one must take $k_3 > k_1$ to guarantee the boundedness and the convergence to zero of $\frac{y_2}{y_4}$.
The previous results can be summarized in the following theorem.

***Theorem 1.*** *Consider the following control law defined over* $\Omega_4 = \{(y_1, y_2, y_3, y_4) \in \Re^4 / \; y_4 \neq 0\}$

$$\begin{cases} u_1 = -k_1 y_4 \\ u_2 = 2k_1 y_2 y_4 - (k_2 + k_3 + k_4) y_3 - k_2 (1 + \frac{k_4}{k_1}) \frac{y_1}{y_4} - \frac{k_2(k_1 + k_3)}{k_1} (2 + \frac{k_4}{k_1}) \frac{y_1}{y_4^2} + (2k_2 + k_3 + \frac{k_2 k_3}{k_1} + \frac{k_4(k_2 + k_3)}{k_1}) \frac{y_2}{y_4} \end{cases}$$

*(27)*

*with* $y_i = x_{4-i+1}, 1 \le i \le 4,\ k_1 > 0,\ k_2 > 2k_1,\ k_3 > k_1,$ *and* $k_4 > 0,$

*Then,*

*(i) the whole state remains in* $\Omega_4$ *provided that* $y_4(0) \neq 0,$
*(ii) the closed loop system (5)-(27) is exponentially stable over* $\Omega_4,$
*(iii)the control law is bounded and well defined over* $\Omega_4.$

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### 2.2. The Unicycle-like Mobile Robot Case

Let us consider system (3) written under the form (6) with $u_1 = -k_1 y_3$

$$\begin{cases} \dot{y}_1 = -k_1 y_2 y_3 \\ \dot{y}_2 = u_2 \\ \dot{y}_3 = -k_1 y_3 \end{cases} \tag{28}$$

Proceeding as in section 2.1, one can find the control law $u_2$ which stabilizes system (28), and the following theorem can be stated.

***Theorem 2.*** *Consider the following control law defined over* $\Omega_3 = \{(y_1, y_2, y_3) \in \Re^3 / y_3 \neq 0\}$

$$\begin{cases} u_1 = -k_1 y_3 \\ u_2 = k_1 y_1 y_3 - (k_2 + k_3) y_2 + k_2 (1 + \frac{k_3}{k_1}) \frac{y_1}{y_3} \end{cases} \tag{29}$$

*with* $y_i = x_{3-i+1}, 1 \le i \le 3,\ k_1 > 0,\ k_3 > 0$ *and* $k_2 > k_1.$
*Then*
*(i) the whole state remains in* $\Omega_3$*., provided that* $y_3(0) \neq 0,$
*(ii) the closed loop system (3)-(29) is exponentially stable over* $\Omega_3$*.,*
*(iii) the control law is bounded and well defined over* $\Omega_3.$

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## 3. Simulation Results

In this section we present some simulation results carried out using MATLAB.

### *3.1 A Car-like Mobile Robot Case*

In this example, the car-like mobile robot is asked to reach the origin starting from the following initial conditions $(x_0 = -2, y_{20} = 2, \theta_0 = 0, \phi_0 = 0)$. See Figure 1 for the plots of the system state evolution, and Figure 2 for the plots of the control variables $v$ and $\omega$. Figure 3 shows the vehicle motion, in the parking maneuver, under the proposed controller.

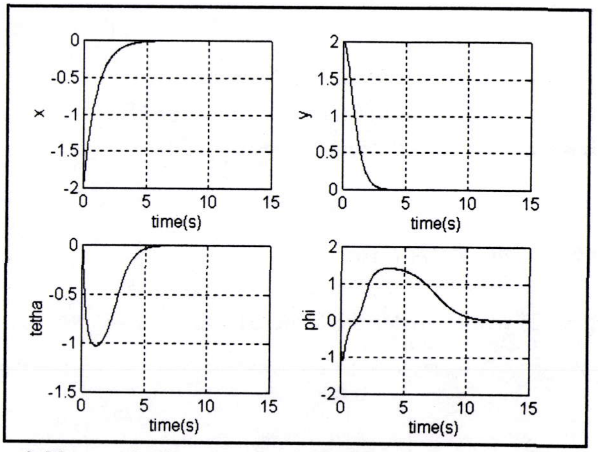


**Figure 1. State variables evolution for the initial conditions** $(x_0 = -2, y_{20} = 2, \theta_0 = 0, \phi_0 = 0)$

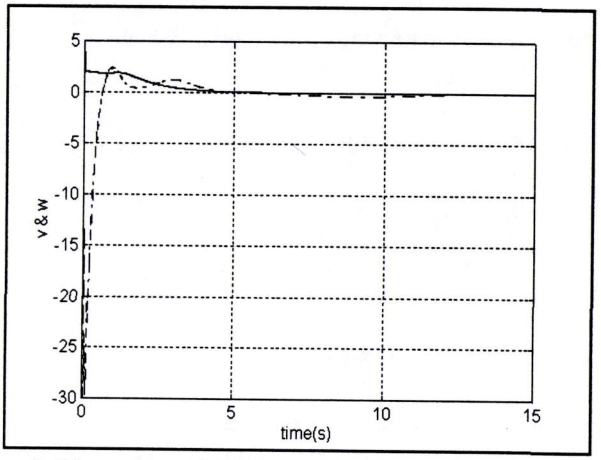


**Figure 2. Time plots of the control variables** $v$**(— ).and** $\omega$**(-.-.)**

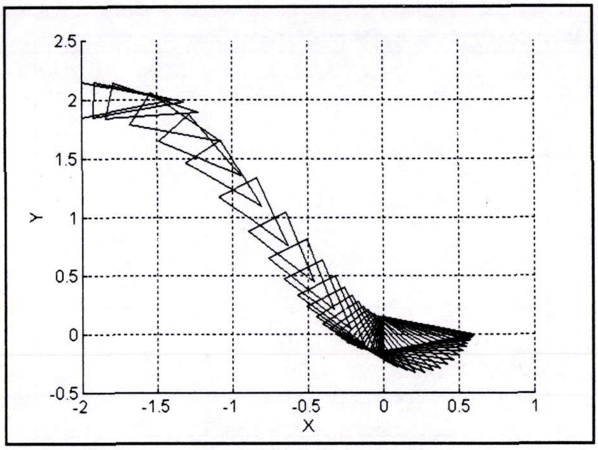


**Figure 3. Steering the vehicle to the origin, stating from** $(x_0 = -2, y_{20} = 2, \theta_0 = 0, \phi_0 = 0)$

***3.2. A Unicycle-like Mobile Robot Case***

In this example, the unicycle-like mobile robot is asked to reach the origin starting from the following initial conditions in Cartesian space $(x_0 = 2, y_0 = 2, \theta_0 = \frac{\pi}{2})$. See figure 4 for the plots of the state variables and generated trajectory in the Cartesian plane. See figure 5 for the time evolution of the control variables $v$ and $\omega$. Figure 6 shows the vehicle motion, in the parking maneuver, under the proposed controller.

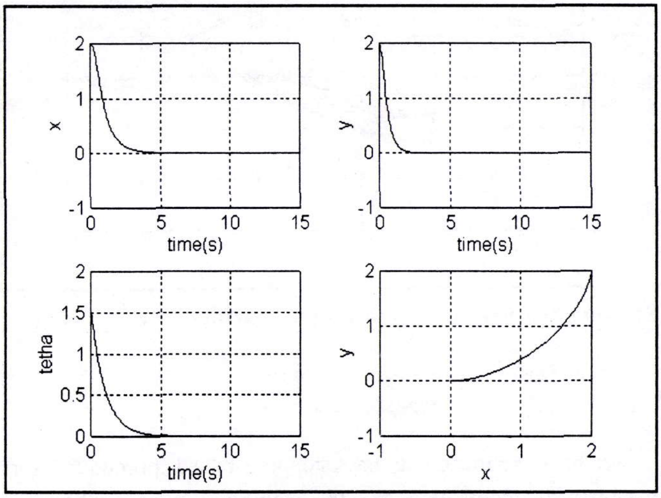


**Figure 4. Time plots of states variables and the generated trajectory, starting from** $(x_0 = 2, y_0 = 2, \theta_0 = \pi / 2)$

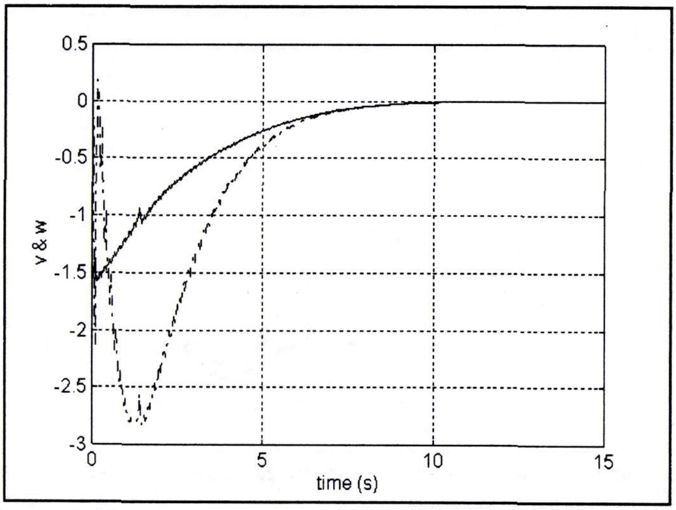


**Figure 5. Time plots of the control variables $v$(-.-.-)and $\omega$(—).**

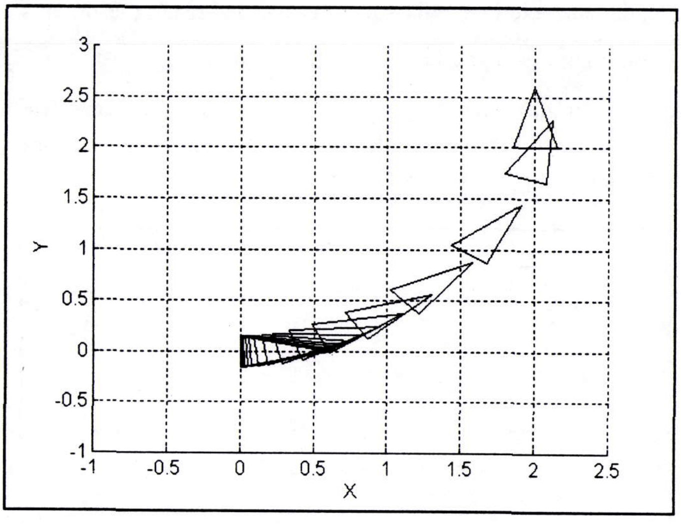


**Figure 6. Steering the vehicle to the origin, starting from** $(x_0 = 2, y_0 = 2, \theta_0 = \frac{\pi}{2})$

## *4. Conclusion*

In this paper, we have proposed a backstepping-based procedure for the design of discontinuous time-invariant controllers for the stabilization of nonholonomic systems in chained form. This procedure is then applied for the stabilization of a car-like mobile robot and a unicycle-like mobile robot. The discontinuity surface, for the control, is not very restrictive since we have just to avoid it at $t = 0$. Finally, it worth noticing that this work may be considered as a new way to systematically design time-invariant discontinuous controllers for nonholonomic systems.

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