Investigation of Complex Multivalued Solutions in Discrete Dynamical Systems With Anticipation.

Sergiy Lazarenko, Alexander Makarenko
Institute for applied system analysis NTUU „KPI",
Prospect Pobedy 37, 03056, Kiev-56, Ukraine
makalex@i.com.ua

Abstract
Since the introduction of strong anticipation by D.Dubois the numerous investigations of concrete systems had been proposed. Discrete dynamical systems with anticipation constitute one of such system class. But not very many investigations of such objects exist recently. More intensive investigations of counterparts to properties of common systems need – namely to stability of solutions, bifurcation diagrams, chaotic behavior. So the investigation of one modification of well known logistic equation by anticipatory property is considered. One of the most interesting properties in such systems is presumable multivaluednes of the solutions. The next issues are described: the examples of periodic and complex solutions, attractor's properties, and dependence on the parameters.
Keywords: Discrete maps, strong anticipation, multivalued solutions, chaos, attractors

1. Introduction

Since the introduction by D. Dubois the definitions and first examples of strong anticipation [1-4] many aspects of strong anticipation have been investigated, especially for discrete models for socio-economical and systems, traffic problems, biology-inspired models (game 'ALife') [1-8]. Many interesting properties and interpretations have been discussed. But now more vide investigations of anticipative systems behavior is necessary. Of course some general theoretical mathematical results exist which concerned the equations with anticipation. First of all we may remark some theoretical results on retarded-advanced differential and difference equations (with some names of researchers: C. Corduneanu, V. Lakshmicantam, L. Elsgoltz, S. Norkin, R. Bellman, K. Cooke). Also some examples of simple discrete equations with anticipation have been considered in papers from CASYS conferences (see the names D. Dubois, G. Weber, P. Beda, M. Burke, E. Otlacan, L. Leydesdorff, S. Holmberg). But such investigations have been implemented with some simple equations and with relatively simple behavior.
Further development of the theory and applications of strongly anticipative systems depends partially on searching of new kind of anticipation manifestation, on more detailed analysis of possible behavior, on further development of mathematical tools for complex multivalued solutions investigation and on development of recognitions and interpretations of solutions peculiarities.

So, any investigations of new equations with anticipation are interesting. Because of this we describe in proposed paper the investigation of complex behavior of one difference equation with anticipation. We propose the equation, computer investigations of solutions behavior and results of corresponding periodic solution investigations, including long-periods circles.

2. One dimensional discrete time equations with anticipation.

2.1 Analytical investigations of equation

Let us consider the discrete dynamics equation with anticipation which is the modification of well-known logistic equations. The proposed equation has the form:

$$x_{n+1} = \lambda \cdot x_n \cdot (1 - x_n) - \alpha \cdot x_{n+1}^2 \qquad (1)$$

where $\alpha \neq 0$ (for $\alpha = 0$ we have a classical logistic map) is an anticipatory factor.
So, our anticipatory equation is reduced to the two-branch evolution operator

$$x_{n+1} = \varphi(x_n) = \frac{-1 \pm \sqrt{1 + 4\lambda\alpha x_n (1 - x_n)}}{2\alpha} \qquad (2)$$

First of all we would like to get the fixed points of our anticipatory equation. Obviously, there are exist two fixed points satisfying the following equation $x = \lambda \cdot x \cdot (1 - x) - \alpha \cdot x^2$. They are $x_1^* = 0$ and $x_2^* = \frac{\lambda - 1}{\lambda + \alpha}$.
Now, when will they be stable? To answer this question we have to get the derivations of each branch: $\frac{dx_{n+1}}{dx_n} = \frac{\lambda(1 - 2x_n)}{2\alpha \cdot x_{n+1} + 1} = \pm \frac{\lambda(1 - 2x_n)}{\sqrt{1 + 4\lambda\alpha \cdot x_n (1 - x_n)}}$, that is

$$|\varphi'(x)| = \frac{|\lambda(1 - 2x)|}{\sqrt{1 + 4\lambda\alpha x(1 - x)}}$$

As we well know the fixed point will be stable when its multiplicator is in $(-1; 1)$. In this way, the fixed point x_1^* is stable if $\frac{|\lambda(1 - 2x)|}{\sqrt{1 + 4\lambda\alpha \cdot x(1 - x)}} < 1 \Leftrightarrow \lambda \in (-1; 1)$ and the one x_2^* if

$$|\varphi'(x_2^*)| < 1 \Leftrightarrow |2\lambda + \lambda\alpha - \lambda^2| < \sqrt{(\lambda - \alpha)(\lambda - \alpha + 4\alpha\lambda) + 4\alpha^2\lambda^2} \Leftrightarrow$$

$$\Leftrightarrow (1 - \lambda)(\lambda - 1/3)(\alpha + \lambda)\left(\alpha - \lambda\frac{\lambda - 3}{3\lambda - 1}\right) < 0$$

So, resolving this inequality, we can say that there are the different areas of the pair-wise stability of x_1^* and x_2^*.

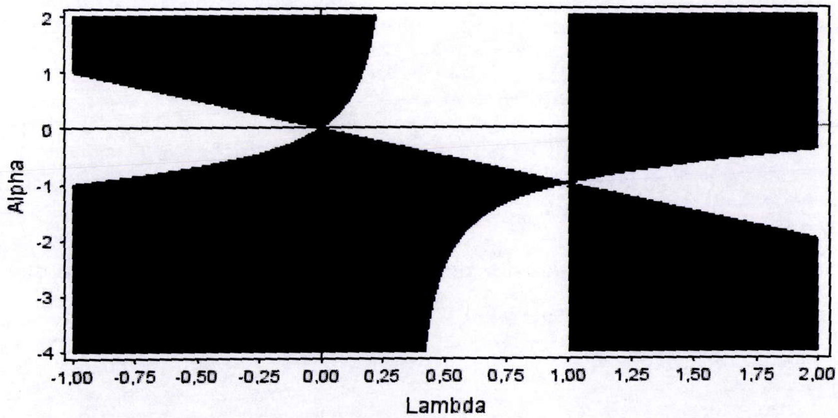


Figure 1. The stability area of x_2^*

Let's consider the conditions (on α and λ) under which x does not leave $(x_1^*; x_2^*)$ interval or leaves it using only one of the branches or both of them.

Denote $y_\lambda(x_n) = \lambda x_n(1 - x_n)$ and $y_\alpha(x_{n+1}) = \alpha x_{n+1}^2 + x_{n+1}$, so our equation (1) becomes $y_\alpha(x_{n+1}) = y_\lambda(x_n)$. Having obtained y_λ from x_n, we get x_{n+1} from $y_\alpha = \alpha x_{n+1}^2 + x_{n+1}$. Our iterations are following $x_0^1 \to (x_1^1; x_1^2) \to (x_2^1; x_2^2; x_2^3; x_2^4) \to \ldots$

Consider $\alpha < 0, \lambda > 0$ case (because of existing of the areas of hyperincursion in this part of (λ, α) plane, we will concentrate on the strict inequality). Considering other cases is similar. The roots of $y_\lambda = 0$ (at $x = 0, x = 1$) and $y_\alpha = 0$ (at $x = 0, x = -1/\alpha$) will be useful for our investigation.

The map $y_\alpha(x)$ has maximum value $-1/4\alpha$ at $x = -1/2\alpha$. And as we know $y_\lambda(x)$ has maximum value $\lambda/4$ at $x = 1/2$.

So, depending on

1) the location of the maximums of y_λ and y_α
2) the location of the roots $1/-\alpha$ and 1
3) the derivation of $y_\lambda'(0) = \lambda$ (under or over 1)

we divide $\alpha < 0, \lambda > 0$ case on 8 (2^3) sub-cases.

1) $\begin{cases} -1<\alpha<0 \\ \lambda\in(0;1) \\ \lambda\le 1/-\alpha \end{cases}$ In this case and $x_2^*>0$ the sequences $\{x_n\}$ will stay in $(x_1^*;x_2^*)$ it's explained by the stability of the both fixed points. If $x_2^*<0$, there exists the sequences witch start from $(x_1^*;x_2^*)$ and leave it (Figure 2a).

2) $\begin{cases} -1<\alpha<0 \\ \lambda>1 \\ \lambda\le 1/-\alpha \end{cases}$ In this case, according to the Figure 1 x_1^* fixed point is unstable but the other one may be stable or unstable. So, there is exists a sequence $\{x_n\}$ that leaves $(x_1^*;x_2^*)$ (e.g., started from a negative x_0).

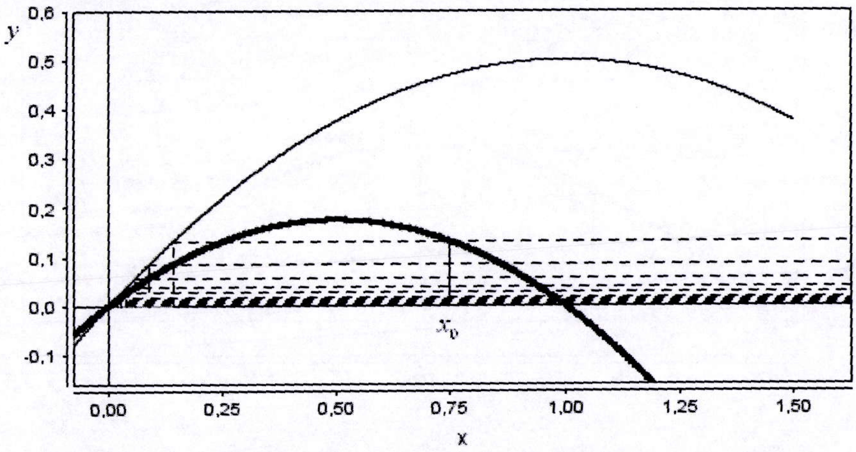
3) $\begin{cases} -1<\alpha<0 \\ \lambda>1 \\ \lambda>1/-\alpha \end{cases}$ In this case, according to the Figure 1 both of fixed points are unstable.

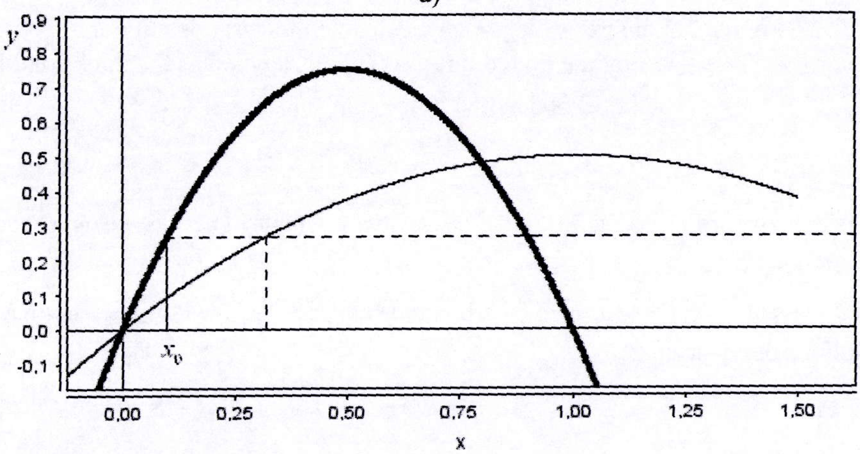
So, the sequences $\{x_n\}$ of each branch from the equation (2) leave $(x_1^*;x_2^*)$. More over, there are the sequences $\{x_n\}$ witch do not have the real solutions of (2) (see Figure 2b).

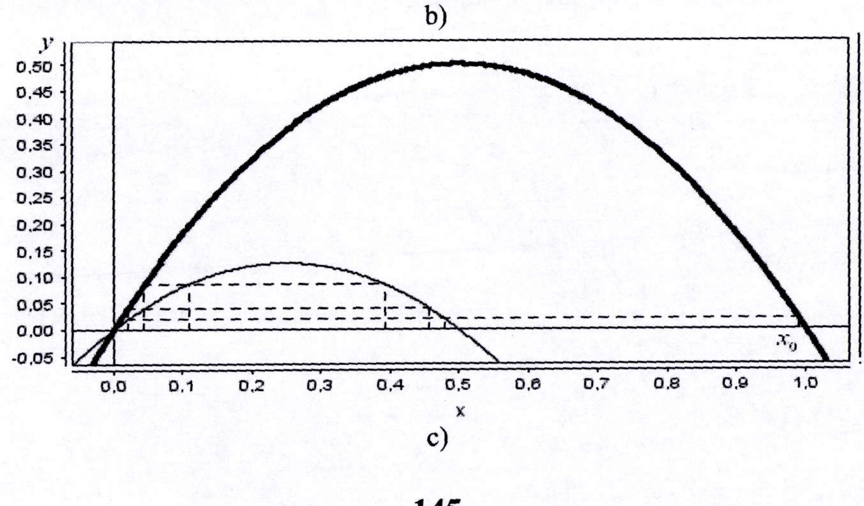
4) $\begin{cases} -1<\alpha<0 \\ \lambda\in(0;1) \\ \lambda>1/-\alpha \end{cases}$ There is not existing of solutions.

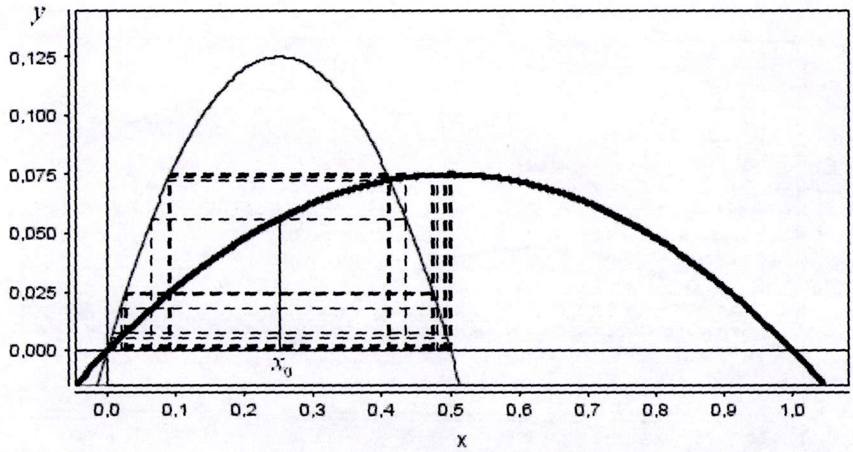
5) $\begin{cases} \alpha\le-1 \\ \lambda>1 \\ \lambda\ge 1/-\alpha \end{cases}$ In this case, according to the fig.1, x_1^* is not stable. So, there is exists a sequence $\{x_n\}$ that leaves $(x_1^*;x_2^*)$. More over, there is the sequence $\{x_n\}$ witch does not have the real solutions of equation (2) (Figure. 2c).

6) $\begin{cases} \alpha\le-1 \\ \lambda\in(0;1) \\ \lambda\le 1/-\alpha \end{cases}$ According to the fig.1, both of fixed points are stable, so the sequences $\{x_n\}$ don't leave $(x_1^*;x_2^*)$. Always there are roots (2) (Figure. 2d) (important that this area is interested cause of potential existing of the fractal structures).


y
0,6
0,5
0,4
0,3
0,2
0,1
0,0
-0,1
0,00
0,25
0,50
0,75
1,00
1,25
1,50
x
x_0
a)


y
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0,0
-0,1
0,00
0,25
0,50
0,75
1,00
1,25
1,50
x
x_0
b)


y
0,50
0,45
0,40
0,35
0,30
0,25
0,20
0,15
0,10
0,05
0,00
-0,05
0,0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
x
x_0
c)



d)

Figure 2. Lamerei's diagrams: we denote the iterations of our equation $y_\alpha(x_{n+1}) = y_\lambda(x_n)$ by the dotted lines, $y_\lambda(x_n)$ is denoted as the thick solid lines, $y_\alpha(x_{n+1})$ is denoted as the thin solid lines.

7) $\begin{cases} \alpha \le -1 \\ \lambda \in (0;1) \\ \lambda > 1/-\alpha \end{cases}$ According to the Figure 1, x_1^* is always stable. There is exists a sequence $\{x_n\}$ that leaves $(x_1^*; x_2^*)$. More over, there is the sequence $\{x_n\}$ witch does not have the real solutions of equation (2).

8) $\begin{cases} \alpha \le -1 \\ \lambda > 1 \\ \lambda < 1/-\alpha \end{cases}$ There are not the real solutions of (2).

Now, (2) will have both roots when $1 + 4\lambda\alpha x_n(1 - x_n) > 0$

$$4\lambda\alpha x^2 - 4\lambda\alpha x - 1 < 0$$

These roots are $x_{1,2}^d = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{1}{\lambda\alpha}}$

Let's consider three cases in which we have both roots of anticipatory equation (1)

$$\begin{cases} \alpha\lambda > 0 \\ x_n \in (x_1^d; x_2^d) \end{cases}$$

$$\begin{cases} \alpha\lambda < -1 \\ x_n \in \left(-\infty; x_1^d\right) \cup \left(x_2^d; +\infty\right) \end{cases}$$

$$\begin{cases} -1 < \alpha\lambda < 0 \\ x_n \in R \end{cases}$$

It is simple to show that:

a) in case of $\alpha\lambda > 0$ we have $\left(x_1^d; x_2^d\right) \supset (0;1)$;

b) in case of $\alpha\lambda < -1$ we have $\left(x_1^d; x_2^d\right) \subset (0;1)$;

Now, we are interested in that when each anticipatory solution branch (2) will be contractive. Each branch of (2) will be contractive on x if

$$|\varphi'(x)| < 1 \Leftrightarrow \begin{cases} |\lambda(1-2x)| < \sqrt{1+4\lambda\alpha x(1-x)} \\ 1+4\lambda\alpha x(1-x) > 0 \end{cases}.$$

Consider first $4\lambda x^2(\lambda+\alpha) - 4\lambda x(\lambda+\alpha) + \lambda^2 - 1 < 0$

$$x = \frac{4\lambda(\lambda+\alpha) \pm \sqrt{16\lambda^2(\lambda+\alpha)^2 - 16\lambda(\lambda+\alpha)(\lambda^2-1)}}{8\lambda(\lambda+\alpha)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{\lambda^2-1}{\lambda(\lambda+\alpha)}} = x_{1,2}$$

It will have the roots when $\dfrac{\lambda\alpha+1}{\lambda(\lambda+\alpha)} > 0$.

The above inequality implies 8 cases (four couple of the similar cases)

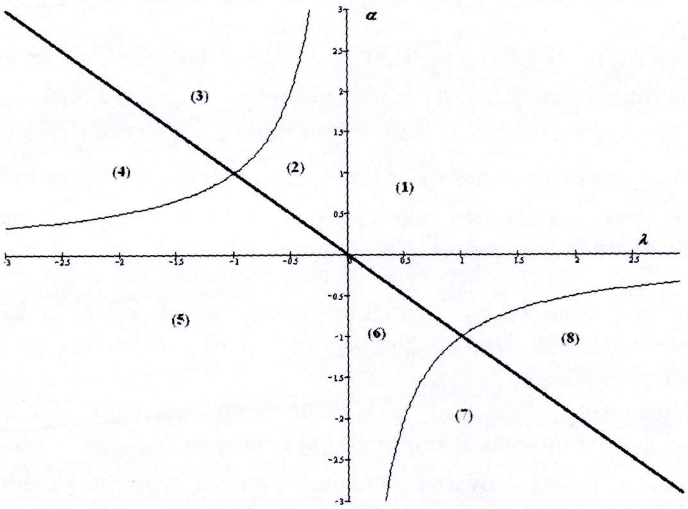


Figure 3. The division of parameter space

(1) = (5)

if x don't leave $\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{\lambda^2-1}{\lambda(\lambda+\alpha)}};\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{\lambda^2-1}{\lambda(\lambda+\alpha)}}\right)$, then this interval will be

contractive one for all x from $1+4\lambda\alpha x(1-x)>0$.

(2) = (6)

For all x for $1+4\lambda\alpha x(1-x)>0$ the equation (2) will be contractive.

(3) = (7)

This interval will be contractive one for all x from $1+4\lambda\alpha x(1-x)>0$.

(4) = (8)

There isn't a contraction.

2.2 Results of computer investigations of solutions

Finally, let's sow a behavior of the dynamical system given by anticipatory equation. The Figure 4 presents the atlases of the charts of the dynamical regimes. This figure obtained by computing modeling of the dynamical behavior of the anticipatory equation (1).

For each parametric point $(\lambda;\alpha)$ we start few trajectories (as rule, not more than 10) from different initial points. At some instant n, we have the set of points $\left(x_n^{(1)},x_n^{(2)},...,x_n^{(m)}\right)$ $m\leq 2^n$ achievable from initial point x_0 after n iteration of (2). After next instant $n+1$, some branches of $\left(x_n^{(1)},x_n^{(2)},...,x_n^{(m)}\right)$ leave the initial defined interval, and some of them may have not the following points of the trajectory branch.

At our figure, distinguish regions corresponds to the different periods of found trajectory orbits.

So, first region corresponds to one-periodic trajectory orbit (the area is denoted by (1)), second implies two-periodic orbit (denoted by (2)), third – 4 (denoted by (4)), forth – 8 (denoted by (8)), fifth – 3, sixth – 5 and so forth …

As we see in this case, periodic-doubling leads to chaotic regime. Periodic-doubling starts after λ leaves the interval of stability $(1;3)$ (in the simple case $\alpha\equiv 0$). So the most interesting in our computer investigations are may be two things. First is that in proposed modification of logistic equation we have the manifestation of period – doubling scenarios of transitions for regimes which may be recognized multi-valued chaos. Second is founded stabilizing role of anticipation which is seen from Figure 4.

Of course many question are interesting for investigation concerning the definitions of multi-valued regimes and their development (see for example [9, 10]).

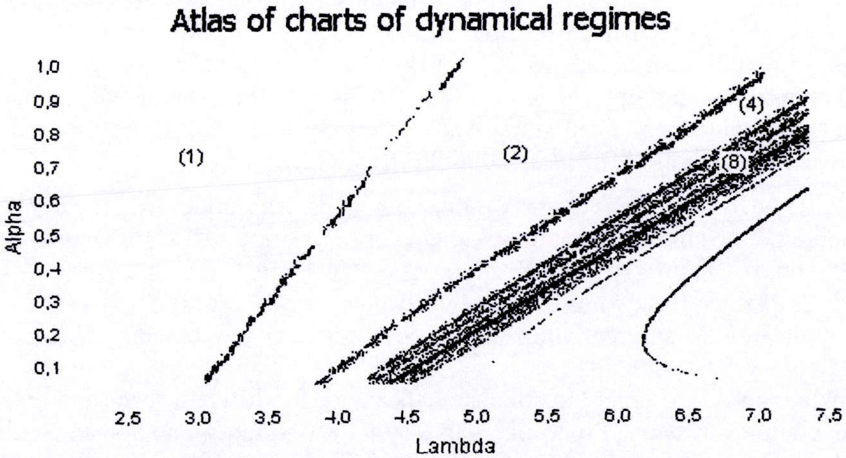


Figure 4. Atlas of charts of dynamic regimes

3. Conclusions

Thus in given paper we propose one simple equation with anticipation. Period-doubling mechanism of transition to multi-valued analog of chaotic behavior had been found. Such investigation are one of the steps in understanding of multi-valued counterparts of single-valued objects such as periodic and chaotic solutions; scenarios of transition to chaos etc. Also interesting is the stabilizing role of anticipation (that is suppressing the development of complex behavior in common logistic equation).

References

[1] Dubois D., Introduction to computing Anticipatory Systems. International Journal of Computing Anticipatory Systems, (Liege), 1998, Vol. 2, pp.3-14.

[2] Dubois D. Review of incursive, hyperincursive and anticipatory systems – foundation of anticipation in electromagnetizm Computing Anticipatory Systems, AIP Conf. Proceed. Series Vol. 517, ed. D. Dubois, AIP N.- Y. 2000. pp. 3-30.

[3] Antippa A., Dubois D. The dual incursive system of the discrete harmonic oscillator. Computing Anticipatory Systems, AIP Conf. Proceed. Series Vol. 839, ed. D. Dubois, AIP N.- Y. 2006. Pp. 11-64

[4] Leydesdorf L., Dubois D., Anticipation in Social Systems: the Incursion and Communication of Meaning, Int. Journal of Computing Anticipatory Systems, 15, 2004. pp. 203 – 216.

[5] Makarenko A. Anticipating in modelling of large-scale social systems – neuronets with internal structure and multivaluedness. Int. Journal of Computing Anticipatory Systems, 13, 2002. pp. 77-92.
[6] Makarenko A. Anticipatory agents, scenarios approach in decision-making and some quantum – mechanical analogies. Int. Journal of Computing Anticipatory Systems, 15, 2004. pp. 217-225.
[7] Makarenko A., Stashenko A. Some two-step discrete-time anticipatory models with 'boiling' multivaluedness. Computing Anticipatory Systems, AIP Conf. Proceed. Series Vol. 839, ed. D. Dubois, AIP N.- Y. 2006. Pp. 265-272.

[8] Makarenko A., Krushinskiy D. Cellular automata with anticipation: examoles and presumable applications. Computing Anticipatory Systems, AIP Conf. Proceed. Series Vol.1303, ed. D. Dubois, AIP N.- Y. 2010, Pp. 246 – 254.
[9] Andre J., Fisher J., Juyyner L. On a multivalued version of the Sharkovskii theorem and ist application to differential inclusions. Set-valued Analysis. 2002. vol. 10, n. 1. pp. 1-14.
[10] Makarenko A. Different type of chaotic behavior for different space and time scales in complex ayatems. Proceedings INDS'09 Second Int. Workshop on Nonlinear Dynamics and Synchronization, Klagenfurt, AUSTRIA, 2009. PP. 60-64.