From Standard Representations to a Global Representation with a N-Dimensional Metric in Quantum Theory

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Abstract
In this paper we discuss the several standard representations of Quantum Theory. The requirements of time order and operator self-adjointness are too much restrictive. To describe quantum correlations of N particles with a possible not causal order we propose a new formalism which include all conjugate quantities in the same configuration space where we define a "global state function" Ψ of $6N+1$ scalar variables.
The function Ψ is $3N$-dimensional (non relativist Ψ) or $4N$-dimensional (relativist Ψ) when spin variables are separable. Usual kinematic functions are computed as implicit functions, and new quantum operators are introduced.
Keywords: quantum correlation, quantum representation, time operator, causal order, self-adjoint extension.

1. Introduction

The idea that a cause should precede its effects in time is an ancient philosophical concept [3] of naïve intuition. In Newton mechanics the observed time order was assumed to always match the causal order. According to the Zeeman theorem [4] the Lorentz transformation is required to preserve the time order. The so-called "causality principle" has been much discussed by many authors [5,6,7,8] although it never appears explicitly in any equation of Physics.

The pioneers of the early Quantum Mechanics assumed that the time order should match the causal order of the evolution of any quantum system. Therefore they introduced [9] the time evolution operator[i] $U(t,t_0)$ computed as:

$$\hat{U}_x(t,t_0)=e^{-\frac{i}{\hbar}(t-t_0)\,\hat{H}(x_1,x_2,x_3)} \tag{1}$$

and satisfying the two following conditions:

$$\hat{U}(t_0,t_0)=1 \quad \text{and} \quad \hat{U}\,\hat{U}^{+}=\hat{U}^{+}\,\hat{U}=1 \tag{2}$$

where $\hat{U}^{+}$ is the transposed conjugate of the operator $\hat{U}$. They also introduced the regular time dependency functions[ii] $f_n(t)$ defined as:

$$f_n(t)=e^{-\frac{i}{\hbar}E_n t} \tag{3}$$

[i] often shortly named "evolution operator"; reference [9] , p. 222-225.
[ii] also called "universal time dependency functions".

International Journal of Computing Anticipatory Systems, Volume 27, 2014
Edited by D. M. Dubois, CHAOS, Liège, Belgium, ISSN 1373-5411 ISBN 2-930396-16-4

and the chronological ordering operator[i] [10] defined by the equation (37).

In a previous communication [1] we have shown how the space-time structure is basically built from space-time derivative operators which are related to momentum-energy operators, to SO(3,R), SO(3,1,R) rotation groups and to 2-spinors.

In an other paper [2] we have shown that the time order might not match always the causal order. A true causal order would be the order of quantum states in a sequence of quantum transitions: it would define the proper time order of the quantum system and it has not to match the time order which is observed in a privileged referential frame.

On the contrary, Carlo Rovelli [11] asked to "forget time" and proposed to build a timeless quantum theory.

For Ognyan Oreshkov et al. [12] the pre-existing space-time is a causal structure where events take place and thus the standard Quantum Theory is only valid locally, and some more general correlations are possible with no causal order.

Thus a new quantum formalism should be proposed: it is the purpose of this paper.

2. The Several Types of Representations in Quantum Mechanics

To describe the evolution of a quantum system in time, the early Quantum Mechanics introduced configuration spaces which do not need to be the space-time manifold (see sections 2.2., 3.). There are two major representations [9,13,14] which we recall in next sections:

1. the representation in coordinates,
2. the representation in momentum,

and there are three versions of these two representations [9,13,14]:

- the Schrödinger representation.
- the Heisenberg representation,
- the Dirac representation (theory of perturbation).

In the Schrödinger representation, the evolution of a quantum system in time since an initial time t_0, is described by the evolution of the wave function in time, as:

$$\Psi_t(x_1, x_2, x_3, t) = \hat{U}_x(t, t_0)\, \Psi(x_1, x_2, x_3, t_0) \tag{4}$$

In the Heisenberg representation, the evolution of a quantum system in time since an initial time t_0, is described by the evolution of the operators in time and the wave function does not depends on time:

$$\Psi(x_1, x_2, x_3, t) \equiv \Psi(x_1, x_2, x_3, t_0) \tag{5}$$

In the Dirac representation the evolution in time of the Hamiltonian $\hat{H}$ is the sum of the initial Hamiltonian $\hat{H}_0(t_0)$ and a perturbation term $\hat{H}_1(x_1, x_2, x_3, t)$.

Moreover, for spin particles the spin σ has to be included in the configuration space. To describe any quantum system of several inseparable particles, the configuration space was extended to contain the coordinates of all particles.

2.1. The Representation in Coordinates

In the non relativist Quantum Mechanics, the representation in coordinates for a

[i] reference [10], p. 333.

single particle is built on the configuration space Σ_x:

$$\Sigma_x=\{x_1, x_2, x_3, t\} \tag{6}$$

where the index 1, 2, 3 refer to the three axis of a space referential, and the wave or state function is:

$$\Psi(x_1, x_2, x_3, t) \tag{7}$$

The position, momentum and energy operators are usually defined as:

$$\hat{x}_1(t)=x_1\times \tag{8}$$

$$\hat{r}(x_1, x_2, x_3, t)=\vec{r}\cdot \tag{9}$$

$$\hat{p}_1(x_1, x_2, x_3, t)=-i\hbar\frac{\partial}{\partial x_1} \tag{10}$$

$$\hat{p}(x_1, x_2, x_3, t)=-i\hbar\vec{\nabla}\cdot \tag{11}$$

$$\hat{E}(x_1, x_2, x_3, t)=i\hbar\frac{\partial}{\partial t} \tag{12}$$

with the usual notations for operators, vectors, gradients, scalar product of vectors.
For a spin particle, the configuration space with the spin σ is:

$$\Sigma_r=\{x_1, x_2, x_3, \sigma, t\} \tag{13}$$

The Minkowski space-time Σ'_x defined by (36), is a relativist spinless representation in coordinates with the time coordinate x_4 (or x_0) such as:

$$x_4=ict \tag{14}$$

so the relativist wave or state function is:

$$\Psi'(x_1, x_2, x_3, x_4) \tag{15}$$

We usually build the relativist momentum-energy 4-dimensional operator $\hat{P}$ with the following definition:

$$\hat{P}(x_1, x_2, x_3, x_4)=-i\hbar\,{}^4\vec{\nabla}_p\cdot \tag{16}$$

where ${}^4\vec{\nabla}_p$ is the 4-dimensional gradient in the relativist configuration space, in relation with the relativist energy operator:

$$\hat{P}_4(x_1, x_2, x_3, x_4)=\frac{-i}{c}\hat{E}(x_1, x_2, x_3, x_4)=i\hbar\frac{\partial}{\partial x_4} \tag{17}$$

2.2. The Representation in Momentum

In the non relativist Quantum Mechanics the representation in momentum for a single particle is built on the configuration space Σ_p:

$$\Sigma_p=\{p_1, p_2, p_3, t\} \tag{18}$$

and the wave or state function is:

$$\Phi(p_1, p_2, p_3, t) \tag{19}$$

The momentum, position and energy operators are usually defined as:

$$\hat{p}_1(t) = p_1 \times \tag{20}$$

$$\hat{p}(t) = \vec{p} \cdot \tag{21}$$

$$\hat{x}_1(p_1, p_2, p_3, t) = i\hbar \frac{\partial}{\partial p_1} \tag{22}$$

$$\hat{r}(p_1, p_2, p_3, t) = i\hbar \vec{\nabla}_p \cdot \tag{23}$$

$$\hat{E}(p_1, p_2, p_3, t) = i\hbar \frac{\partial}{\partial t} \tag{24}$$

For a spin particle the representation in momentum with the spin σ is:

$$\Sigma_p = \{p_1, p_2, p_3, \sigma, t\} \tag{25}$$

A relativist representation in energy-momentum can also be defined: instead of the time variable t we introduce the energy variable as:

$$P_4 = \frac{-i}{c} E \tag{26}$$

It leads to the configuration space Σ'_F of the representation in momentum-energy:

$$\Sigma'_P = \{p_1, p_2, p_3, p_4\} \tag{27}$$

which has a Minkowski metric, so the relativist wave or state function is:

$$\Phi'(p_1, p_2, p_3, p_4) \tag{28}$$

A relativist event, i.e. a 4-dimensional position in the Minkowski space-time is then defined with the following position operator:

$$\hat{\hat{X}}(p_1, p_2, p_3, p_4) = i\hbar \, {}^4\vec{\nabla}_p \cdot \tag{29}$$

where ${}^4\vec{\nabla}_p$ is the 4-dimensional gradient in the relativist configuration space of momentum-energy. This relativist representation in momentum-energy Σ'_P needs to introduce a time operator $\hat{t}$ (see section 5.1.).

3. The Extended Representations in Quantum Mechanics

When a quantum system contains several interacting particles the configuration space usually contains the coordinates of all particles: when particles are inseparable their coordinates are also inseparable in the state equation of the quantum system.

In the case of two inseparable spin particles $A_{(1)}$ and $A_{(2)}$, the extended configuration space, in the representation in coordinates, is:

$$\Sigma_r = \{\vec{r}_{(1)}, \sigma_{(1)}, \vec{r}_{(2)}, \sigma_{(2)}, t\} \tag{30}$$

where the labels 1, 2 between parenthesis identify the particles. The state function is:

$$\Psi = (\vec{r}_{(1)}, \sigma_{(1)}, \vec{r}_{(2)}, \sigma_{(2)}, t) \tag{31}$$

In the representation in momentum the extended configuration space is:

$$\Sigma_p = \{\vec{p}_{(1)}, \sigma_{(1)}, \vec{p}_{(2)}, \sigma_{(2)}, t\} \tag{32}$$

and the wave or state function is:

$$\Phi=(\vec{p}_{(1)},\sigma_{(1)},\vec{p}_{(2)},\sigma_{(2)},t) \tag{33}$$

As the vectors $\vec{r}$ and $\vec{p}$ are 3-dimensional and σ is a scalar, the configuration space of a system of N spin particles has the dimension:

$$dim(\Sigma)=4N+1 \tag{34}$$

The separability condition defines when the system S can be decomposed into independent subsystems. In the case of two particles we write:

$$\exists\psi_1,\psi_2:\ \Psi(\vec{r}_{(1)},\sigma_{(1)},\vec{r}_{(2)},\sigma_{(2)},t)=\psi_1(\vec{r}_{(1)},\sigma_{(1)},t)+\psi_2(\vec{r}_{(2)},\sigma_{(2)},t) \tag{35}$$

After the Bell theorem [15] there is no local (hidden) variables which help to reduce the dimension of the configuration space. So the separability condition has to be demonstrated for every quantum system which is studied.

When the Relativist Quantum Theory uses the Minkowski space-time manifold as the unique configuration space:

$$\Sigma'_x=\{x_1,x_2,x_3,x_4\} \tag{36}$$

it does not allow to properly describe quantum correlations between inseparable particles, and it naturally leads to the E.P.R. paradox when the separability condition (35) is not met.

4. Discussion About Time

Time is a physical observable which is measured from the periodic movement of clocks, but it was not considered as a quantum observable by the early Quantum Mechanics because the time operator was not assumed to be self-adjoint[i] as we explain below (see section 5.1.). So the variable t was introduced in every configuration space to represent the time order and also it was assumed to match the causal order.

For n successive perturbations $\hat{V}(t)$ the invariant theory of perturbations writes symbolically:

$$\phi(t_n)=\hat{T}\prod_{k=1}^{n} e^{\frac{-i}{\hbar}\hat{V}(t_{k-1})\ (t_k-t_{k-1})}\phi(t_0) \tag{37}$$

or

$$\phi(t_n)=\hat{T}\,e^{\frac{-i}{\hbar}\sum_{k=1}^{n}\hat{V}(t_{k-1})\ (t_k-t_{k-1})}\phi(t_0) \tag{38}$$

where the so-called "chronological operator" $\hat{T}$ has been artificially introduced to recall that the addition is not commutative within the exponent in the equation (38).
According to the official interpretation[ii] [10] $\hat{T}$ is supposed to reset the product of operators in the "correct order" (SIC!). Moreover the equation (37) is dubious[iii] because mathematically [16,17] we cannot obtain $e^{i(A+B)}$ from e^{iA},e^{iB} when the operators A, B do not commute.

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[i] see the definition and properties of self-adjoint operators in the appendix.
[ii] reference [10] page 335, text and notes 1, 2.
[iii] see the Lie-Trotter product formula in references [16,17].

Now let's consider two events *a*, *b* seen by the observer O, the equation (37) simplifies into:

$$\phi(t_2)=e^{\frac{-i}{\hbar}\hat{B}(t_1)(t_2-t_1)}\,e^{\frac{-i}{\hbar}\hat{A}(t_0)(t_1-t_0)}\,\phi(t_0) \tag{39}$$

If they are separated by a timelike interval:

$$c(t_2-t_1)^2-(\vec{r}_b-\vec{r}_a)^2>0 \tag{40}$$

the time order is preserved by any Lorentz transformation, but the product of exponential operators in equation (39) is not commutative.

If the events *a*, *b* are separated by a spacelike interval:

$$c(t_2-t_1)^2-(\vec{r}_b-\vec{r}_a)^2<0 \tag{41}$$

the time order is not preserved by any Lorentz transformation so a second observer O' may see the corresponding events *a'*, *b'* at times $t'_1 \geq t'_2$ in the inverse order of the observer O. In this case the chronological operator may define a privileged referential frame.

Does the chronological operator $\hat{T}$ introduced in equation (37) by the so-called "invariant theory of perturbations" define the proper time order of a quantum system or does it privilege the referential frame of the laboratory ?

Moreover in the CPT transformation group the ***T*** transformation shown below:

$$t \overset{(T)}{\rightarrow} -t \tag{42}$$

reverse the time order and "causal loops" can be "formed by using bradyon and anti-bradyon signals" [18] as a positron may be an electron going backward in time [19, 20].

5. Boundary Conditions for Introducing Quantum Operators

In usual treaties on Quantum Mechanics all quantum operators are formally introduced with the equations (8) to (12) or (20) to (24), all demonstrations are done in a mere formal way and they let the reader think that all operators are self-adjoint.

To associate the wave function of a particle with expected values of kinematic measurements, the corresponding operator must have real eigenvalues, and thus the operator must be self-adjoint or have a self-adjoint extension. In the contrary, the wave function cannot be normalized, and then both the De Broglie double solution and the Copenhagen interpretation fail.

5.1. Time as a Quantum Observable

Since W. Pauli's works [21] time was not assumed to be a quantum observable as the time operator $\hat{t}$ was not considered to be a self-adjoint because of the semi-boundness of the continuous kinetic energy spectrum:

$$E_c \geq 0 \tag{43}$$

with however some exceptions as e.g. electrically charged particles.

Nevertheless there is a strong requirement of a quantum time operator for measuring tunneling time [22,23,24], time delay induced by scattering [25,26], collision duration [27,28], nuclear lifetimes, etc ...

The introduction of a time operator $\hat{t}$ has already been proposed by several authors [29,30,31].
It is defined in the representation in time as:

$$\hat{t} = t \tag{44}$$

and in the representation in energy as:

$$\hat{t} = -i\hbar \frac{\partial}{\partial E} \tag{45}$$

Although we can define formally the time operator, we must demonstrate that the time operator is self-adjoint or has a self-adjoint extension for the known boundary conditions of a given quantum system.

5.2. Some Position and Momentum Operators may be Semi-Bounded

The problem of boundary conditions is not specific to time, as it has been shown by J. von Neumann [32] with the example of a free particle moving on a spatial semi-axis $x_1 > 0$ bounded by a rigid wall at $x_1 = 0$: its momentum p_1 cannot be represented by the operator (10) which is not self-adjoint. So position and momentum are not always quantum observables (with the standard definition of self-adjoint operators).

5.3. Choose a Quantum Representation According to Operators Properties

Which quantum representation can we choose to describe a given quantum system ? The answer was definitely given in the mathematical studies of J. von Neumann [32,33] and Marshal Stone [34]: in the chosen representation all operators must be self-adjoint or have a self-adjoint extension. There is no unique standard representation of any quantum system

If the energy or momentum operators (11), (12) are self-adjoint we rather choose the representation in coordinates. If they are not self-adjoint, the representation in momentum (or momentum-energy) is used.

$$\forall\, j=1,2,3 \quad \hat{r}_j \text{ bounded} \;\Rightarrow\; \hat{p}_j(r_j) = \hat{p}_j^{+}(r_j) \qquad \Sigma_r = \{r_1, r_2, r_3, t\} \tag{46}$$

$$\forall\, j=1,2,3 \quad \hat{p}_j \text{ bounded} \;\Rightarrow\; \hat{r}_j(r_j) = \hat{r}_j^{+}(r_j) \qquad \Sigma_p = \{p_1, p_2, p_3, t\} \tag{47}$$

If any operator is just symmetric (not self-adjoint) a self-adjoint extension has to be found – if it ever exists.

When both r_1 and p_2 are semi-bounded the operators $\hat{p}_1$ and $\hat{r}_2$ are not self-adjoint. If they have no self-adjoint extension, no standard representation can be used.

So standard representations do not allow to build a complete quantum theory.

J. von Neumann [32] has shown that considering only self-adjoint operators is too much restrictive to build a complete quantum theory. Operators which are only symmetric[i] must also be considered.

Furthermore Ognyan Oreshkov et al. [12] claimed that the usual Quantum Theory is only valid locally in the pre-defined space-time structure. They predicted quantum

[i] see the definition of symmetric operators in the appendix.

correlations which have no causal order (ruled by some "causal inequalities"). The authors predicted non causal correlations "which are not included in the standard formalism" and require a more general formalism [12].

6. A Global Representation of any Quantum System

6.1. Definition of a Global Representation

To lay the foundations of a more general quantum theory, we propose a global representation in a multi-dimensional configuration space by merging the two conjugate representations.

For a quantum system of two spin particles in the non relativist theory we merge the representations (30) and (32) as:

$$\Sigma_{(2)}=\Sigma_r \cup \Sigma_p \tag{48}$$

For a quantum system of N spin particles the global configuration space is:

$$\Sigma_{(N)}=\{t,\vec{r}_{(1)},...,\vec{r}_{(N)},\vec{p}_{(1)},...,\vec{p}_{(N)},\sigma_{(1)},...,\sigma_{(N)}\} \tag{49}$$

where the label between parenthesis in the right hand side of the equation is the number of every particle. For better convenience we write the variable t in first position.

To describe all relations in the system, we introduce a real or complex vector function Ψ and we call it the "global state function". The state equation is then:

$$\Psi(t,\vec{r}_{(1)},...,\vec{r}_{(N)},\vec{p}_{(1)},...,\vec{p}_{(N)},\sigma_{(1)},...,\sigma_{(N)})=0 \tag{50}$$

When the spin variables are separable from other variables as:

$$\Psi=\Psi(t,\vec{r}_{(1)},...,\vec{r}_{(n)},\vec{p}_{(1)},...,\vec{p}_{(N)})\Psi_{\sigma}(\sigma_{(1)},...,\sigma_{(N)}) \tag{51}$$

we have a state equation without spins:

$$\Psi(t,\vec{r}_{(1)},...,\vec{r}_{(N)},\vec{p}_{(1)},...,\vec{p}_{(N)})=0 \tag{52}$$

In the non relativist case $\vec{r}_{(k)},\vec{p}_{(k)}$ are 3-vectors and $\sigma_{(k)}$ is a scalar, so $\Sigma_{(N)}$ has $7N+1$ scalar dimensions (with spins) or $6N+1$ (without spins).

From the state equation (50) or (52) kinematic equations can be computed as implicit functions. As an example we may compute the momentum of the k^{th} particle, if the momenta $p_{(k)}$ are independent, as:

$$\vec{p}_{(k)}=f_k(t,\vec{r}_{(1)},...,\vec{r}_{(N)}) \tag{53}$$

and, if also the variables $r_{(k)}$ are separable, as:

$$\vec{p}_{(k)}=f_k(t,\vec{r}_{(k)}) \tag{54}$$

Let's apply the implicit function theorem in the real domain (sections 6.2., 6.3.) and in the complex domain (section 6.5.).

6.2. Case of one Free Spinless Particle

In the case of a free spinless particle moving straight forward on the x axis, the equation (52) is known to reduce to the simplest function:

$$\Psi(t,x,p_x)=0 \tag{55}$$

Let us consider Ψ as a real scalar function of real variables, from the well known

implicit function theorem [35] we obtain the time derivatives of the real functions $x(t)$ and $p_x(t)$ expressed as:

$$\frac{\partial x}{\partial t}=-\frac{\dfrac{\partial \Psi}{\partial t}}{\dfrac{\partial \Psi}{\partial x}};\quad \frac{\partial p_x}{\partial t}=-\frac{\dfrac{\partial \Psi}{\partial t}}{\dfrac{\partial \Psi}{\partial p_x}} \tag{56}$$

6.3. Case of a Real State Function of Several Particles

Let's consider a non relativist system of N particles and omit spin variables for simplicity. In the equation (52) we replace every 3-vector by its scalar components, so we have $3N$ scalar variables of momentum and $3N+1$ scalar variables of space-time coordinates.

For better convenience we further use the following notation:

$$q_0=t;\ q_j=r_j\ (j=1,\cdots,3N) \tag{57}$$

so the global state function can be formally written as:

$$\Psi(q_0,...,q_{3N},p_1,...,p_{3N})=0 \tag{58}$$

Considering Ψ as a real vector function of real variables, let's apply the multi-dimensional implicit function theorem [36], we can write the "total derivative" as the Jacobian bi-matrix below:

$$D\Psi=\left\langle \begin{array}{ccc|ccc} \dfrac{\partial \Psi_1}{\partial q_0} & \cdots & \dfrac{\partial \Psi_1}{\partial q_{3N}} & \dfrac{\partial \Psi_1}{\partial p_1} & \cdots & \dfrac{\partial \Psi_1}{\partial p_{3N}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \dot{\Psi}_{3N}}{\partial q_0} & \cdots & \dfrac{\partial \dot{\Psi}_{3N}}{\partial q_{3N}} & \dfrac{\partial \dot{\Psi}_{3N}}{\partial p_1} & \cdots & \dfrac{\partial \dot{\Psi}_{3N}}{\partial p_{3N}} \end{array} \right\rangle \tag{59}$$

If there is a point $(\mathbf{a},\mathbf{b})$ in the configuration space $\Sigma_{(N)}$ of N particles such as:

$$\exists(\boldsymbol{a},\boldsymbol{b})=(a_0,...,a_{3n},b_1,...,b_{3N})\in\Sigma_{(N)}:\ \Psi(\boldsymbol{a},\boldsymbol{b})=0 \tag{60}$$

where the right hand side matrix of the Jacobian (59) is invertible:

$$J_{[p]}=\begin{bmatrix} \dfrac{\partial \Psi_1(\boldsymbol{a},\boldsymbol{b})}{\partial p_1} & \cdots & \dfrac{\partial \Psi_1(\boldsymbol{a},\boldsymbol{b})}{\partial p_{3N}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \Psi_{3N}(\boldsymbol{a},\boldsymbol{b})}{\partial p_1} & \cdots & \dfrac{\partial \Psi_{3N}(\boldsymbol{a},\boldsymbol{b})}{\partial p_{3N}} \end{bmatrix}:\ \exists J_{[p]}^{-1} \tag{61}$$

then there is an open set S containing the point $(\mathbf{a},\mathbf{b})$ where exists a vector function $\vec{g}(q_0,...,q_m)$ defined by its vector derivative as:

$$\frac{\partial \vec{g}}{\partial q_k}=-\frac{\dfrac{\partial \Psi}{\partial q_k}}{\dfrac{\partial \Psi}{\partial \vec{p}}} \tag{62}$$

and the function $\vec{g}$ represents the following set of momenta:

$$\vec{g}(q_0,...,q_{3N})=(p_1,...,p_{3N}) \tag{63}$$

As $J_{[p]}$ is a square matrix, the global state function Ψ must have $3N$ real dimensions.

6.4. Generalization to a Complex State Function of Several Particles

To represent a quantum state, the function Ψ is required to be a complex vector function. The generalization is done with the "analytic implicit function theorem" [37] which has been demonstrated [38,39]. If the function Ψ is holomorphic, all rules and computations presented in the previous section still hold. $J_{[p]}$ is still a square matrix and so the global state function must have $3N$ complex dimensions.

6.5. Generalization to a Relativist Complex State Function

In the relativist quantum theory the variable q_0 is the time coordinate $x_0 = ct$ and the N particles are described with N momentum 4-vectors. The particles positions in space-time still have $3N+1$ scalar dimensions (real or complex), but the N momenta have $4N$ scalar dimensions (real or complex). So the relativist state function becomes:

$$\Psi(q_0,...,q_{3N}, p_1,..., p_{4N})=0 \tag{64}$$

The relativist global state function has a total derivative which is also represented by an invertible Jacobian bi-matrix similarly to (59). It is:

$$D\Psi=\left\langle \begin{array}{ccc|ccc} \frac{\partial \Psi_1}{\partial q_0} & \cdots & \frac{\partial \Psi_1}{\partial q_{3N}} & \frac{\partial \Psi_1}{\partial p_1} & \cdots & \frac{\partial \Psi_1}{\partial p_{4N}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Psi_{4N}}{\partial q_0} & \cdots & \frac{\partial \Psi_{4N}}{\partial q_{3N}} & \frac{\partial \Psi_{4N}}{\partial p_1} & \cdots & \frac{\partial \Psi_{4N}}{\partial p_{4N}} \end{array} \right\rangle \tag{65}$$

where the right hand side matrix $J_{[p]}$ is still a square matrix, and so the global relativist state function must have $4N$ dimensions.

6.6. New Quantum Operators

We introduce operators which are acting on the global state function Ψ defined by the equation (50). These operators are the following:

$$\frac{\partial}{\partial q_k}\,;\ \frac{\partial}{\partial p_k}\,;\ \frac{\partial}{\partial \sigma_k} \tag{66}$$

They look like usual derivative operators, but they are quite different as they are related to the global configuration space $\Sigma_{(N)}$ defined by (49). They are used to build the invertible Jacobian bi-matrix (59) or (65).

We also introduce an integral operator defined by:

$$\vec{g}(q_0,...,q_m)=\int \frac{\partial \vec{g}}{\partial q_k} dq_k \tag{67}$$

which gives the implicit function $\vec{g}$ from the equation (62).

7. Appendix: Recall of Basic Properties of Linear Operators

A linear operator $\hat{A}$ defined in a domain $D(\hat{A})$ on a Banach space H, i.e.

$$D(\hat{A}) \subseteq H \tag{68}$$

is said to be "symmetric" if:

$$\forall x, y \in D(\hat{A}): \langle \hat{A}x \mid y \rangle = \langle x \mid \hat{A}y \rangle \tag{69}$$

Here the space H is not required to be a Hilbert space.

The adjoint $\hat{A}^+$ of a linear operator $\hat{A}$ is defined as:

$$\forall x, y \in D(\hat{A}): \langle \hat{A}x \mid y \rangle = \langle x \mid \hat{A}^+ y \rangle \tag{70}$$

A linear operator is said to be "self-adjoint" if and only if:

$$\hat{A}^+ = \hat{A} \tag{71}$$

In any vector basis its associated matrix is Hermitian.

A linear operator $\hat{A}$ is said to be "bounded" if its norm is bounded by a real number m over all non-zero vectors:

$$\exists m \in \mathbb{R}, \forall x \in D(\hat{A}) \ \|x\| \neq 0 : \|\hat{A}x\| \leqslant m\|x\| \tag{72}$$

and a linear operator $\hat{A}$ is said to be "semi-bounded from below" if

$$\exists m \in \mathbb{R}, \forall x \in D(\hat{A}) : \langle \hat{A}x \mid x \rangle \geqslant m\|x\|^2 \tag{73}$$

An operator which is not bounded may be semi-bounded or unbounded.

If the operator $\hat{A}$ is defined everywhere:

$$D(\hat{A}) = H \tag{74}$$

the operator $\hat{A}$ is bounded after the Heelinger-Toeplitz's theorem[i] [40] and thus it is self-adjoint (it is called Hermitian). Only self-adjoint operators have real eigenvalues.

If the operator $\hat{A}$ is partially defined, i.e.:

$$D(\hat{A}) \subset H \tag{75}$$

it may be or not self-adjoint. A partially defined symmetric operator may have no or several self-adjoint extensions [41,42] and it may have no eigenvalues at all. In many problems of quantum physics, the Hamiltonian is only symmetric: to solve such problems we have to find self-adjoint extensions of the operator in relation to some boundary conditions.

Unbounded operators[ii] [40] can have an empty spectrum.

8. Conclusion

The standard Quantum Theory which has been developed with self-adjoint operators on a Hilbert space, is valid locally but it is not complete. A more general theory should be able to use quantum operators which are not bounded and have no self-adjoint extension. It should be able to describe quantum correlations with no causal order.

This paper proposes a most general formalism with complex implicit functions, to describe non relativist or relativist quantum systems.

[i] reference [40], section III.5, page 84.

[ii] reference [40], chapter 8.

In future works we will develop expressions of the global state function of some quantum systems.

References

1. Gilles Nibart, "Building the space-time structure in theories of physics", AIP Conf. Proc., vol. 839, pp. 198-211 (2006).
2. Gilles Nibart, "Théorie de l'horloge et temps propre aux systèmes organisés", Revue de Bio-Mathématique, Congrès de Biomathématique septembre 1983, vol; n°100, pp. 5-46, 4° trim. (1987).
3. Aristotle, Physics, book #2 (-384 to -322).
4. E.C. Zeeman, J. of Math. Phys., vol. 5, n°4, p. 490 (1964).
5. John L. Mackie, "The cement of the universe: a study in causation", Clarendon Press, Oxford, England (1988).
6. Judea Pearl, "Causality: models, reasoning, and inference", Cambridge University Press, (2000).
7. P. Spirtes, C. Glymour, and R. Scheines, "Causation, prediction, and search", Springer-Verlag, New York (1993).
8. W. Salmon, "Scientific explanation and the causal structure of the world", Princeton University Press (1984).
9. Lev V. Tarassov, "Physique quantique et opérateurs linéaires", éditions MIR, Moscou (1978), translated from Russian by S. Medvédev (1980).
10. V. Berestetski, E. Lifchitz, L. Pitayevski, "Théorie quantique relativiste", coll. "Physique théorique" (L. Landau anf E. Lifchitz), tome IV, éditions MIR, Moscou (1967), translated from Russian by Édouard Gloukhian (1967).
11. Carlo Rovelli, "Forget time", (Essay written for the FQXi contest on the nature of time), arXiv:0903.3832v3 [gr-qc] (27 Mar 2009).
12. Ognyan Oreshkov, Fabio Costa & Časlav Brukner, "Quantum correlations with no causal order", Nature Communications, vol. 3, p. 1092 (2012); DOI:10.1038/ncomms2076.
13. L. Landau anf E. Lifchitz, "Mécanique quantique, théorie non relativiste", coll. "Physique théorique" (L. Landau anf E. Lifchitz), tome III, éditions MIR, Moscou (1967), translated from Russian by Édouard Gloukhian (1967).
14. Albert Messiah, "Mécanique quantique", édition Dunod, Paris (1995), nouvelle édition (2003); tome 1: ISBN 2 10 007361 3, tome 2: ISBN 2 10 004695 0.
15. J.S. Bell, "On the Einstein Podolsky Rosen paradox", Physics Publishing Co, Physics vol. 1, n°3, pp. 195-200 (1964).
16. H. F. Trotter, "On the product of semi-groups of operators", Proceedings of the American Mathematical Society, vol. 10, n°4, pp. 545–551 (1959); DOI: 10.2307/2033649, ISSN 0002-9939; http://www.ams.org/mathscinet-getitem?mr=0108732.
17. Tosio Kato, "Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups", topic in "Functional analysis" (essays dedicated to M. G. Kreĭn on the occasion of his 70th birthday), Adv. in Math. Suppl. Stud., vol. 3, pp.

185–195, Academic Press, Boston, MA (1978).
18. Adel F. Antipa, "Limiting Velocity at the Interface Between Causality and Virtuality", AIP Conf. Proc., vol. 573, pp. 61-76 (2001).
19. Stückelberg, E.C.G., Helv. Phys. Acta, 14, 588 (1941) and 15, 23 (1942).
20. Feynmann, R. P., Phys. Rev., 74, 939 (1948).
21. W. Pauli, "General principles of quantum mechanics" in Handbuch der Physik, vol.5, part 1, ed. Fluegge, Springer, Berlin, Germany (1926); re-edited: W. Pauli, "General principles of quantum mechanics", Springer-Verlag, Berlin (1980) translated from the German by P. Achuthan and K. Venkatesan; ISBN: 3540098429.
22. V. S. Olkhovsky, E. Recami, "Recent developments in the time analysis of tunneling processes", Physics Reports, vol. 214, n°6, pp. 339–356 (1992); http://dx.doi.org/10.1016/0370-1573(92)90015-R.
23. V. S. Olkhovsky1, E. Recami and G. Salesi, "Superluminal tunneling through two successive barriers", Europhys. Lett., vol. 57, pp. 879-884 (2002); doi:10.1209/epl/i2002-00592-1.
24. V. Petrillo, V.S. Olkhovsky, "Tunnelling time of a Gaussian wave packet through two potential barriers", Central European Journal of Physics, vol.3 n°3, pp. 339-350 (2005).
25. B. A. Lippmann, "Operator for time delay induced by scattering", Physical Review, vol. 151, n°4, pp. 1023–1024 (1966).
26. T. T. Gien, "On the operators for time of motion and delay time induced by scattering", Canadian Journal of Physics, vol. 47, n°3, pp. 279–289 (1969).
27. V. S. Olkhovsky and E. Recami, "About a space-time operator in collision descriptions", Lettere AlNuovo Cimento, vol. 4, n°24, pp. 1165–1173 (1970).
28. V. S. Olkhovsky, "On the problem of time operator and collision duration", Ukrainskiy Fizicheskaya Zhurnal, vol. 18, p. 1910 (1973).
29. V. S. Olkhovsky, E. Recami and A. J. Gerasimchuk, "Time operator in quantum mechanics, I: Nonrelativistic case", Il Nuovo Cimento A, vol. 22, n°2, pp. 263-278 (1974); DOI: 10.1007/BF02813438.
30. V. S. Olkhovsky and E. Recami, "New developments in the study of time as a quantum observable", International Journal of Modern Physics B, vol. 22, n°12 (2008).
31. V. S. Olkhovsky, "Time as a quantum observable, canonically conjugated to energy, and foundations of self-consistent time analysis of quantum processes", Advances in Mathematical Physics, Hindawi Publishing Corporation, vol. 2009, article ID 859710, 83 pages (2009); DOI: 10.1155/2009/859710.
32. J. von Neumann, Mathematische Grundlagen der Quantummechanik, Springer, Berlin, Germany (1932).
33. J. von Neumann, "Über Adjungierte Funktionaloperatoren", Ann. Math. (2), vol. 33, n°2, pp. 294-310 (1936); DOI:10.2307/1968331.
34. M. Stone, "Linear transformations in Hilbert spaces and their applications to analysis", Amer. Math. Soc. Colloq. Publ., New York, vol. 15 (1932).
35. Stewart, James, "Calculus concepts and contexts", § 11.5, Brooks/Cole Publishing Company (1998); ISBN 0-534-34330-9.

36. S. Kumagai, "An implicit function theorem: comment", Journal of Optimization Theory and Applications, vol. 31, n°2, pp. 285-288 (June 1980).
37. Ludger Kaup and Burchard Kaup, "Holomorphic functions of several variables", de Gruyter Studies in Mathematics, vol. 3 (1939); ISBN 3-11-004150-2.
38. Klaus Fritsche and Hans Grauert, "From holomorphic functions to complex manifolds", coll. Graduate texts in Mathematics, Springer, vol. 213 (2002).
39. Raymond O. Wells, "Differential analysis on complex manifolds", coll. Graduate texts in Mathematics, Springer, vol. 25 (2008).
40. Michael Reed, and Barry Simon, "Methods of mathematical physics", vol. 1: "Functional analysis", Academic Press (1980).
41. A.I. Loginov, V.S. Shul'man, "Extension of an operator", Encyclopedia of Mathematics; ISBN 1402006098; http://www.encyclopediaofmath.org/index.php?title=Extension_of_an_operator&oldid=17795.
42. M.G. Krein, "Theory of selfadjoint extensions of semibounded Hermitian transformations and its applications" (in Russian), part I: Mat. Sb., vol. 20, pp. 431–498 (1947); part II: Mat. Sb., vol. 21 pp. 365–404 (1947).