# The Kottler-Whittaker Metric of the Homogeneous Gravitational Field and the Assumption of the Conservation of Spacetime Areas

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**Abstract**
In the Kottler-Whittaker metric of the homogeneous gravitational field the spacetime volume is conserved. The spacetime area is conserved, as well, if only the space dimension parallel to the field is taken into account. The latter property leads to an intuitive account of geodesic motion. Starting from the so-called Rindler hyperbolae, which describe constant proper acceleration in special relativity, the coordinate transformations for the equivalent homogeneous gravitational field are developed and applied to some elementary gravitational scenarios. The geodesic equations of planar fields are analyzed and compared for the two cases of volume conserving metrics and "isotropic" metrics conserving spacetime areas. An isotropic counterpart of the Schwarzschild metric is given by Broekaert's "scalar gravitation model", which is shortly discussed.
**Keywords:** gravitation, Riemannian spacetime, homogeneous field, Kottler-Whittaker metric

## 1 Introduction

In General Relativity Theory, Riemannian spacetime metrics conserving the infinitesimal "volume element" are of special interest. As repeatedly stressed by Einstein in his foundational article [1], such metrics lead to considerable mathematical simplifications. The static solutions of the field equations which are volume conserving metrics do not exhibit gravitational effects on lengths perpendicular to the field. For this reason, spatially one-dimensional scenarios involving only motion parallel to the field can be modeled by metrics conserving spacetime areas.

In the first part of this article it is shown that area-conserving metrics provide an intuitive understanding of geodesic motion. This will be elaborated by the analysis of the Rindler scenario of accelerated motion, which amounts to the so-called homogeneous gravitational field after appropriate coordinate transformations: After graphically reconstructing the Kottler-Whittaker metric [2, 3] of the homogeneous field from the assumption of area conservation, two spacetime diagrams will be discussed which enhance a qualitative understanding of gravitational effects.

The second part deals with two types of planar static field metrics. The geodesic equations of volume conserving metrics are compared to the respective equations of a class of "isotropic" metrics, which only conserve the spacetime area in the direction parallel to the field, but not the volume.


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We conclude by relating the results for planar fields to the case of spherically symmetric fields, where an isotropic alternative to the volume conserving Schwarzschild metric is given by Jan Broekaert's "scalar gravitation model".

## 2 Rindler Hyperbolae and Spacetime Diagrams

The planar field solution of Einstein's equations describes the so-called homogeneous gravitational field, which is equivalent to a well-known special relativistic scenario involving accelerated objects.

### 2.1 Accelerated Light Clocks

In special relativity, an object that is being accelerated at a constant rate (from its own perspective) is described by a hyperbola in the spacetime diagram. The family of Rindler hyperbolae (Rindler [4]) consists of such hyperbolae, whereby the proper acceleration $g$ of the objects depends on the location $x=1/g$ at time $t=0$. Equation (1) describes a Rindler hyperbola for a chosen proper acceleration $g$ (the light speed $c$ is set to 1).

$$x^2 - t^2 = \frac{1}{g^2} \qquad (1)$$

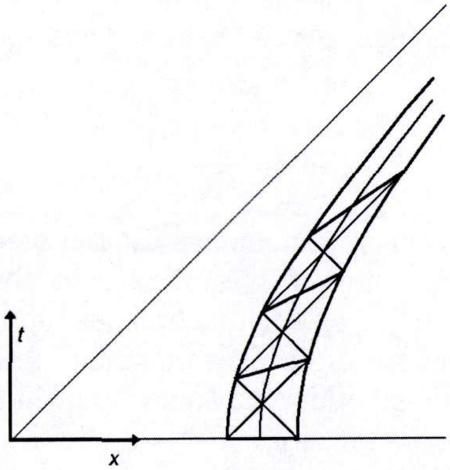


**Figure 1.** Two objects following Rindler hyperbolae form a "light clock" by exchanging photons. The meetings of the photons take place on a third Rindler hyperbola.

Fig. 1 shows the spacetime diagram of a light clock which is built of three Rindler hyperbolae: At time $t=0$ two photons are emitted from the objects described by the left and right hyperbolae and keep bouncing between the two objects. In this construction, the meetings of the two photons are located on the "middle" hyperbola. A measure for the proper time of the light clock is given by the respective number of spacetime cells[1], which are formed by parts of the left and right hyperbolae and the respective lines of simultaneity. As will become clear in the following, the spacetime cells have equal spacetime areas. Another interesting property is the fact that the lines of simultaneity go through the origin (c.f. Fig. 2). The 45° line describes the world line of a photon emitted from the origin. An object following a Rindler hyperbola cannot interact with what is on the left side of this line.

## 2.2 Coordinate Transformations

In this section it is outlined how the coordinate transformation from the Rindler scenario depicted in Fig. 1 to the view of an accelerated light clock can be calculated from geometrical considerations in Euclidean geometry - the special restriction of the transformation being the conservation of spacetime areas.

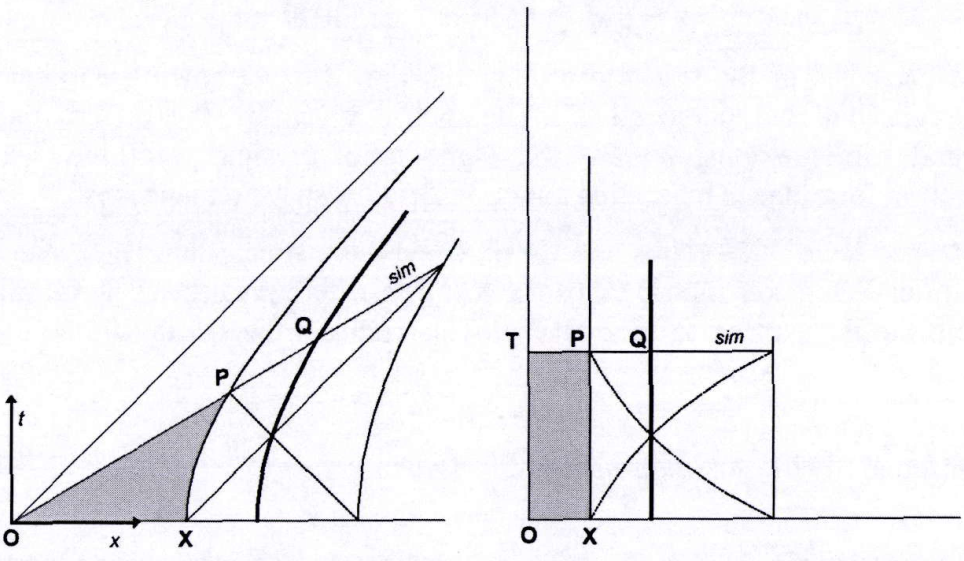


**Figure 2.** A Rindler light clock before and after the area-conserving coordinate transformation.

Figure 2 shows a spacetime cell of a Rindler light clock and the area-conserving transformation (indicated by the gray areas) of the scenario. The following steps lead to the transformation of some event $P$ to the perspective of the Rindler light clock (represented by the thick middle hyperbola):

a) The Rindler hyperbola on which $P$ is located follows from Eq. (1). The $x$-coordinate of this hyperbola at time $t=0$ is the provisional $x$-coordinate of $P$ in the transformed diagram.
b) The line of simultaneity (*sim*) on which $P$ is located goes through the origin. The intersection of this line with the middle hyperbola gives the event $Q$, from which the proper time coordinate is taken as the transformed time coordinate of $P$.

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1 Light clocks and spacetime cells have been suggested as useful means to explain most of special relativity (Winkler [5]). The concepts are quite naturally extended to the case of accelerated motion in this place.

c) Using a standard formula, the area of the "triangle" *OXP* is calculated.
d) In the transformed diagram, the area of the rectangle *OXPT* is calculated.
e) It turns out that the two areas are not identical. Therefore the provisional *x*-coordinate in the transformed diagram has to be rescaled.
f) The rescaled *x*-coordinate and the already calculated time coordinate give the required area-conserving transformation.
g) An appropriate shift of the transformed *x*-coordinate brings the middle of the light clock to the location $x=0$ (this is not drawn in the diagram, but is necessary to finally derive the Kottler-Whittaker metric).
h) The Kottler-Whittaker metric follows after calculating the differentials for space and time extensions leading to the metric coefficients.

Performing steps (a) to (f) gives the following coordinate transformation to the perspective of the accelerated observer located at $x=1/a$ with proper acceleration *a*.

$$x' = \frac{a}{2}\sqrt{x^2 - t^2}, \quad t' = \frac{1}{2a} Log[\frac{x+t}{x-t}] \tag{2}$$

This is the basis of the spacetime diagrams Fig. 2 to 4 showing Rindler scenarios from the "gravitational perspective". The metric given by the differentials for the lengths and time extensions in the "gravitational field" follows from these transformation formulae. The scaling function $A(x)$ for space distances is

$$A = \sqrt{2ax}. \tag{3}$$

The Kottler-Whittaker metric takes the location where proper acceleration is *a* to be at $x=0$, and the light speed to be *c* at $x=0$. This changes the scaling function for space distances to

$$A = \sqrt{1+2ax} \tag{4}$$

and consequently the scaling function for time intervals to

$$\overline{A} = A^{-1} = 1/\sqrt{1+2ax} \tag{5}$$

Under the assumption that there is no gravitational effect on the perpendicular extensions, the line element of the Kottler-Whittaker metric follows.

$$\begin{aligned} ds^2 &= \overline{A}^{-2} c^2 dt^2 - A^{-2} dx^2 - dy^2 - dz^2 \\ ds^2 &= (1+2ax)c^2 dt^2 - (1+2ax)^{-1} dx^2 - dy^2 - dz^2 \end{aligned} \tag{6}$$

## 2.3 Two Gravitational Scenarios

Fig. 3 shows two accelerated light clocks in the Rindler scenario which each send two photons in the direction of the other light clock. The emission time interval is one "tick" from the perspective of the emitting light clock. The transformed diagram shows both light clocks at rest and makes clear why the right light clock measures a redshift and why the left clock measures a blueshift.

The right hand side of Fig. 4 shows a resting light clock and a light clock in constant motion from the perspective of a Rindler light clock. Depending on speed and location, gravitational attraction becomes "gravitational repulsion". This phenomenon has been analyzed in the context of the Schwarzschild metric[2] and will be derived later from the geodesic equations (Eq. (20)).

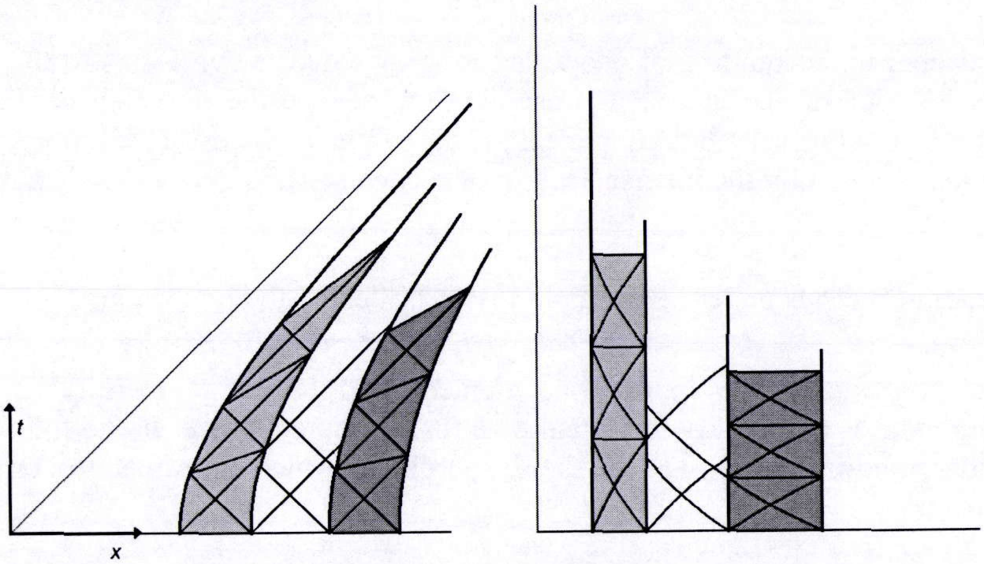


**Figure 3.** Two Rindler light clocks exchange photons. In the transformed diagram the measured time distances explain the effect of gravitational redshift.

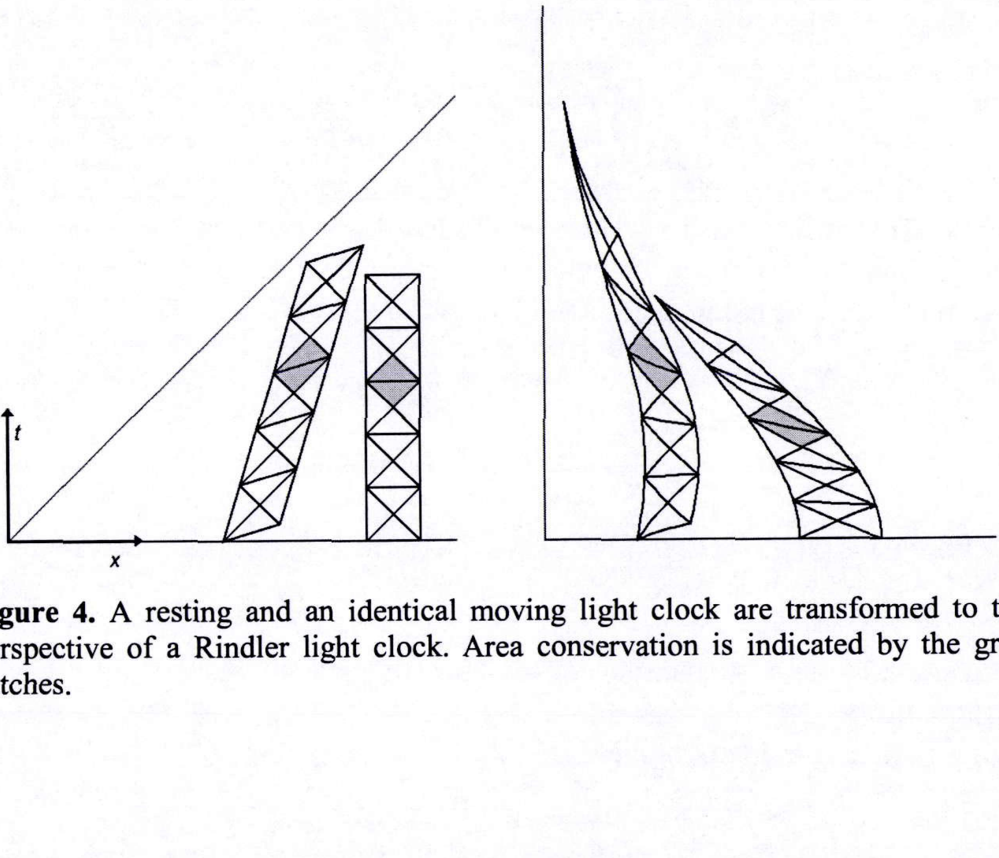


**Figure 4.** A resting and an identical moving light clock are transformed to the perspective of a Rindler light clock. Area conservation is indicated by the gray patches.

2 A comprehensive analysis can be found in McGruder [6]; an early calculation has been performed by Hilbert [7].

## 3 Geodesic Analysis

For a deeper understanding of geodesic motion we derive two forms of the geodesic equations for a class of static area-conserving metrics (in the direction parallel to the field). The following calculations are restricted to planar fields. The property of area conservation is given by the inverse scaling of space and time extensions (parallel to the field).

The class of metrics under investigation is described by the line element

$$ds^2 = A^2 c^2 dt^2 - A^{-2} dx^2 - B^{-2} dy^2 - B^{-2} dz^2 . \tag{7}$$

The scaling function $B$ in the line element will be set to identity for volume conserving metrics and will be set equal to $A$ for isotropic metrics. In the following, we assume that motion takes place in the $(x, y)$-plane, which allows us to omit the $z$-coordinate.

### 3.1 Two Forms of the Geodesic Equations

For the analysis of timelike geodesics we use the Langrangian

$$L = \frac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}, \tag{8}$$

which leads us to

$$L = \frac{c^2}{2} = \frac{1}{2} \frac{ds^2}{d\tau^2} = \frac{1}{2}\left( A^2 c^2 \left(\frac{dt}{d\tau}\right)^2 - A^{-2}\left(\frac{dx}{d\tau}\right)^2 - B^{-2}\left(\frac{dy}{d\tau}\right)^2 \right). \tag{9}$$

Equation (9) represents a first conservation law for geodesics. The variables $t$ and $y$ are cyclic which allows us to calculate two more conservation laws (the notations $\dot{t} = dt/d\tau$ and $\dot{y} = dy/d\tau$ being used).

$$p_t = \frac{\partial L}{\partial \dot{t}} = c^2 A^2 \frac{dt}{d\tau} = const \tag{10}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = -\frac{1}{B^2} \frac{dy}{d\tau} = const \tag{11}$$

Exposing the derivatives of the two cyclic variables leads to

$$\frac{dt}{d\tau} = \frac{p_t}{c^2 A^2}, \qquad \frac{dy}{d\tau} = -p_y B^2 . \tag{12}$$

The differentiation of the three conservation laws with respect to $\tau$ leads to the geodesic equations.

Deriving $p_t$ with respect to $\tau$ gives

$$\frac{d^2 t}{d\tau^2} = -2 \frac{A'}{A} \frac{dx}{d\tau} \frac{dt}{d\tau}. \tag{13}$$

Deriving $p_y$ with respect to $\tau$ gives

$$\frac{d^2y}{d\tau^2}=2\frac{B'}{B}\frac{dx}{d\tau}\frac{dy}{d\tau}. \tag{14}$$

Deriving $L$ with respect to $\tau$ and inserting from Eqs. (13) and (14) gives

$$\frac{d^2x}{d\tau^2}=-c^2A^3A'\left(\frac{dt}{d\tau}\right)^2+\frac{A'}{A}\left(\frac{dx}{d\tau}\right)^2-\frac{A^2B'}{B^3}\left(\frac{dy}{d\tau}\right)^2. \tag{15}$$

The geodesic equations (13) to (15) are formulated with respect to the proper time $\tau$ of an object in free fall. In order to be applicable to the spacetime diagrams of the presented kind, however, they have to be reformulated with respect to coordinate time $t$.

We write

$$\frac{dt}{d\tau}=\frac{p_t}{c^2A^2},\quad \frac{dy}{d\tau}=\frac{dy}{dt}\frac{dt}{d\tau}=\frac{p_t}{c^2A^2}\frac{dy}{dt},\quad \frac{dx}{d\tau}=\frac{dx}{dt}\frac{dt}{d\tau}=\frac{p_t}{c^2A^2}\frac{dx}{dt}, \tag{16}$$

and get from Eqs. (10) and (11)

$$\frac{dy}{d\tau}=-p_yB^2=\frac{p_t}{c^2A^2}\frac{dy}{dt},\quad p_y=-\frac{p_t}{c^2A^2B^2}\frac{dy}{dt}. \tag{17}$$

Differentiating $p_y$ with respect to $t$ leads to

$$\frac{d^2y}{dt^2}=2\left(\frac{A'}{A}+\frac{B'}{B}\right)\frac{dx}{dt}\frac{dy}{dt}. \tag{18}$$

According to Eq. (16) $L$ can be written as

$$L=\frac{c^2}{2}=\frac{p_t^2}{2c^2A^2}\left(1-\frac{1}{c^2A^4}\left(\frac{dx}{dt}\right)^2-\frac{1}{c^2A^2B^2}\left(\frac{dy}{dt}\right)^2\right). \tag{19}$$

Differentiating with respect to $t$ and substituting Eq. (18) for $d^2y/dt^2$ leads to

$$\frac{d^2x}{dt^2}=-c^2A^3A'+3\frac{A'}{A}\left(\frac{dx}{dt}\right)^2-\frac{A^2B'}{B^3}\left(\frac{dy}{dt}\right)^2. \tag{20}$$

We call the representation of the geodesic equations given by Eqs. (18) and (20) the "*t*-form" as opposed to the usual "$\tau$-form".

### 3.2 Speed Dependence of Acceleration

A well-known result for the $\tau$-form of the geodesic equations in the Schwarzschild metric, which is an area conserving metric in the spatially one-dimensional case, is the fact that gravitational acceleration for mere radial motion does not depend on the speed of the falling body. This result will be reconstructed for the Kottler-Whittaker metric and will be extended for isotropic metrics. The dependence on the perpendicular speed will be made explicit. For this purpose we have to reformulate Eq. (15).

The Lagrangian (9) allows us to write

$$\left(\frac{dt}{d\tau}\right)^2=A^{-2}+A^{-4}c^{-2}\left(\frac{dx}{d\tau}\right)^2+A^{-2}B^{-2}c^{-2}\left(\frac{dy}{d\tau}\right)^2. \tag{21}$$

Inserting into Eq. (15) gives an expression for the gravitational acceleration that is free of $dt/d\tau$ and consequently makes the dependence of gravitational acceleration on the speed components explicit.

$$\frac{d^2x}{d\tau^2} = -c^2 A A' - \left(\frac{dy}{d\tau}\right)^2 \left(\frac{A A'}{B^2} + \frac{A^2 B'}{B^3}\right) \tag{22}$$

For volume conserving metrics (*B=1*) the dependence on the perpendicular speed component is given by

$$\frac{d^2x}{d\tau^2} = -A A' \left(c^2 + \left(\frac{dy}{d\tau}\right)^2\right). \tag{23}$$

For isotropic metrics (*B=A*) the dependence is even stronger.

$$\frac{d^2x}{d\tau^2} = -A A' \left(c^2 + 2\left(\frac{dy}{d\tau}\right)^2\right) \tag{24}$$

Given the secondary meaning of coordinate systems in general relativity, the present analysis of the $\tau$-forms of the geodesic equations may not seem very relevant. There is, however, a significant difference between the volume conserving metrics and the isotropic metrics which can now be formulated: If and only if A=B the radial motion equation can be derived from a kind of energy conservation law including potentials.

In the following definitions, *P(x)* can be regarded as the "potential of the linear momentum perpendicular to the field" which is invariant in Newtonian physics. *V(x)* represents the "potential of the gravitational force".

$$P(x) \equiv B^4 p_y^2 = \left(\frac{dy}{d\tau}\right)^2, \quad V(x) = c^2 A^2 \tag{25}$$

Using these definitions, an energy expression can be formulated.

$$E \equiv \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + V(x) = \left(\frac{dx}{d\tau}\right)^2 + P(x) + V(x) \tag{26}$$

Assuming $E$ to be constant and deriving Eq. (26) with respect to $\tau$ leads to Eq. (24) in the case of *B=A*, but not to Eq. (23) in the case *B=1*.

### 3.3 The Kottler-Whittaker Metric

The results can now be applied to the Kottler Whittaker metric.

$$A = (1 + 2ax)^{1/2}, \quad A' = a(1 + 2ax)^{-1/2}, \quad B = 1, \quad B' = 0 \tag{27}$$

The geodesic equations follow from Eqs. (13), (14) and (15).

$$\frac{d^2t}{d\tau^2} = -\frac{2a}{1 + 2ax}\frac{dt}{d\tau}\frac{dx}{d\tau} \tag{28}$$

$$\frac{d^2x}{d\tau^2} = -c^2 a(1+2ax)\left(\frac{dt}{d\tau}\right)^2 + \frac{a}{1+2ax}\left(\frac{dx}{d\tau}\right)^2 =$$
$$= -ac^2 - a\left(\frac{dy}{d\tau}\right)^2 \tag{29}$$

$$\frac{d^2y}{d\tau^2} = 0 \tag{30}$$

The *t*-form of these equations follows from Eqs. (18) and (20).

$$\frac{d^2x}{dt^2} = -c^2 a(1+2ax) + \frac{3a}{1+2ax}\left(\frac{dx}{dt}\right)^2 \tag{31}$$

$$\frac{d^2y}{dt^2} = \frac{2a}{1+2ax}\frac{dx}{dt}\frac{dy}{dt} \tag{32}$$

The world lines of the free falling objects in Fig. 4 and all the world lines for light in the transformed diagrams follow after using the scaling function from Eq. (3).

$$A = B = (2ax)^{1/2}$$
$$\frac{d^2x}{dt^2} = -2c^2a^2x + \frac{3}{2x}\left(\frac{dx}{dt}\right)^2 \tag{33}$$

### 3.4 An Isotropic Metric for the Planar Field

A possible isotropic metric with some similarity to the homogeneous gravitational field of GRT is given by setting

$$A = B = e^{ax}, \quad A' = B' = a e^{ax}. \tag{34}$$

The geodesic equations follow from Eqs. (13), (14), and (15).

$$\frac{d^2t}{d\tau^2} = -2a\frac{dt}{d\tau}\frac{dx}{d\tau} \tag{35}$$

$$\frac{d^2x}{d\tau^2} = -c^2 a e^{4ax}\left(\frac{dt}{d\tau}\right)^2 + a\left(\frac{dx}{d\tau}\right)^2 - a\left(\frac{dy}{d\tau}\right)^2 =$$
$$= -c^2 a e^{2ax} - 2a\left(\frac{dy}{d\tau}\right)^2 \tag{36}$$

$$\frac{d^2y}{d\tau^2} = 2a\frac{dx}{d\tau}\frac{dy}{d\tau} \tag{37}$$

The *t*-form of these equations follows from Eqs. (18) and (20).

$$\frac{d^2x}{dt^2} = -c^2 a e^{4ax} + 3a\left(\frac{dx}{dt}\right)^2 - a\left(\frac{dy}{dt}\right)^2. \tag{38}$$

$$\frac{d^2y}{dt^2} = 4a\frac{dx}{dt}\frac{dy}{dt}. \tag{39}$$

## 4 Schwarzschild Metric Versus Broekaert Metric

The calculations of the preceding section refer to planar fields, only. We would like to mention, though, that the results apply to the case of spherically symmetric fields, as well. It has been shown by the author (Winkler [8]) that the so-called "scalar gravitation model" suggested by Jan Broekaert [9] is a metric model in Riemannian spacetime. Broekaert's model fits the present definition of area conserving isotropic metrics and it explains the experimental tests of GRT: gravitational redshift, precession of orbital perihelia, radar echo delay, and light deflection (Broekaert [9]).

For the following comparison of Schwarzschild and Broekaert metrics we use the functions *A* and *B* for radial and tangential differential lengths.
The Schwarzschild metric is given by setting

$$\kappa=\frac{GM}{c^2},\quad A=\left(1-\frac{2\kappa}{r}\right)^{1/2},\quad A'=\frac{\kappa}{r^2}\left(1-\frac{2\kappa}{r}\right)^{-1/2},\quad B=1,\quad B'=0. \tag{40}$$

Performing calculations analogous to those in sections 3.1 and 3.2, the geodesic equations lead to the radial acceleration in $\tau$-form.

$$\frac{d^2r}{d\tau^2}=-\frac{c^2\kappa}{r^2}+(r-3\kappa)\left(\frac{d\varphi}{d\tau}\right)^2 \tag{41}$$

The *t*-form of this equation is

$$\frac{d^2r}{dt^2}=-\frac{c^2\kappa(r-2\kappa)}{r^3}+\frac{3\kappa}{r(r-2\kappa)}\left(\frac{dr}{dt}\right)^2+(r-2\kappa)\left(\frac{d\varphi}{dt}\right)^2. \tag{42}$$

The Broekaert metric is given by setting

$$\kappa=\frac{GM}{c^2},\quad A=B=e^{-\kappa/r}\quad A'=B'=\frac{\kappa}{r^2}e^{-\kappa/r}, \tag{43}$$

leading to the radial acceleration in $\tau$-form

$$\frac{d^2r}{d\tau^2}=-\frac{c^2\kappa}{r^2}e^{-2\kappa/r}+(r-2\kappa)\left(\frac{d\varphi}{d\tau}\right)^2. \tag{44}$$

The *t*-form of this equation is

$$\frac{d^2r}{dt^2}=-\frac{c^2\kappa}{r^2}e^{-4\kappa/r}+\frac{3\kappa}{r^2}\left(\frac{dr}{dt}\right)^2+(r-\kappa)\left(\frac{d\varphi}{dt}\right)^2. \tag{45}$$

For the $\tau$-forms of radial equations, the analysis of the potentials leads to the same difference between volume conserving and isotropic metrics as in the planar case. *P(r)* stands now for the "potential of the angular momentum" and the "potential of the gravitational force" is written as *V(r)*.

$$P(r)=B^4\frac{p_\varphi^2}{r^2}=r^2\left(\frac{d\varphi}{d\tau}\right)^2,\quad V(r)=c^2A^2 \tag{46}$$

Differentiating the energy expression

$$E=\left(\frac{dr}{d\tau}\right)^2+r^2\left(\frac{d\varphi}{d\tau}\right)^2+V(r)=\left(\frac{dr}{d\tau}\right)^2+P(r)+V(r) \tag{47}$$

leads to the radial acceleration (Eq. (44)) for the Broekaert metric, but not to Eq. (41) for the Schwarzschild metric.

## 5 Conclusions

The intuitive power of area conserving metrics has been revealed by transforming Rindler scenarios involving accelerated light clocks to equivalent gravitational scenarios. A distinction between volume conserving and area conserving isotropic metrics has been made and two forms of the geodesic equations for the two classes have been derived. Both representations have their merits: The coordinate space formulation provides the basis for spacetime diagrams and the proper time formulation exhibits a classical energy law including potentials for the isotropic metrics.

The relevance of the present results is given by the applicability to the spherically symmetric field, where an isotropic counterpart to the Schwarzschild exists in the form of Broekaert's "scalar gravitation model", which explains the known experimental evidence for GRT. While in full agreement with the Riemannian view of spacetime provided by GRT, the assumption of area conserving isotropic metrics might lead to the formulation of a new set of field equations. The planar field from section 3.4 and the spherically symmetric Broekaert metric would appear as simple solutions.

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