Minkowski Space: Tick Here

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Abstract We explore a two-photon model of a digital clock. The model implicates the use of spacetime algebra to describe Minkowski space on large scales, but suggests that its scale independent use in special relativity presupposes reference frames of infinite mass. A close look at the finite-frequency digital clock shows that either classical special relativity or Dirac propagation emerges from the clock depending on how the continuum limit is taken. If one smoothly interpolates the tick sequence the clock remains classical. If one extrapolates the inter-tick behaviour, wave propagation is implicated.
Keywords : Special Relativity, Quantum Mechanics.

1 Introduction

At the beginning of the 20'th century, physics changed radically following two separate revolutions. Special relativity and quantum mechanics not only changed how physicists predict and calculate the outcomes of experiments, they changed the very concepts of space, time and object. The two revolutions took place in distinct arenas. Special relativity was about events in space and time, and how these would be perceived from different inertial frames of reference. Quantum mechanics was about submicroscopic objects and how they would evolve in time, usually under restrictions to small length scales.

Both revolutions involved a gestation period in which the new physics was written in a language that accentuated the similarities with classical mechanics. Thus, Minkowski took Einstein's component-viewed special relativity and embedded it in a four-vector formalism, setting the stage for more modern spacetime algebra formulations. Similarly old quantum theory with its ad hoc rules, gave way to matrix mechanics from Heisenberg and ultimately wave mechanics from Schrödinger. The latter allowed Hamiltonian mechanics a central place in the study of quantum systems.

In 1928 Dirac fused the two pictures in the equation that would bear his name. He essentially took wave propagation from quantum mechanics and embedded it in Minkowski space.

A curious feature of the two revolutions is that both are related to classical mechanics through a form of analytic continuation. Minkowski space can be obtained from a four dimensional Euclidean space by a judicious replacement of t by it.

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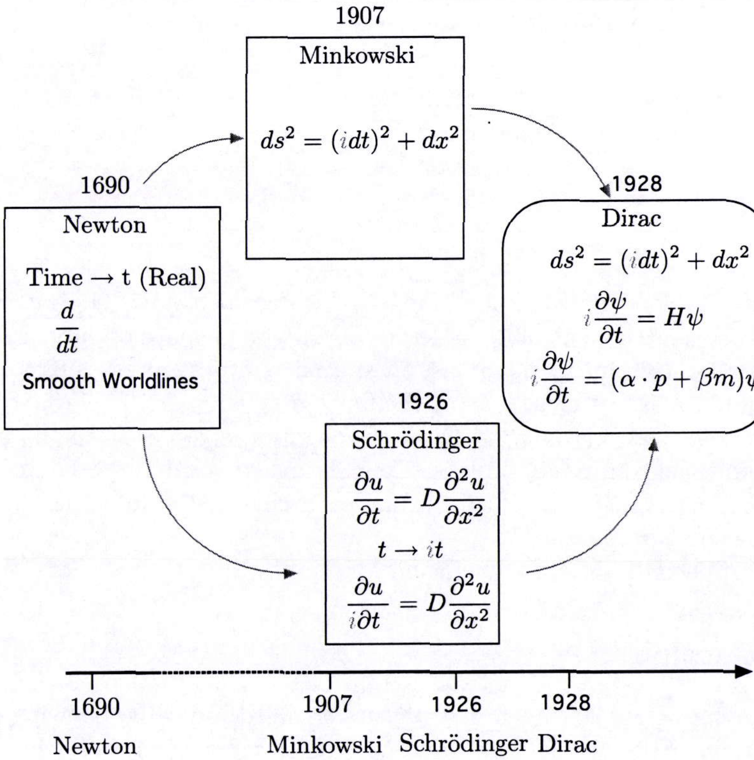
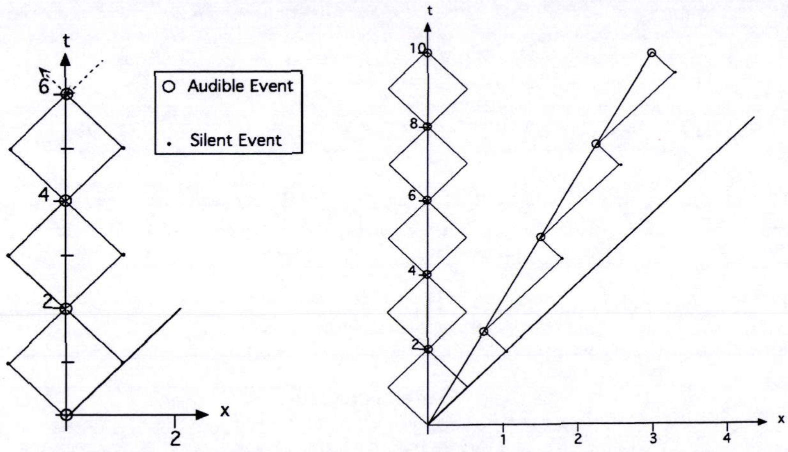


Fig. 1: Both special relativity and quantum mechanics inherited the representation of time as a real number. The two revolutions utilized a formal analytic continuation in two different ways. Minkowski used *it* to make spacetime pseudo-Euclidean. Schrödinger effectively used the same formal replacement to convert the diffusion equation to a wave equation

Similarly, non-relativistic quantum propagation may be obtained from a diffusion equation by the same device Fig[1]. In this paper we point out that the two analytic continuations arise from the same source. The difference between the two is a result of a difference in the choice of which aspects of time are chosen to be represented on large scales, the number of events or the inter-event region.

To illustrate this clearly, we build a classical digital two-photon clock and examine its properties. Over long time-scales we see how Minkowski space is implicated and why the odd signature of spacetime is a natural consequence of the assumption of infinite mass reference frames. On short time scales we see that the Dirac equation is implicated. Given the classical context of the clock, this is something of a surprise. Further investigation reveals that for massive particles, the behaviour that gives rise to Minkowski space on large scales, gives rise to wave propagation on



(a) A two photon clock. (b) Viewed in a moving frame.

Fig. 2: (a) A two-photon clock at rest provides a chain of spacetime areas. Here $c = 1$ and the clock audibly 'ticks' at the crossing points. The corner points in the paths are considered 'silent' events. The sequence of crossing points and a smooth interpolant between them provides an analog of the world line of the clock in classical special relativity. (b) In a moving reference frame the areas between events become rectangular, however the area is preserved and is calculated as the product of the lengths of the two sides of the rectangles.

small scales.

In the first section we introduce an idealized two-photon clock confined to a two-dimensional spacetime, Fig[2]. We notice that in order for the clock to be relativistically correct, the sequence of events it produces has to be determined by a mechanism that relies on the production of a fixed spacetime *area* in all reference frames.

The following section develops an algebraic method for counting events using a transfer matrix. Observing how the transfer matrix handles higher frequency events gives a direct connection between event counting and the introduction of pseudo-Euclidean spaces for the description of special relativity.

Section 4 extends the model to a four dimensional spacetime and the final section reviews and discusses the results.

2 A Planar Two-Photon Clock

Classical mechanics starts with the assumption that time is a real number, but both special relativity and quantum mechanics essentially replace t by it. Since all measured time intervals in Nature are finite we shall start with a finite frequency digital

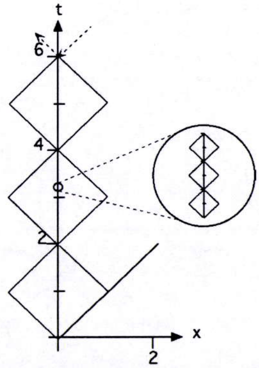


Fig. 3: To interpolate between ticks of the two-photon clock, we assume we have access to higher frequency two-photon clocks to partition the inter-event time. We associate the high-frequency clocks with a spacetime frame.

clock and explicitly take the limit of infinite frequency to see what peculiarities arise when we take the limit. The purpose of this is to see whether there is a geometrical origin for the i in Minkowski space.

We note that any clock, regardless of size, must involve accelerations to facilitate periodicity. In a relativistic universe with a finite upper bound on speeds, high frequency small-scale clocks necessarily involve rapid accelerations and these accelerations are likely to be intrinsic to time measurement.

Consider a hypothetical 2-photon planar clock in a two dimensional spacetime as in Fig.[2(a)]. The clock 'ticks' at the crossing points of the two photons and keeps time by simply counting ticks. Thus a clock that begins and ends at specific crossing points as in the figure counts time discretely according to the number of ticks. In Fig.[2(a)] the first tick is at the origin and the second is at $t = 2$. We are using units in which the speed of light c is 1 and our clock has been chosen for convenience so that the fundamental period is 4 units. The corners at the odd integers we call 'silent' events. They are not aligned with the crossing points that form the event sequence and are not accounted for in an interpolation of the audible event sequence. However they are important in determining how the clock progresses between adjacent audible events.

The enclosed areas between ticks are oriented, with successive areas having opposite orientation as indicated by the two different path colours in Fig.[2(a)]. The reason for this shall appear when we consider higher frequency clocks.

To be used as a clock to measure time we imagine embedding the clock in a spacetime frame. We shall subsequently have to reconcile the discrete nature of the clock with the continuous labelling of the frame however, for the time being, we shall

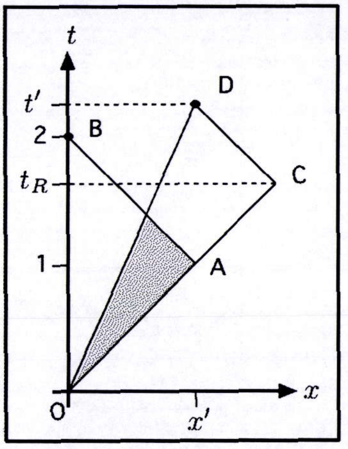


Fig. 4: The time between clock events in the moving reference frame, t' is determined by requiring that the area in the moving frame, $\Delta(OCD)$ is equal to the rest frame area $\Delta(OAB)$.

assume that some higher frequency clocks exist and are used to interpolate between the ticks of the clock in the frame. So for example, in Fig[2(a)], the two-photon clock has events on the t-axis only at $t \in \{0, 2, 4, \ldots\}$ but we assume that the associated frame has events arbitrarily close together, interpolating between these special times Fig[3].

For the clock to exhibit a trail of events in arbitrary inertial frames in a fashion that is consistent with special relativity, the clock needs a rule that will translate the distance between events into time intervals. Noting that the clock actually counts *spacetime areas* rather than a direct Euclidean length, we can use geometry to see that the Euclidean spactime area between crossing points is invariant under inertial transformations Fig[4]. In this figure the triangular area $\Delta(OAB)$ is half the area of the first half-cycle of the stationary clock (between $t = 0$ and $t = 2$) and we see that $\Delta(OAB) = 1$. The area $\Delta(OCD)$ is half the area of the first half-cycle of the clock in a frame moving with velocity $-v$ with respect to the clock. In this frame the first clock event is at (x', t') where $x' = vt'$. Noting that $x' = 2t_R - t'$, the length of the line OC is

$$|OC| = \sqrt{2}t_R = \sqrt{2}t'(1+v)/2 \tag{1}$$

Similarly the length of CD is

$$|CD| = \sqrt{2}(t' - t_R) = \sqrt{2}t'(1-v)/2 \tag{2}$$

The area of $\Delta(OCD)$ is then

$$\Delta(OCD) = t'^2(1-v^2)/4. \tag{3}$$

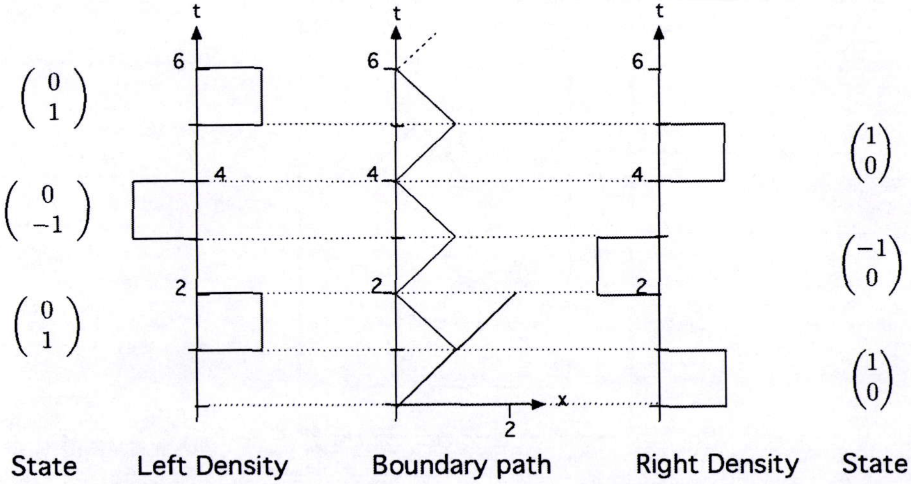


Fig. 5: Counting with a Transfer matrix. The right hand boundary of the two photon clock is in one of four states corresponding to being parallel or antiparallel to the left or right light cone. The left and right densities are signed characteristic functions that turn on when the boundary is in the appropriate state. The left and right densities are orthogonal since the boundary can only be in one of the four states.

Equating the two areas $\Delta(OAB) = \Delta(OCD)$ we then get

$$t' = \frac{2}{\sqrt{1-v^2}} = 2\gamma = \gamma t \tag{4}$$

where γ is the time dilation factor. Notice that it arises from the requirement that the enclosed spacetime *areas* between events is invariant under an inertial transformation. We can calculate the area between events by taking the products of the lengths of the sides of the projections of the areas onto the two orthogonal light cones. If those lengths are l and r respectively for the left and right cones, then the resulting proper time interval is the geometric mean

$$t = \sqrt{lr}. \tag{5}$$

A clock that measures time based on the square root of its inter-event spacetime area will then always yield its proper time. However, in order to do this there has to be at least one silent event between audible events in order to determine an area. The necessity of this off-axis silent event to determine proper time is the first hint that a continuum limit from finite frequency clocks may involve more than a simple extension of time intervals to real numbers.

In the next section we formulate the counting of events in the two photon clock by introducing a transfer matrix.

3 The two photon clock as a transfer matrix

In the previous section we proposed a 2-photon clock that measured time through the crossing-point events of the entwined paths of two photons. We noted that the clock ticks by producing chains of rectangular regions of spacetime, the proper time being the number of links in the chain multiplied by the proper time corresponding to a single area. In this section we assemble a matrix method for counting the events in the chain of areas.

Consider the 'boundary path' of a clock illustrated in Fig[5]. This is just the right-hand boundary of the chain of areas that constitutes the clock. A boundary path segment can be in one of four states depending on its direction in spacetime and its orientation. A column vector with four states

$$s_k \in S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \tag{6}$$

can represent the path. The states represent an indicator function that registers the presence or absence of an oriented boundary of a rectangle with the accompanying orientation Fig.(5). If the two photon clock is in state s_k at time k then the subsequent state, after the next event is

$$s_{k+1} = T s_k \tag{7}$$

where T is the transfer matrix:

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

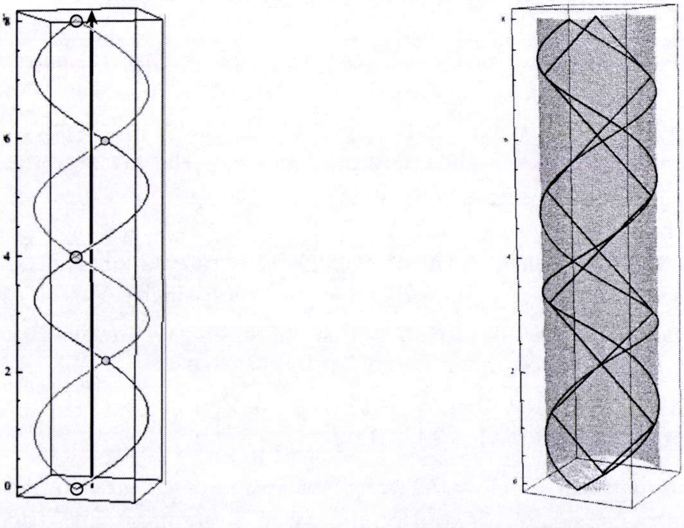
T operating on a state transfers us through the next event and T^2 takes us through two events, returning to the same direction but opposite orientation. The two-photon clock ticks cyclically through the set S. Thus powers of the transfer matrix give us the number of ticks of the clock.

If all we have is a single clock, we could use it to sequence events that happen over time scales much greater than 1 by counting time in unit intervals. We *cannot* use it as a reference frame clock to measure events happening on time scales much less than one, since multiple events could occur between ticks and not be sequenced. We can however imagine a reference 2-photon clock working in the same way as T but at a much higher frequency, say N, where N is an integer much greater than 1. If T_ν is the transfer matrix for the high-frequency N-clock, and it is synchronized with T, we must have an equivalence between the N-th power of T_ν and T itself. Thus we can write

$$T_\nu^N = T \tag{9}$$

giving T_ν as an Nth root of T or

$$T_\nu = \begin{pmatrix} \cos\left(\frac{\pi}{2N}\right) & -\sin\left(\frac{\pi}{2N}\right) \\ \sin\left(\frac{\pi}{2N}\right) & \cos\left(\frac{\pi}{2N}\right) \end{pmatrix} \tag{10}$$



(a) High frequency clocks, synchronized with the two-photon clock to give events at the integers, interpolate between the integers by mapping the four states onto a circle and using phase to refine partitions of the time interval between clock events. The resulting paths representing spacetime area boundaries, analogous to those in Fig. 1(a) form a pair of spirals.

(b) The planar 2-photon clock shown in comparison to(a). The events of the two clocks both consist of crossing points of paths. The silent events of the two-photon clock correspond to turning points in the projection of the spiral paths onto the plane.

Fig. 6: (a) The reference frame clock. (b) Comparison with two-photon paths.

Taking a continuum limit gives a transfer matrix representing a reference clock with arbitrarily high precision. That is we define T_R as

$$
\begin{aligned}
T_R(t) = \lim_{N\to\infty} T_\nu^{Nt} &= \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) \\ \sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix} \\
&= \cos\left(\frac{\pi t}{2}\right) \mathrm{I}_2 + \mathrm{T} \sin\left(\frac{\pi \mathrm{t}}{2}\right) \qquad (11) \\
&= \cos\left(\frac{\pi t}{2}\right) \mathrm{I}_2 - \mathrm{i}\,\sigma_{\mathrm{y}} \sin\left(\frac{\pi \mathrm{t}}{2}\right) \qquad (12)
\end{aligned}
$$

where I_2 is the 2×2 identity matrix and σ_y is the second Pauli matrix. This clock agrees with our period four clock T at the integers but uses a rotation matrix to interpolate between events (Fig.[6]). That is $T_R(n) = T^n$ for integer n.

Note that when we use T_R as an operator on the states s_k, these no longer *appear* to behave as *discrete* characteristic functions that indicate presence or absence of a boundary of a rectangle. Initially, for our two-photon clock, the s_k clearly indicate that a boundary is in one of four *mutually exclusive* states. However, unless t is

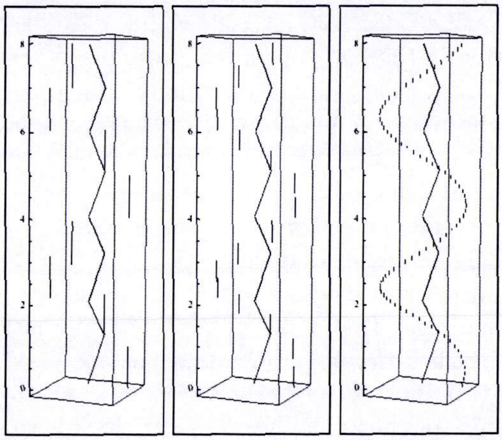


Fig. 7: Increasing the frequency of events. The Transfer matrix can be used to increase the frequency of events. It does this by 'rotating' the indicator functions for the four states s_k. The first frame shows a single boundary of the two-photon clock and the associated indicator function as a two-component function of t (vertical lines). The path is in a single state at a given time so only one of the s_k is non-zero. The second frame shows the result of a two-photon clock running at double the frequency. The indicator function is still showing the path is always only in a single state but now the new events are shown rotated from the four states in S. The third frame indicates a reference clock running at 10 times the photon-clock freqency and one can see the emergence of one of the reference spirals of Fig[3].

an integer, $T_R(t)s_k \notin S$. For example $T_R(1/2)s_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It appears that our indicator function is in a 'superposition' state. Since we are only using the transfer matrix to count objects, the 'superposition state' must have a simple interpretation.

Indeed, notice that the original s_k are orthogonal because a boundary can only be in a single state. State 1 is orthogonal to states 2 and 4. State 2 is orthogonal to states 1 and 3. However the two dimensional space spanned by S is *not the coordinate space of the boundary* but an algebraically constructed space whose orientation has been chosen so that successive events 'rotate' the s_k through the elements of S making the role of the states as characteristic functions obvious. Using the transfer matrix to increase the frequency of events simply uses the extra degree of freedom in the choice of basis set to map the extra states onto a spiral. Observing Fig[7] we see that the higher frequency clock T_ν in eqn(9) behaves just as the two-photon clock except that it expresses the indicator function as a rotation of *the entire set* S. The orthogonality that is an expression of the fact that the clock must be in a single state is preserved in the rotated basis. This is illustrated in the second frame of Fig[7]. There the frequency is doubled and the indicator function for the path splits so that the second half of every inter-event interval is rotated by $\pi/4$. This is just an accommodation of the fact that at this resolution there are eight possible states and the refined clock visits each sequentially. The last frame of the figure shows a refinement of the clock by a factor of 10. At this resolution one can see the outline of the spiral limit of Fig[6] while still seeing the digital aspect that the clock has an inner scale between ticks.

$T_R(t)$ is an infinite precision idealization of our two photon clock obtained from an ensemble of such clocks of higher frequency. It can be used for analog conversion since $T_R(\lfloor t \rfloor) = T^{\lfloor t \rfloor}$ and the two clocks agree at the events at integer times. $T_R(t)$ is a prototype of a frame clock that gives us an arbitrarily large number of events between integers and justifies the use of the real parameter t to describe position along the t-axis of the two-photon clock.

One interesting feature of the analog form of our clock is that although we start with the planar path of Fig[2(a)] as an idealization of a clock mechanism restricted to a plane, the refinement of that clock to allow for higher precision *automatically* invokes a complex structure that we shall ultimately recognize as Minkowski space. The planar structure of a real two dimensional spacetime is not big enough to support a description of synchronized clocks of higher precision than the one we started with, and *the result is an expansion in dimension through the use of complex numbers.*

This is, in part, illustrated by the fact that the analog approximation to our digital clock satisfies a differential equation. If $U(t)$ is a column vector, then using (11) we see that the analog of the difference equation (7) is

$$\frac{dU}{dt} = \left(\frac{\pi}{2}\right) T_R\, U \tag{13}$$

a version of the zero-momentum Dirac equation in two dimensions. What has happened here is that the periodic structure of our clock has to be accommodated by any reference clock that would give a synchronized but more refined partition of the t-axis. This is accomplished by the transfer matrix that exploits a representation of complex numbers to map sequences of events onto rotations. *The real numbers, by themselves, do not have enough structure to encode this periodic process.*

We can use the analog clock to reproduce event counting from a reference frame that is moving with constant velocity $-v$ with respect to T. It is assumed that $v < 1$ and in anticipation of the fact that a clock may tick at different rates in different frames, we shall need the extra precision of a high frequency 'frame' clock to detect differing frequency and phase. From Fig[2.b] we can see that the column vector counting states stays in the upper component by a factor of v longer than the stationary case, with the lower component reducing residency time by a factor of v (Eqn.(1, 2)). The transfer matrix will have to reflect this. It will have to modify the residency times in the two states so the phase of the silent event is shifted (Fig[8]). Consider the reference clock T_R at time $\epsilon << 1$. To lowest order in ϵ we have

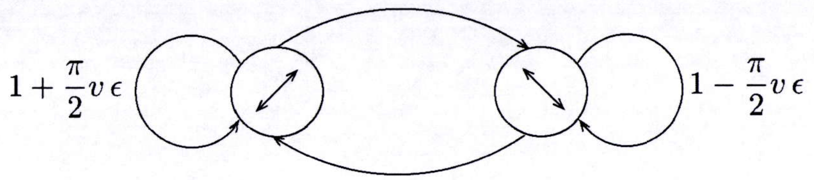


Fig. 8: The transfer matrix alters the residency time of the two states according to the relative velocity of the frame. The residency times are a manifestation of a Doppler effect of the relative motion of source and frame. For the two-photon clock of period 4, the upper state has a residency time increased by a factor of $(\pi/2)v\epsilon$ over the stationary case. Similarly the lower state is decreased by the same proportion. See equations.(1, 2, 15)

$$\begin{aligned} T_R(\epsilon) &= \begin{pmatrix} 1 & -\frac{\pi\epsilon}{2} \\ \frac{\pi\epsilon}{2} & 1 \end{pmatrix} \\ &= \mathrm{I}_2 + (\frac{\pi\epsilon}{2})\,\mathrm{T} \end{aligned} \tag{14}$$

Increasing the residency time in the upper state and decreasing the residency time in the lower state by v gives

$$T_M(\epsilon) = \begin{pmatrix} 1+\frac{\pi}{2}v\epsilon & -\frac{\pi}{2}\epsilon \\ \frac{\pi}{2}\epsilon & 1-\frac{\pi}{2}v\epsilon \end{pmatrix} = \mathrm{I}_2 - \frac{\pi}{2}\epsilon\,(\mathrm{i}\sigma_\mathrm{y} + \mathrm{v}\sigma_\mathrm{z}) \tag{15}$$

If we take the limit of the transfer matrix raised to the power t/ϵ as ϵ goes to zero we get

$$\begin{aligned} T_M(t) &= \begin{pmatrix} \cos\left(\frac{\pi t}{2\gamma}\right) - v\gamma\sin\left(\frac{\pi t}{2\gamma}\right) & -\gamma\sin\left(\frac{\pi t}{2\gamma}\right) \\ \gamma\sin\left(\frac{\pi t}{2\gamma}\right) & \cos\left(\frac{\pi t}{2\gamma}\right) + v\gamma\sin\left(\frac{\pi t}{2\gamma}\right) \end{pmatrix} \\ &= \cos\left(\frac{\pi t}{2\gamma}\right) I_2 - \gamma\sin\left(\frac{\pi t}{2\gamma}\right)(v\sigma_z + i\sigma_y) \end{aligned} \tag{16}$$

Let us check this against the sketch in Fig[2(b)]. According to that sketch, for $v=0$ we should get events at $t \in \{0, 2, 4, \ldots\}$ and we can see from eqn(16) that this is the case since for $v=0$, $\gamma=1$ and the odd part of T_M vanishes at the even integers. Furthermore, the frame clock with relative velocity $-v$ should give us events with inter-arrival times modified by the Lorentz factor γ, ie. $t \in \{0, 2\gamma, 4\gamma, \ldots\}$. This also appears to be the case in eqn(16) since the odd component vanishes at precisely these points, ie.

$$\begin{aligned} T_M(2\gamma n) &= \begin{pmatrix} \cos\left(\frac{\pi(2\gamma n)}{2\gamma}\right) - v\gamma\sin\left(\frac{\pi(2\gamma n)}{2\gamma}\right) & -\gamma\sin\left(\frac{\pi(2\gamma n)}{2\gamma}\right) \\ \gamma\sin\left(\frac{\pi(2\gamma n)}{2\gamma}\right) & \cos\left(\frac{\pi(2\gamma n)}{2\gamma}\right) + v\gamma\sin\left(\frac{\pi(2\gamma n)}{2\gamma}\right) \end{pmatrix} \\ &= \cos(n\pi)\, I_2 \end{aligned} \tag{17}$$

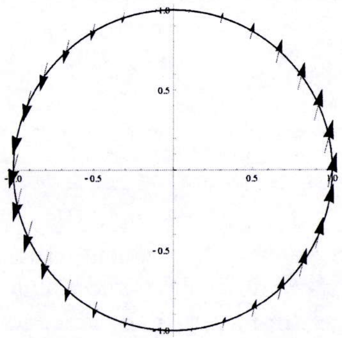


Fig. 9: Minkowski's clock mixes scalar and vector. The vector portion points in the direction of the t-axis in the inertial frame of reference and its length is modulated through the clock cycle.

The transfer matrix for our two-photon clock correctly gives us the locus of points corresponding to events in the frame-clock and it is interesting to see what happens at the silent events. When $v = 0$ the silent events occur at $t \in \{1, 3, 5, \ldots\}$ and for non-zero v we would expect the silent events to occur at $t \in \{\gamma, 3\gamma, 5\gamma, \ldots\}$ and indeed the even term in eqn(16) vanishes there, leaving the odd term

$$r = \pm\gamma(v\sigma_z + i\sigma_y). \tag{18}$$

Notice that if we take the matrix product of this expression with itself we get $-I_2$. If we think of $i\sigma_y$ as a unit vector representing time and σ_z as a unit vector representing space then r is a unit vector with γ representing a normalization factor. The ensemble of all vectors r over all possible values of v (ie. $\{v : |v| < 1\}$) gives the locus of points of unit distance from the origin Fig[10]. The region between the light cone and the hyperbola is the 'unit cell' for our two photon clock. Points within the unit cell correspond to events that our two-photon clock *cannot resolve.* It is interesting to note that the area itself is unbounded and foreshadows the non-locality of quantum mechanics. Note also that the two unit vectors $i\sigma_y$ and σ_z have norms respectively $(i\sigma_y)^2 = -I_2$ and $(\sigma_z)^2 = I_2$ and they are orthogonal under the symmetric product $A \cdot B = (AB + BA)/2$.

It is interesting to note that our transfer matrix T_R is very much like a conventional clock with a rotating hand marking time. In eqn(16), had the coefficients of both trigonometric functions been scalars, the clock would indeed have been a rotating vector. However note that while the even component is a scalar, the odd component is a vector. The clock that we subsequently call Minkowski's clock mixes a scalar and a vector Fig[9].

The reason that Minkowski's clock mixes the two types of objects is that the clock has to accommodate the Lorentz transformation at events. We can see that the reference frame clock does this explicitly at all events. Noticing that r in equation(18)

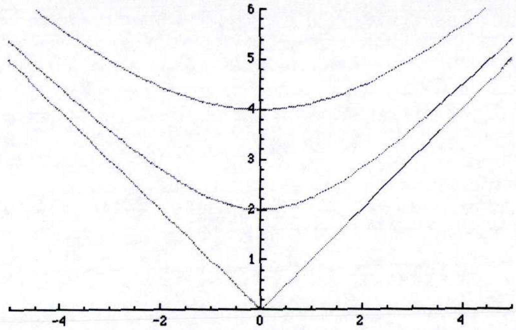


Fig. 10: An ensemble of reference frames with different relative speeds leads to a locus of points of first and second events from the two-photon clock.

is the unit vector of the time axis of the two-photon clock in the moving frame, we should be able to obtain T_M by using a 'rotation' facilitated by the bivector $B = \sigma_z(i\sigma_y)$[15]. In fact we should be able to obtain T_M as

$$T_M = \exp[-B\theta/2]\, T_R \exp[B\theta/2]. \tag{19}$$

Completing the matrix multiplication it can be seen that this is the case provided

$$\theta = \text{arctanh}(\text{v}) \tag{20}$$

as might be expected.

If we are working on length scales much greater than one and can ignore discrepancies in event location of the order of one length unit, we can label points in the plane by treating σ_z and $i\sigma_y$ as unit vectors in the x and t directions respectively, labelling points as $\mathbf{r} = x\sigma_z + it\sigma_y$ giving $r^2 = x^2 - t^2$. This is a version of the Minkowski metric. Our two-photon clock only gives us an event at $\mathbf{r}$ when $\sqrt{t^2 - x^2} = 2n$ for integer n. Using the metric under circumstances where the square root is not an integer is a mathematical convenience appropriate on scales where n would normally be very large.

To get a perspective on scales, the two-photon clock in the simplified units chosen ticks at a frequency $\nu = \frac{\pi}{2}$. Turning to the situation for particles, the relation $E = mc^2$ suggests that all massive particles somehow 'know' their own mass and the invariant speed c. Since c is common to all particles, if we regard an elementary particle of mass m as being a clock described by the two-photon clock model then the appropriate frequency is the Compton frequency $\nu = \frac{mc^2}{\hbar}$ ($\approx 10^{21}$ radians per second for the electron). The clock-particle generates a sequence of periodic events with the inter-event 'distance' being a fixed positive number. The events themselves will be the intersection of two light-like paths as described by the two-photon clock model. Replacing the units in eqn(15) with standard units gives the transfer matrix

$$T_M(t) = \cos\left(\frac{mc^2t}{\gamma\hbar}\right) \mathrm{I}_2 - \gamma \sin\left(\frac{mc^2t}{\hbar\gamma}\right) (\frac{v}{c}\sigma_z + i\sigma_y) \tag{21}$$

where m is the particle mass. If we write $p = m\gamma v$ and $E = m\gamma c^2$ the above is

$$T_M(t) = \cos\left(\frac{mc^2 t}{\gamma\hbar}\right) \mathrm{I}_2 - \sin\left(\frac{mc^2 t}{\hbar\gamma}\right)(pc\sigma_z + iE\sigma_y)/mc^2 \tag{22}$$

and the vector part of the transfer matrix eqn(21) becomes

$$\mathbf{r} = -\gamma(v\sigma_z/c + i\sigma_y) \rightarrow -(pc\sigma_z + iE\sigma_y)/mc^2 \tag{23}$$

so that the invariant squared length is:

$$\mathbf{r}^2 = \left(p^2c^2 - E^2\right)/m^2c^4 = -1 \Rightarrow E^2 = m^2c^4 + p^2c^2 \tag{24}$$

Notice here that the energy and momentum are components of an invariant vector that arises without the usual appeal to conservation principles through dynamics. Mass and energy here are not background attributes but manifestations of the clock mechanism that recognizes the fundamental frequency and the invariant spacetime area.

It is also worthwhile noticing here exactly *why*, according to this model, conventional treatments of special relativity have to put in the mass-energy relation by hand. When we extracted the frame clock T_R eqn.(16), we did this by taking a limit in which the frequency of events between two ticks of the two-photon clock became arbitrarily large. We see from (21) that this is effectively an infinite mass limit!

Discussions about the relations between reference frames are discussions about relations between infinite mass clocks. Finite mass clocks(particles) have a finite number of events in their paths where reference frame and particle clocks agree. Our reference frame has been assigned an infinite mass to interpolate between the ticks of the two-photon clock. However, if we compare two reference frames with each other through a limit of increasing mass for both frames, the invariant proper time relation eqn(18) remains as a relation for clocks of all frequencies. The Lorentz transformation then becomes a relation between frames that has to be augmented with the addition of a background mass whose behaviour is then deduced by dynamics.

Finally, it is worth noting that Minkowski's clock, eqn(22) satisfies the Dirac equation. This may be seen through differentiation, or its relation to the Feynman chessboard model[6],[9].

4 The embedding in 3-D

We started out with the idea of a planar 2-photon clock. Embedding the clock in a reference frame of clocks that tick at higher frequency suggested that the planar mechanism of the clock is neither unique nor necessary. The clock operates using a local rule that allows the clock to keep proper time via an invariant area, however the rectangular area itself is just a convenience. The frame version of the clock sketched in Fig(6), were we to interpret the spiral as an actual path, is an alternative picture of a clock that has the same projected densities as the original if the indicator functions just indicate 'left moving' vs. 'right moving'. Both versions work by sweeping out a fixed spacetime area per unit time, one over integer time steps the other at a continuous rate. In both cases the transfer matrix that keeps track of time evolution does so through projection onto two light-cones corresponding to the spacetime plane defined by the time axes of the clock and the moving frame. The generalization of the transfer matrix is then straightforward. Suppose the relative speed of the frame and the clock is

$$\mathbf{v} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z \tag{25}$$

where the σ_k are the Pauli matrices representing unit vectors. We then have the scalar

$$\mathbf{v}^2 = (v_x^2 + v_y^2 + v_z^2)\mathrm{I}_2 \tag{26}$$

and the transfer matrix for an infinitesimal step, eqn(15) becomes

$$T_M(\epsilon) = \begin{pmatrix} \mathrm{I}_2 + \frac{\pi}{2}\mathbf{v}\,\epsilon & -\frac{\pi}{2}\epsilon\,\mathrm{I}_2 \\ \frac{\pi}{2}\epsilon\,\mathrm{I}_2 & \mathrm{I}_2 - \frac{\pi}{2}\mathbf{v}\,\epsilon \end{pmatrix} = \mathrm{I}_4 - \frac{\pi}{2}\epsilon\,(\sigma_\mathrm{z} \otimes \mathbf{v} + \mathrm{i}\sigma_\mathrm{y} \otimes \mathrm{I}_2) \tag{27}$$

This transfer matrix functions in exactly the same way as the previous version, except that it codes for a reference frame that moves at constant velocity in a three dimensional space. The (1,1) and (2,2) blocks code for the Doppler shifts of the clock, as did the (1, 1) and (2, 2) elements in the two dimensional version. The tick frequency of the clock is unaffected and only the relative velocity between the clock and the frame affects the distribution of events. The continuum limit of this may be taken as in the previous case, using an eigenvalue expansion:

$$T_M(t) = \lim_{\epsilon\to 0} T_M(\epsilon)^{t/\epsilon} = \cos\left(\frac{\pi t}{2\gamma}\right)\mathrm{I}_4 - \frac{\pi}{2}\gamma\sin\left(\frac{\pi \mathrm{t}}{2\gamma}\right)(\mathrm{v_x}\alpha_\mathrm{x} + \mathrm{v_y}\alpha_\mathrm{y} + \mathrm{v_z}\alpha_\mathrm{z} + \mathrm{i}\beta) \tag{28}$$

where the α_k and β are Kronecker products $\alpha_k = \sigma_z \otimes \sigma_k$, $\beta = \sigma_y \otimes \mathrm{I}_2$. The α_k and β are also orthogonal vectors. They are all anticommuting and satisfy $[\alpha_i, \alpha_j] = \delta_{ij}$, $[\alpha_i, \beta] = 0$ and $\beta^2 = \mathrm{I}_4$. The analog of the tick matrix T eqn(8) is

$$i\beta = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{29}$$

It is this matrix that transfers us from event to event and pertains to the actual two-photon clock with period 4. As before, two successive ticks change the sign of the initial state. Similarly we can look at the transfer matrix and 'read off' representations of unit vectors for a reference frame in Minkowski space. The space

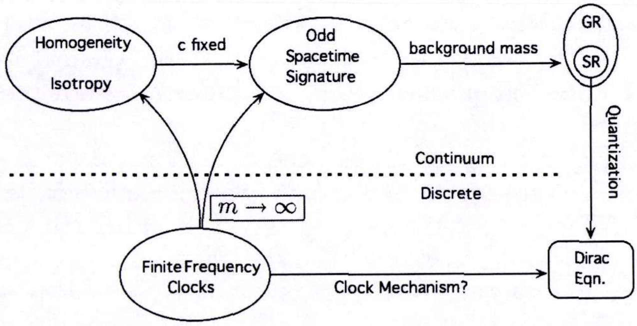


Fig. 11: Discrete vs. Continuum. The conventional route through relativity to quantum mechanics is to assume homogeneity and isotropy (top left) both for physical and mathematical appeal. Adding Einstein's second postulate to fix an invariant speed leads to an odd spacetime signature and ultimately special and general relativity on large scales. On small scales classical mechanics fails and a quantization procedure is necessary to make contact with the Dirac equation. The picture suggested by the two photon clock is that everything above the dotted line is a form of infinite mass limit. The odd spacetime signature is a manifestation of a local rule for finite clocks made global by the infinite mass limit (lower left in figure moving up). The relation to the Dirac equation is easier to see if you start from the discrete perspective, since 'quantization' in this picture arises from the fact that finite mass objects are finite frequency clocks.

vectors are the α_k and the time vector is $i\beta$. The spacetime signature is then $[-,+,+,+]$. Use of these vectors provides a coordinate system with integer spacing at $v = 0$ that corresponds to events of a two photon clock. Giving these vectors real coefficients interpolates between events based on 'large scales' in comparison to the tick sequence. This gives a background frame for the smooth paths of classical particles. Alternatively, keeping the discrete aspect of the vectors and interpolating based on the time interval between ticks allows a very direct path to the Dirac equation, as we shall illustrate in a subsequent paper.

5 Discussion

Any clock, regardless of size, requires some component that accelerates in order to work.The two-photon clock in this paper is a 'minimalist' clock that provides a periodic process giving rise to discrete events. It is minimalist in the sense that it keeps time by using a single local rule of area invariance and calculates the area from only two linear elements of a pair of simple paths. Two successive events then conform to special relativity by being tied to paths that cross periodically, enclosing an invariant area. Using this simple clock, we can take a high frequency limit to

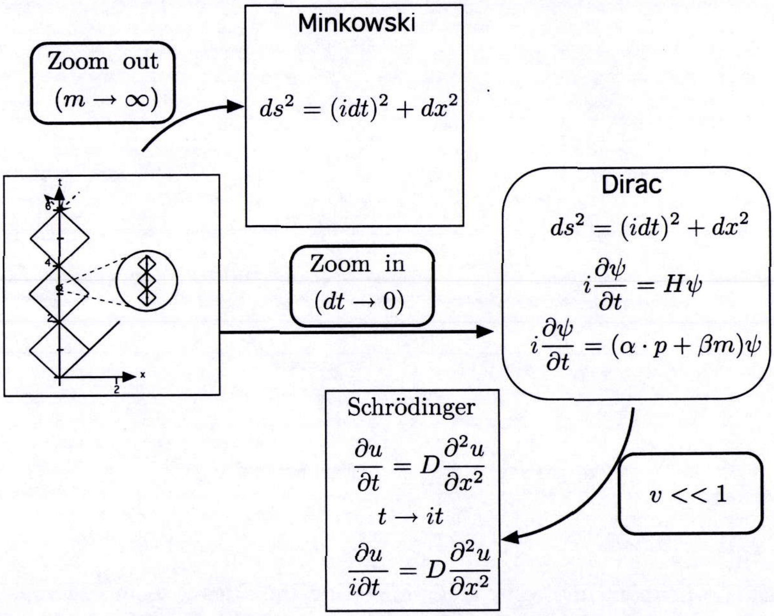


Fig. 12: The clock picture of the relation between special relativity and quantum mechanics. Comparing this to Figure 1, the clock model delays the continuum limit that initiates the conventional picture.

produce a reference frame clock corresponding to an object with arbitrarily large mass. The transfer matrix that counts events at higher frequency does so by simply rotating a three value $\{0, \pm 1\}$ characteristic function in a two-dimensional space, essentially mapping inter-arrival times onto a spiral (Fig[7]). *We can then extract the unit vectors of the rest-frame of the particle as elements of the transfer matrix.*

This route to Minkowski space is longer than the conventional invocation on the basis of a comparison of ds^2 for a light wave-front from two different frames. However, the virtue of this approach is that it give a clearer picture of the role of differential operators and world-lines in both relativity and quantum mechanics.

As suggested by Fig[11], the conventional route implicitly invokes homogeneity and isotropy at the start by using differential operators. This, in combination with the invariant speed c gives rise to the Lorentz transformation which is then augmented through dynamics to give special, and ultimately, general relativity. The Dirac equation is then reached from special relativity by a quantization procedure.

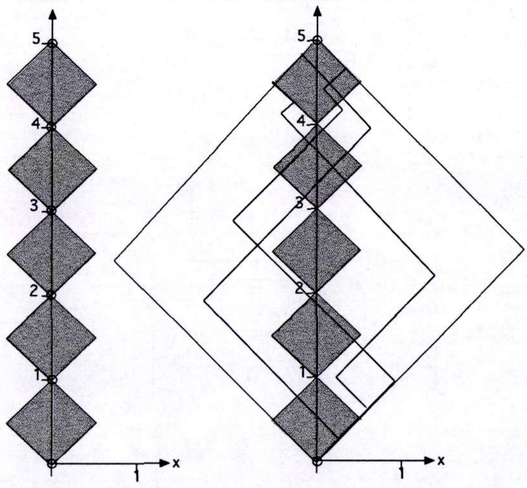
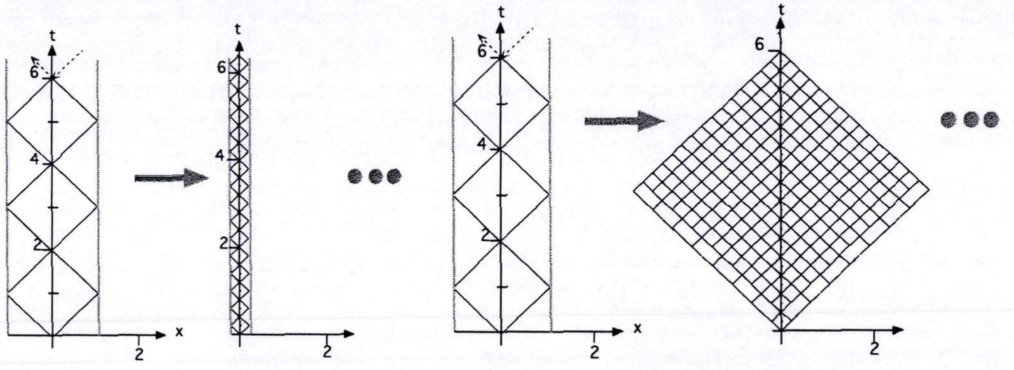


Fig. 13: Minkowski's clock works by encoding both the frequency of events and the direction of the proper time axis. This encoding includes a form of 'wave propagation' between events. In terms of clocks, this wave propagation is a manifestation of the indefinite number of silent ticks between audible ticks in the continuum limit. If audible ticks in a path are made silent, in the sense of not being observed, the spacetime area between events is scaled up to observational levels, providing a basis for quantum propagation.

In contrast, the above paper starts with clocks that are frequency limited. It can be seen that to get to the Lorentz transformation, an infinite mass limit is needed Fig[12]. With that limit as an attribute of a reference frame, the finite frequency clock satisfies a form of the Dirac equation *without any appeal to quantum mechanics* (eqn.(13)). The connection to quantum mechanics should not be a complete surprise since the requirement that clocks be finite frequency is itself a form of quantization. For example if we wrap Minkowski's clock on a cylinder and require that the events lost due to time dilation are integer in number, we get the Bohr angular momentum quantization rule:

$$L = n\hbar \qquad n = 1, 2, \ldots \tag{30}$$

to first order in v^2. This rule played a significant part in old quantum theory and the evolution of modern non-relativistic quantum mechanics. However, here it appears as a consequence of finite frequency relativistic clocks. Notice however it only makes sense in the context of a real particle if we *do not require the audible events around the cylinder to be heard.* If we did we would have 'which way' information about the paths and the clock would stay 'classical'. If the intervening ticks are silent, the



(a) Classical Limit.

(b) Quantum Limit.

Fig. 14: (a) The classical limit of Minkowski's clock keeps the intermediate events audible. As a result, the continuum limit shrinks the width of the clock so that in the continuum limit it becomes a smooth worldline or a clock with infinite frequency. Since the frequency is the analog of mass, this is an infinite mass limit and becomes a 'reference frame clock' in the conventional approach. (b) In the 'quantum' limit, the audible events between observations are made silent, so the spacetime area between first and last event expands. Silent events far from the classical worldline are then part of the clock and the continuum limit expands, rather than contracts the analog of the classical worldline.

Bohr rule just expands the region between the ticks of a clock! Fig[13] Ultimately the difference between quantum and classical behaviour is the form of continuum limit used, one that scales up the region between audible ticks or one that scales it down Fig[14].

6 Conclusions

The objective of this paper was to uncover a statistical mechanics underneath Minkowski space. In doing so we have seen that the Dirac equation appears to be implicated as a result of finite frequency clocks. For this reason the model intersects the work of many authors seeking a better understanding of the Dirac equation. Feynman's Chessboard model is visible in the transfer matrix (19) [1]-[10]. The 'zig-zag' model of Penrose[11] is evident in the Kronecker product version of the transfer matrix(27). The use of zitterbewegung as the basis on which the Dirac equation rests has been emphasized by Hestenes[12] and advocates of geometric algebra [15]. The use of discrete processes to show the parallel emergence of probability density functions and wavefunctions has been shown by Dubois[13, 14]. The replacement of world-lines by chains of *areas* is reminiscent of 'Fractal Spacetime' approaches that exploit the fact that the uncertainty principle implies paths with fractal dimension

$D = 2$ [16]-[19]. The use of events as the alternate convergence and divergence of paths is reminicent of nilpotent quantum mechanics [20].Finally, the partitioning of spacetime into cells by the two photon clock, giving rise to an ensemble average that shares characteristics with wavefunctions is reminicient of the Bohmian pilot-wave picture where the quantum potential that helps guide a particle is created by the particle itself [21].

Although many approaches intersect in this model, they do so primarily from the perspective of quantum mechanics where quantization and mass are put in by hand. The novelty of this picture is that it assigns a different logical status to the relationship between quantum mechanics and special relativity. 'Events' in this picture are discrete and the conventional kinematics of special relativity are seen as idealizations corresponding to infinite mass objects. In this view 'spacetime' is a smoothed idealization arising from massive particles rather than an abstract container, and quantum propagation results from a scaling up of a clock mechanism to observer scales Fig[14].

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