Visual Perception of Polychromatic Flows.
A Systemic Approach Essay. Application to Pattern Recognition

Bonnardot Jean(*) Sabatier Philippe(**), Commerçon Jean-Claude(*)

(*) Institut National des Sciences Appliquées de Lyon, Laboratoires RFV & LISPI,

20, Avenue Albert Einstein F-69621 Villeurbanne Cedex

E-mail : bonnardot@cipcinsa.insa-lyon.fr

(**) Ecole Vétérinaire de Lyon, Unité BioInformatique,

1, Avenue Bourgelat, F-69280 Marcy l'Etoile

E-mail : sabatier@clermont.inra.fr

Summary

Working on the pre-attentive stage φ_1 of perception by the human visual system (HVS), we propose both a modelling of chromatic perception (1) by means of 4 attractors associated to a mathematical symbolism induced by branching theory, and some structures for the shapes -or coloured shapes- processing . This research involved by the computer graphics and neuro-physiology results about recognition of optical signal processing circuits have necessitated to reinvest the visual chain : object (emitter or re-emitter), light information vehicle, the colour receiver, the processors of the SVH. The local physico-mathematical formula set very developed about propagation, has brought solutions for monochromatic waves and for obstacle with smooth edge reveal neither various curvatures (2) nor a fortiori the texture of re-emitting object U(3). A resort to a formula-set about interaction matter radiation, extended to a systemic approach, give an algorithmic stage to take account the spatial character (2) and (3). Taking account on one side this scalar conjecture about visual information, and on another side a retinal functional specialisation, we propose a modelling of C^1 occluding edge detection of 3D-object. C^1 except in isolated points by envelop method. The elaboration of a valuation table of morphemes associating the mathematical characteristics of curves with their pregnance treat the more cognitive stage of processing, associating edge and colour. Some pictures illustrate the abilities of curves family to suggest 3D-shape. The chromaticity of delimited areas strengthen, or rather create, psycho-affective suggests associated to logos.

Keywords :

Flow, energy, spectrum, edge, forms, convexity, envelope, retinal spécialization, bases 3D learning frames.

I. Physical characteristics of visual information carrier: electromagnetic waves $\nu \in [4.28, 6.88]\ 10^{14}$ Hz

1. Undulating/Geometric characters

A. In a point of a direct emitter, the Hertz's experiments had justified a modelling of the phenomenon by means of two vibrating vectors $\vec{E}(t)$, $\vec{B}(t)$, superposition of ω sinusoidal laws of n-arches. The classical propagation laws of each components E^i_ω in a medium and through some dispersive elements are set from the two coupled vectorial equations of Maxwell of three conservative equations and behavioural relations:

$$\text{rot}\ \vec{E}(m,t) = -\frac{\partial \vec{B}}{\partial t},\ \text{rot}\ \vec{H} = j + \frac{\partial D}{\partial t},\ \text{div}\ \vec{D} = \rho_v,\ \text{div}\ \vec{B} = 0\ ;$$

$$\vec{D} = f_1(E),\ \vec{B} = f_2(H),\ \vec{j} = f_3(\vec{E}),\ \text{div}\ \vec{j}_m + \frac{\partial \rho}{\partial t} = 0$$

The various methods of resolution concatenate the tools elaborated from the scalar transversal waves, initiated from the mechanics : phase, wave front, normal propagation speed. The vectorial and random nature of $\vec{E}$ imply an over-investment for which we have evaluated the limits, in order to preserve the modelling faculty for the visual image processing : (1) wave pulsation, (2) phase lag from the emitter, (3) wave front at (M,t), (4) position of $\vec{E}$ in the wave plan.

The retinal photo-receptive layer of the Human Visual System (HVS) is sensitive to pulsation interval categories, to the intensity of the vibrating vector, but neither to the instantaneous phase nor the field direction.

The interferential optics, in a Linear, Homogeneous and Isotropic medium (LHI) constrained to homogeneous wave, hides the point (4) above, and reduce the vectorial wave to a scalar propagation function : $\Psi(M,t)$, progressive wave $\Psi(M,\Delta(M,t))$, where $\Delta(M,t)$ is the propagation term. This solution gives : (a) a positive point : luminance distribution on the 2D-image with the discrete Fourier transform ; (b) a negative point : restriction to a 2D-modelling, and complexity of the connection between chromaticity and geometry.

We will begin with a scalar t-sinusoidal wave : $\Psi(M,t) = A(M)\cos(\omega t - \varphi(M))$, with the prior hypothesis: $A(M)$ varies more slowly than $\varphi(M)$ in order to mix the wave surface and the equi-phase surface.

The function $\Psi(\vec{M},t)$, classically satisfying the same evolution equation spatio-temporal, with the six field components $\square E^i_{\bullet}$ ($\square$ d'Alembertien), is split in:

- an informative equation on φ,

$$\left\|\overrightarrow{\text{grad}}\,\varphi\right\| - k^2(M) = \frac{\Delta^2 A}{A}\ \text{(I), with}\ k = \frac{\omega}{v(M)},\ v(M) = (\varepsilon(M)\mu(M))^{-1/2}.$$

For $\frac{\Delta A}{A} << \frac{\Delta\lambda}{\lambda}$, this equation is reduced to $\overrightarrow{\text{grad}}\,\varphi(M,t) = \vec{k}(M,t)$; this equation will give the geometry of the equi-phase surface, by integrating three partial differential equations.

- a coupling equation,

$$\Delta^2\varphi + \frac{2}{A}\overrightarrow{\text{grad}}\,\varphi\,\overrightarrow{\text{grad}}\,A(M) = 0 \quad \text{(II)}.$$

This equation, associating the amplitude of the vibration to the wave vector $\vec{k}(M)$, implies the conservation of the flux $\operatorname{div} \vec{k}\, A(M)^2 = 0$. The identification of the vector $\vec{k}\, A(M)^2$ with a vector of energy needs to take into account the bi-vectorial structure of the field.

At a microscopic scale and in the vacuum, $\vec{r}$ associates to $\vec{E}, \vec{B}$, the vectorial field normal to the wave $\vec{r}(M,\tau) = \vec{E} \wedge \frac{\vec{B}}{\mu}$. At a sub-microscopic scale, and travelling in a LHI medium, the field of the wave is $\vec{r}(M,\tau) = \overline{\vec{E}}^{e} \wedge \frac{\overline{\vec{B}}^{e}}{\mu}$, its velocity is conditioned by the interaction with the medium.

$$\vec{B} = \frac{\vec{u}}{v} \wedge \vec{E} \Rightarrow \vec{r}(M,\tau) = \frac{1}{v\mu}(\|E(M)\|^2 \vec{u})$$

$\vec{E}(M) = \vec{E}_0(M)\cos(\omega\tau - \varphi(M))$ where $\vec{E}_0(M)$ is a non-homogeneous and non-polarised wave.

The flow expression with a macro-local energetic variable implies to sum up N wave trains : $\frac{\vec{u}_0}{v(M)\mu} \frac{1}{N\Delta\tau} \int_{\tau}^{\tau+N\Delta\tau} \|\vec{E}_0(M)\|^2 \cos(\omega\alpha - \varphi(M))\mathrm{d}\alpha =$

$$\frac{\vec{u}}{2v\mu_0}\|\vec{E}_0(M)\|^2[1 + \frac{1}{2N\omega\Delta\tau}\sin(2\omega(\tau + N\Delta\tau) + \varphi)...] \sim$$

$$\frac{1}{2v\mu_0}\frac{\|\vec{E}_0(M)\|^2}{v} v\vec{u} = \frac{1}{2}\varepsilon(M)\|\vec{E}_0(M)\|^2 v(M)\vec{u}(M) = \frac{1}{2}\rho_E(M)\vec{w}(M)$$

in which ρ_E is the electromagnetic density of energy for a non-polarised wave.

$\overline{\vec{r}_\omega(M,t)}^{N\Delta\tau} = \frac{1}{2}\varepsilon\|\vec{E}_0(M)\|^2 \vec{w}(M)$ for a straight polarisation.

$\overline{\vec{r}_\omega(M,t)}^{N\Delta\tau} = \frac{1}{2}\varepsilon(\|\vec{E}_{10}(M)\|^2 + \|\vec{E}_{20}(M)\|^2)\vec{w}(M)$ for an elliptic polarisation.

In a linear homogeneous isotropic medium (l, h, i) the collinearity between $\overrightarrow{r}(M,t)$ and $\overrightarrow{k}(M,t)$, therefore the tangency of their trajectories, associates energy transmission with those of shape information correlative to shade, but lowers these of polychromatic radiation re-emitted by ∂U itself. The modelling of wave front break caused by an opaque object, superposed to the chromatic scatter by an interaction matter radiation, and so the going through an anisotropic medium imply the bi-vectorial character of fields, and bring us the choice through many methods of resolution.

For propagative equation : $\boxed{\overrightarrow{E}(M,t)}$, with $\Box$ = d'Alembertien,

(III) in an isotropic medium, or $A_{(X)}\overset{\circ}{\circ}d^2E_\alpha + H_\alpha(x^\alpha, \partial_j E_\alpha, t) = 0$,

(IV) in an anisotropic medium $\vec{E}(M,t)=\sum_{1}^{3} E_\alpha \vec{e}_\alpha$

someone research the characteristic surfaces $G_S(x^\alpha,t)=0$, (or $t=g(x^\alpha)$) along which the Cauchy's problem have no solution or no unicity. Their implicit equations satisfies :
${}^{\tau}[dG]_X A[dG]=0$ (F),

that is to say $v^2 \sum_\alpha \left(\frac{\partial g}{\partial x^\alpha}\right)^2 - \left(\frac{\partial g}{\partial t}\right)^2 = 0$ for *(III)*,

or $\sum_\alpha v_\alpha^2 \left(\frac{\partial g}{\partial x^\alpha}\right)^2 - \left(\frac{\partial g}{\partial t}\right)^2 = 0$ for *(IV)* in a reducing base.

The partial differential equation *(F)* of first order (PDE) is obtained from *(III)*, and the PDE of second order by concatenating the first order Maxwell's equations, using a Taylor's solving approach. We prefer to express, in extended Maxwell's equation the relations between spatial and temporal jumps for $\partial\vec{E},\partial\vec{H}$ resulting from propagation.

$$\left[\vec{rot}\,\vec{E}\right]=-\mu[\partial_t H]\Rightarrow\frac{\vec{n}_s}{v_n}\wedge\left[\partial_t\vec{E}\right]=-\mu\left[\partial_t\vec{H}\right]\left[\vec{rot}\,\vec{H}\right]=\varepsilon\left[\partial_t\vec{E}\right]\Rightarrow\frac{\vec{n}_s}{v_n}\wedge\left[\partial_t\vec{H}\right]=\varepsilon\left[\partial_t\vec{E}\right]$$

$$\Rightarrow\left[\left|\vec{n}_s\right|^2-\varepsilon\mu v_n^2\right]\left[\partial_t\vec{E}\right]=0\ ;\ \vec{n}_s\,\partial_t\vec{E}=0\ ;\ [\ ]=\text{jump}$$

if $G(x^\alpha,...,t)=0$ is the equation of front wave (discontinuity surface) :
$\vec{n}_s=\frac{\nabla G}{\|\nabla G\|}$; $v_n=-\frac{\partial_t G}{\|\nabla G\|}$ therefore $\|\nabla G\|^2-\varepsilon\mu(\partial_t G)^2=0$ *(V)*.

This equation (V) gives again the approximation (I') for (I), when $\frac{\Delta A}{A}$ is a negligible quantity, and in the case of a t-sinusoidal wave (C). We have putted attention upon the significative weight of discontinuity on the fine optic formulation. But the choice (C), in modelling perception, bring us the equation (V) for its abilities to reach some framework shape. The shape perception from a contour by the HVS begins in the cellular layers organize its process through area V1, in the visual cortex, layer 4cb perhaps with a feed back toward interchromatic clusters, than the action potential reach pale stripes of area V2 (cf. M. Livingstone). This process seems at this level independent from stereo process (large stripes) and chromatic process (thin stripes). Correlatively to an shape equation, the perception of colours and more of these chromatic properties of the objects, must be modelled in the following parts.

B. The geometry and the graphical art have brought out some key features for the modelling of an opaque 3D object in two dimensions :

(1) the visible true or occluding edge (Γ) is the locus of points where the tangent plan of S is parallel to direction view $\vec{V}$ $\Gamma=S\cap S_C$ S_C defined by $\nabla\vec{F}.\vec{V}=0$ if $F(x,y,z)=0$ is implicit equation of S The projections γ of such Γ on to receiving

surfaces give information for identify Ω. The γ singularities, stable through Ω movements, permit to detect significant convexity, or internal concavity on $\partial\Omega$.

(2) Some line family on $\partial\Omega$ (crest line, thalweg) can be obtained by algorithms [cf. INSA-RFV laboratory]

For sufficiently smooth surfaces (defined by parameters), we retain the lines of curvature, that is to say, the extremum of the normal curvature $c_n(\vec{\tau}) = \frac{1}{H}\frac{[\vec{\tau}]D[\vec{\tau}]}{[\vec{\tau}]G[\vec{\tau}]}$, where $[\vec{\tau}] = \begin{bmatrix} u \\ v \end{bmatrix}$ is the contravariant components of $\vec{\tau}$ in a local basis $\vec{M}'_u, \vec{M}'_v$, and G is a matrix giving the metrics in the tangential plane. The local values $\overline{c_n}, \underline{c_n}$ are the eigenvalues of an endomorphism $G^{-1}D$: $\det[D - \lambda G] = 0$, $\overline{c_n}\,\underline{c_n} = \frac{1}{H^2}\frac{\det D}{\det G} = \frac{\det[d^2 f]}{[1 + f_x'^2 + f_y'^2]}$, if $z = f(x,y)$ in the referential $(O, \vec{e}_1, \vec{e}_2)$

$$\tfrac{1}{2}(\overline{c_n} + \underline{c_n}) = \tfrac{1}{2}\frac{\det\begin{bmatrix} D & F \\ D' & G \end{bmatrix} + \det\begin{bmatrix} E & D' \\ F & D'' \end{bmatrix}}{(\det G)^{3/2}} = \frac{Trace\ d^2 f}{H^2}$$

, if $(\vec{e}_1, \vec{e}_2)$ is parallel to tangent plane at point M.

If $\underline{c}_n \overline{c}_n > 0$, M is an elliptic point (<0, an hyperbolic point). But mainly $\underline{c}_n \overline{c}_n = 0$ imply either an inflexion point on one of the both curvature line l_c crossing at M, or a rectitude of l_c. The parametric equations of l_c in a curvilinear pattern is obtained by resolution of $\det\left[D\begin{pmatrix} du \\ dv \end{pmatrix}, G\begin{pmatrix} du \\ dv \end{pmatrix}\right] = 0$. For an ellipsoid $\{l_c\}$ suggest very well a 3D shape : for they surround a focal area Γ as edge wavelets emitted by punctual source F, and their asymmetry so that tangency at occluding contour suggest a scan with refection (see Fig. 1 ; 2). The two families of geodesic curves (osculator plan orthogonal to S) seems informative only local convexities. Their metric, built with extremal methods, give modelling of proximity from colour A to colour B, on 3D pattern colorimetry.

2. Pre-quantum features : receiving a radiation

We adapt the balance equation of motion quantity for a gas of particles to a gas of photons $(\frac{\partial}{\partial t} + \vec{W}_\varphi \vec{\nabla}_r)(f_\varphi(\vec{r}, \vec{p}_\varphi, t)) = (\frac{\partial f_\varphi}{\partial t})_C$; the collision term is difficult to express.

We integrate on the distribution of $\vec{p}_\varphi$:

$$\int \vec{p}_\varphi \frac{\partial f_\varphi}{\partial t} dp = \frac{\partial}{\partial t}\int pf\, dp - \int f \frac{\partial p}{\partial t} dp = \frac{\partial}{\partial t} n(\vec{r}, t) < p_\varphi > I \quad (\frac{\partial p}{\partial t} = 0, \nabla_r P = 0)$$

$$\int (\vec{W}_\varphi \nabla_r f) p\, dp = \vec{W}_\varphi \int p \nabla_r f(r, p, t) dp = \vec{W}_\varphi \int \nabla_r pf dp = \vec{W}_\varphi \nabla_r n < p >$$

also $\frac{\partial}{\partial t} n < p > + \vec{W}\nabla_r n < p > = \frac{\partial}{\partial t}[n < p >]_C$; $p = \frac{h\nu}{c} \Rightarrow n < p > = \frac{n}{c} < h\nu > = \frac{1}{c}\rho_E$

who is the radiative density of energy linked to macroscopic energetical intensity by

$$\int I_\varphi d\Omega = c\rho_E = c^2 n < p > \frac{\partial}{\partial t} I_\varphi(\vec{r},\vec{\omega},t) n < p > + \vec{W}_\varphi \nabla_r I_\varphi = (\frac{\partial I_\varphi}{\partial t})_C$$

As the retinal receptor level these energetical relations of radiative transfer base the microscopic filiation of I_φ with the kinetics $\vec{W}_\varphi$ of photons and justify the reject of vectorial character in visual information.

3. The colour solution for the polychromatic flux perception by the Human Visual System (HVS)

The experiment on the trichromatic synthesis of a white reference stimulus, and of an any stimulus C , are quantified on the visual luminance (lumen by steradian.m^2):

$L_w = L_{WR} + L_{WG} + L_{WB}$, $\qquad L_C = L_{CR} + L_{CG} + L_{CB}$ $\qquad$ (I)

They lead:

- to a scalar modelling $L_w = R.L_{WR} + G.L_{WG} + B.L_{WB}$ (I') by means of three non-dimensional numbers: R, G, B ; the chromatic components of the C stimulus shows the visual luminance proportions of each of the three reference components which are necessary to the synthesis of C ;
- to a vector modelling $\vec{OC} = R.\vec{OL_R} + G.\vec{OL_G} + B.\vec{OL_B}$ (II) in an oblique referential $(\vec{OL_R}, \vec{OL_G}, \vec{OL_B})$ of ε^3. This vector $\vec{OC}$ has a length associated to the global information, luminance of C: $L_C = \int l_e(\lambda)v(\lambda)K_m d\lambda$, but $\|\vec{OC}\|_2$ is equal to L_C only if C is one of the three reference stimulus. Equation (II) induces on the ε^3 space the norm 1, $\|\vec{OC}\|_1 = |R.L_{WR}| + |G.L_{WG}| + |RL_{WB}|$, so L_C if R,G,B are positive. The synthesis of (quasi) monochromatic light by step of 10nm allows to build an open (relatively to $\vec{OL_B}, \vec{OL_R}$) and convex modelling cone Σ, limited to $\lambda_1 = 0.38\mu$m and $\lambda_2 = 0.78\mu$m. The angular position of $\vec{OC}$ relatively to the referential axes characterizes the chromaticity . We prefer to compare the stimulus of the same global luminance $L_C = L_{W_0}$, defining a plan Π, and where W_0 is white reference. The various chromaticities are modelled by the interior points of the curve $\gamma_\lambda = \Sigma \cap \Pi$ which is closed by the purple line BR which cannot be associated with monochromatic light. The chromaticity diagram $T = \gamma_\lambda \cup BR$ experimentally build is approached by a (rounded) triangle ; its vertices models the directing stimulus associated to the maximum of the sensitivity of the three cone families ; its central region models the more polychromatic stimulus. T is a very suggestive representation of the category perception of the colours by means of four attractors. In an approach adapted from the bifurcation theory, we induce a symbolisation of a leading strategy developed by this perception way.

The trichromatic synthesis of the monochromatic stimulus of equal energetic luminance $l_e(\lambda) =$ constant have allowed the determination of the mean spectral sensitivity $\bar{r}(\lambda), \bar{g}(\lambda), \bar{b}(\lambda)$ related to the trans-spectral sensitivity of the cones in diurnal vision $v(\lambda)$ by the energetic equation equality $l_e(\lambda)v(\lambda)K_m\Delta\lambda = \bar{r}(\lambda)L_{WR}\Delta\lambda + \bar{g}(\lambda)L_{WG}\Delta\lambda + \bar{b}(\lambda)L_{WB}\Delta\lambda$, *i.e.* $\bar{r}(\lambda) + 4.5907\bar{g}(\lambda) + 0.0601\bar{b}(\lambda) = v_\gamma$. The graph of these three colour matching functions have shapes and relative positions very near to the ones giving the photon number absorbed by a cone of each sort [Nathan]. The macro-curves $\bar{r}(\lambda), \bar{b}(\lambda)$ (mainly $\bar{r}(\lambda)$) have a zone of negative values which makes a problem with the energetic relation (equation I above).

The mathematics transformations (changes of the main stimulus basis) throw out the above difficulty, but we move away from the experimental support. We still have a cone and a colorimetric 'triangle' in the space X,Y,Z (CIE). On this triangle, we deduce usually more synthetic information than the one obtained from the luminance curve $l_v(\lambda)$ of a polychromatic stimulus $C, C \notin WBR$ (purple triangle).

i) The C dominant wavelength (tint) $\lambda_d = CW \cap \gamma_\lambda$ is more directly obtained by cutting out of regions under the $l_v(\lambda)$ curve, then taking their barycentre.

ii) The global luminance $L_c = \int_\lambda l_v(\lambda)\mathrm{d}\lambda = K_m \int_\lambda v_\lambda l_e(\lambda)\mathrm{d}\lambda$ is an energetic information; it influence the tint perception.

iii) The purity P_c' or saturation determination use some numerical information taken from i) point (or by direct measuring colorimetric purity $P_c = \tilde{L}_d / L_C$, $\tilde{L}_d$ is obtained by adjusting the λ_d and white luminance to fit the relation $\tilde{L}_d + \tilde{L}_w = L_C$).

$$P_c' = \frac{L_C - \tilde{L}_d}{L_C} \frac{\tilde{L}_d}{\tilde{L}_d - L_w} = \frac{y_d}{y_c} \frac{y_c - y_w}{y_d - y_w}$$

If $C \in WBR$, then C is a purple and have no dominant wavelength, but have a complementary colour, and a 'dominant purple' $p = WC \cap BR$. We define the purple purity by $\tilde{L}_p / L_C$, with $\tilde{L}_p$ being adjusted from $\tilde{L}_p + \tilde{L}_w = L_C$.

The colorimetric analytic experiments have allowed the building of colour matching functions implied in the integral formulation of the trichromatic vision, thanks to the spectral slices obtained by means of filters and thanks to the experimental codified tests. But the SVH at the retinal level, does not know all the details of spectral reflectance laws proper to each object of the scene. At first glance, the SVH gets access to global information of LTS type, and by interactive information exchange with the other information processing systems, the SVH improve its perception progressively.
We focus principally on the interactions between colour and pattern perception. The luminance of the direct sources (1) and the reflectance R of the re-emitting objects (2) depend of the direction of the irradiation and of the wavelength; more, (2) also depends of re-emitting direction D_r captured by the retina. The dependency of D_r relatively to $R(D_i, D_r \lambda)$ allows to reach the 3D perception in the monocular vision. A sufficient understanding of the physical phenomenon occurring in the radiation/matter interactions lead us to make choice through a formula set associating microscopic theory and macroscopic functions.

II. **Retro-diffused fluxes, 3D pattern recognition**

The perception of a bright blob inside a circular contour strongly suggests the spherical property of a bowl Ω, illuminated in a given direction. Moreover, only the re-emissions out of the blob bring information about the Ω colour. So, the pattern and the colour perception seems separated at this level.
The modelling of these facts in image synthesis (ray tracing) have brought out a formula tree. A source emits a field $(\vec{E}, \vec{B})$, that we assume to be a plane wave; its Poynting vector is:

$$P(\tau, M) = \vec{E} \wedge \vec{H} = \sqrt{\varepsilon_0 / \mu_0} n_M \|\vec{E}(\tau, M)\|^2 \vec{v}_\Pi \text{ (I)}$$

For a spherical wave, we keep (I), but we choose for $\|..\|$ the law $\frac{b}{\left\|\vec{OM}\right\|}\cos(\omega t - kr)$.

The generated flux by the Poynting field is:

$$d\phi_{\mu}(\tau, M) = \vec{p}(\tau, M).\vec{N}_{\Sigma}d\Sigma = n_{M}.\frac{\sqrt{\varepsilon_0/\mu_0}.b^2}{\left\|\vec{OM}\right\|^2}.\cos^2(\omega t - kr).d\Sigma$$

By temporal means, we obtain: $\overline{d\phi_{\mu}}^{\tau\_\_\_\_(N+1)\tau=t} = d\phi(t, M)$, which is the captured flux at a macroscopic scale, that is to say: $d\phi(t, M) = \frac{n_M}{2}.\frac{\sqrt{\varepsilon_0/\mu_0}.b^2}{\left\|\vec{OM}\right\|^2}.d\Sigma$. Let $a = \sqrt{\varepsilon_0/\mu_0}.b^2$.

To obtain the macroscopic formula, we must satisfy the two following point:

- put a macroscopic time dependence to a, that is to say, $a(t)$,
- put $\frac{d\Sigma}{\left\|\vec{OM}\right\|^2} = d\Omega$, the solid angle in which we see the patch $d\Sigma$ from the source O.

So, we can write the classical macroscopic result: $d\phi = I\, d\Omega$, where;

- $d\Omega$ is the solid angle in which we see a geometric contour of capture, non necessary orthogonal to the beam axis (we notice that the $1/r^2$ physical dependency is taken into account by the geometrical term $d\Omega$),
- I is the intensity, that is to say a macroscopic entity, *a priori* defined, and for which the dependency from the emitting direction $I(^{\theta}_{\varphi}, \lambda)$ has been experimentally confirmed. This dependency results from the macroscopic structure of the source O, by a subtle process which is not necessary to take into account in this paper.

For an expanded source *ds*, we define two properties :

- 1. Intensity through *dW* by *dI (dSe, De, l)* : $dI(dS_e, D_e, \lambda) = \frac{d^2\phi\left(\begin{matrix}\theta\\ \varphi\end{matrix}, d\Omega\right)}{\underset{I\rightarrow}{d\Omega_i}}$

- 2. Radiance (energetic luminance) $L_M\left(\begin{matrix}\theta\\ \varphi\end{matrix}, \lambda\right) = \frac{d^2\phi}{\underset{\cdot\rightarrow}{d\Omega_i}.dS.\cos\theta_e}$

Whatever the shape of irradiative wave, we define from a macroscopic point of view the irradiation of the surface *ds* under an over-directional flux $d\phi$: $dE(\lambda) = \frac{d\phi(\lambda)}{dS}$

If dS' is irradiated by :

1. a punctual emitter, the intensity of which is *I*,

$d\phi = I(\lambda).\underset{\rightarrow dS'}{d\Omega} = I(\lambda).\frac{dS'.\cos\theta_r}{\rho^2}$: $E(\lambda) = I(\lambda).\frac{\cos\theta_r}{\rho^2}$

2. an expending emitter ds with luminance towards D_i.

$$L(D_i,\lambda)=\frac{d^2\phi}{d\omega.dS.\cos\theta_1} \;;\; \theta_1=\left(\widehat{\vec{N_S},E_R}\right);$$
$\rightarrow I$

The partial irradiance from dS will be

$$dE_{(D_i)}=\frac{d^2\phi}{dS'}=\frac{L(D_i,\lambda)}{dS'}.d\omega.dS.\cos\theta_1$$
$\rightarrow I$

$$dE_{(D_i)}=L(D_i,\lambda).\frac{dS.\cos\theta_1}{r^2}.\cos\theta_2 \text{ with } \underset{\leftarrow}{d\Omega}=\frac{dS.\cos\theta_1}{r^2}$$

The object U (surface ∂U) behave like a re-emitter (emitting source).

We can define its radiance (luminance) but here upon particular conditions of radiation $dL_r(D_i,D_r,\lambda_i,E_{inc})$; with D_r : direction of re-emission. Inducing the linear density of L we define $R(D_i,D_r,\lambda_i)=\frac{dL_r(...E_{inc})}{dE(D_i,\lambda_i))}$, that is the bi-directional distribution function of radiance emitted in D_r. The radiative diagram of $R(D_i,D_r,\lambda_i)$ correlated with D_r has for usual emitter two parts : l_1 a semi-circular lobe center O ; l_2 a maximum for D_r symmetric of D_i, with respect to the normal $\vec{N_S}$ surrounded with a normal lobe l_s.

The functional types of R relatively to its attributes, their relations with macro-experimental lobes result from fine interaction radiation-matter process through upper layers of ∂U. The successive experiments have drawn three reflections process :

(P1). the strict specular emission (reflection) - early geometrically characterised by the Snell-Descartes's laws, and the Fresnel's wave field model for the photometric properties. This rate denote the reflected flux under ($-q_i$) / the incident flux for a monochromatic unpolarised wave). $\bar{R}=1/2(R_{//}+R_{\perp})=\bar{r}^2$ where,

$$R_{//}=\left(\frac{n\cos\theta_i-\cos\theta_t}{n\cos\theta_i-\cos\theta_t}\right)^2;R_{\perp}=\left(\frac{\cos\theta_i-n\cos\theta_t}{n\cos\theta_i-n\cos\theta_t}\right)^2$$

obtained with continuity of fields components $(\vec{E},\vec{B})$ through the first atomic layer C_1 excited at incidence frequency $v_i=\frac{w_i}{2\pi}$ and imply a low dependence of $\bar{R}$ with the optical properties of U. The independence of $\bar{r}=\frac{E_r}{E_i}$ relatively to v_i corroborate this result : for incident light with spectral law $l(\lambda)$: those of specular emission will be homothetic.

(P2). excitations of middle layers, starting up an emission with directional structure, revealed by the specular lobe. This process disrupt lightly $l(\lambda)$. Numerical formulation must be developed.

(P3). diffused re-emissions with deep layers. The modelling initially designed with a differential balance, has been made precise with a diffusion approach (Chandrasekar, Silvy, Ballian). In this study, the classic pattern of *(P3)*, the resonance between vibratory fields and atomic oscillators, will constitute an explication base for the strong disruption of $I(\lambda)$ by re-emission. This one is macro-schematised with a reflection spectral curve : $R(\lambda)$ characteristic of deep colour, and little affected by incident direction.

According to process developed in systemic we extend the independence between *(P1)*, *(P2)*, *(P3)* and the linear equation in a point : $R(D_i, D_r) = sR_s + dR_d$, with splitting some variables.
We propose the following equation *(M)* :

$$R(D_i, D_v, l_i, l_{rm}) = sR_s(D_i, D_v, l_i) + m_r R_{sl}(D_i, D_v, l_i) + dR_{dp}(D_v, l_{rm})$$

$$= I \qquad\qquad + II \qquad\qquad + III$$

λ_m characteristical (wave length) re-emitted by U ;
D_V : collimated re-emitting direction.
sR_s, $m_r R_{sl}$ take into account the geometry of ∂U relatively to incident light, and direction of view, so the convexity of ∂U and the texture (in case of a rough surface). They are $I(\lambda)$ dependant,
R_{dp} is a little variable compared with D_V, and assume the colour of U :

The empirical equation of Phong. (Per, 1988),

$$I + II = \int (sR_s(\lambda_i) + m_r R_d(\lambda_i)) d\lambda_i = k_s \cos(\theta_v - \theta_i)^b ,$$

with $b \in [1,...,200]$ which measured the thickness of specular pick, treat smooth surface, but occult convexity and chromaticity.
A multiplicative modelling of *(P1)U(P2)* (out of spectral property) given by Horn (Cook, 1982).

$$dI_{reemitted} = \left| k\left(\vec{N}.\vec{L}.\right) + (1-k)\frac{b+1}{2}\left(\vec{E}.\vec{V}.\right)^b \right| \rho.E_i$$ (r albedo of S constant on $[\lambda_b, \lambda_r]$)

This equation, computed by Baron (1991), has given the specular reflection direction, by means of iso-intensity contours in an focal plan.

Hata (1992) use a set of monochromatic lights r, o, y, g, b, p surrounding a smooth test object U, to mark chromatically the tilt angle φ of specular normal in convex area on *U*. Then by computing a Sobel's gradient detector on the iso-φ family give out the polar angle θ. He access to the difference of deepness on ∂U, by means of normal direction of specular reflection areas, well oriented for the geometry (source, camera) and distance calibration. Then he deduce the shape of *U*. A formula for R_s including the scattering role of micro-geometry by Cook and Torrance (82) and implemented by computer graphics (Per, 1988) gift good results. The surfaces was characterised by smooth area near textured one :

$$R_S = \frac{1}{4\pi} F(\theta_i, \lambda_i) \frac{D.G}{\left(\vec{N}.\vec{L}\right)\left(\vec{N}.\vec{V}\right)}$$

F : Fresnel's factor linking the geometry of incidence, and the spectral law of the source

$D\left(\widehat{\vec{N},\vec{B}}\right)$ measure of facet distribution following the macroscopic normal

$$D = \frac{1}{m^2 \cos^4 \alpha} e^{-\left(\frac{tg\alpha}{m}\right)^2}$$, m roughness factor

$$G = Min\left(1, 2\frac{\left(\vec{N}.\vec{V}\right)\left(\vec{N}.\vec{B}\right)}{\vec{V}.\vec{B}}, 2\frac{\left(\vec{N}.\vec{L}\right)\left(\vec{N}.\vec{B}\right)}{\vec{V}.\vec{B}}\right) ; \vec{B} = Bisector\left(\widehat{\vec{L},\vec{V}}\right)$$

In visual perception, the first detected property is the global shape (with 3D occluding contour) then internal convexity area, well suggested by specular upper radiance (achromatic for white source). The 3D micro-geometrical details are not perceived straightly, but by a decreased reflectance, and a softened dominant wave length. So we suggest a condensed matter modelling of R with two chromatic components.

$$R(D_i, D_v, \lambda_i, \lambda_m) = \left[\delta(D_v - D-_i)\alpha F + \cos(\theta_v - \theta_i)^b (1-\alpha) F\right] C_s(\lambda_i) + \mu_p(i,e,g) C_p\left(\lambda m / \lambda_i\right)$$ with :

$$i = \left(\widehat{\vec{L},\vec{N}}\right) ; e = \left(\widehat{\vec{V},\vec{L}}\right) ; i = \left(\widehat{\vec{L},\vec{V}}\right)$$

C_S with flat variation in l_i (constant in NIR pattern) m_P geometrical factor of deep diffusion following flat law.

The eye and usual captors detect collimated power fluxes, therefore the density :

$$\frac{d\phi}{d\Omega.dS_0} = dL_r = R(D_i, D_v, \lambda_i, ...).dE(D_i, \lambda_i) ;$$

D_i : incidence direction upon U, R, its bi-directional reflectance

dE : illuminance upon U.

R_λ : part of l reflected by dS (bi-directional reflectance)

$\vec{L}$: direction of the illuminance

$\vec{V}$: direction of the eyesight

b : quantification of the narrowness of the peak of specularity

E : received energy

d : part of the diffuse luminous reflectance

The local instantaneous physical modelling, at the macroscopic scale, must be corrected to assume the three usual temporal constraints :
(C1) response lapse of a retinal network ;
(C2) saturation of such a circuit
(C3) colour remanance, and permit for U to access at his shape in this formulation ... of re-emitted radiation. The separation of local geometry of ∂U from the chromatic properties of U, is an approximation necessitated by limitations of time computation. Otherwise among the three or forth visual treating system drawn in neuro-physiology, and experimental physiology to detect separately shapes from colours. But Zeky have conceived a circuit initialized in parvo-cellular layers ; crossing V1, the inter-blobs spaces, then in folds of V2, than reach the specialised area V4 who associate processing shapes and those of colour. V4 will be correlated with a less analytical area (L.O) proceeding to the shape detection upon fuzzy contours, or coloured picture with interaction to a nearby specific memory area. The identification of object in a scene is activated in the infero-temporal cortex that is the general area of visual memory (Buser, Imbert, 1986)

These two parts have bring out the main physico-mathematical frameworks, used in the modelling of objective, or sub-objectives, properties of human vision phenomenon (e.g. colorimetry). The third part, waiting a possible synthesis, add to these equations, propositions in modelling first level phenomenon of perception act. The temporal dependence will only be implicit. A more explicit approach will find its basis in the systemic theory, and the dynamical systems. Vallée (1979, 1995), working on the evolutional model $dX/dt = -A(t)X(t) + V(t)$ *(S)* has :

- in one hand revealed fine conditions on the resolvant F, convolution kernel of solutions,
- and in another hand quantified the perception by *(S)* of external influences, with functional shapes for $V(t) = \left(\sum_{k=0}^{S} B_k(t).U_t^{(k)},\ \int_{-\infty}^{t} \beta_k(t,\tau) U_{(\tau)}^{(k)} d\tau \right)$

Perceptual interpretation have permitted to bring out the mathematics of their axiomatic neutrality. The adaptation to spatio-temporal dependencies in visual perception, will set up dynamical links through a very diversified formula set.

III. Retinal image : modelling of the treatment by the Human Visual System (HVS)

The numerous steps of the analytical process of simulation, of some 130 $\overline{\overline{M}}$ photoreceivers have bee well modelled by neural networks, whose synaptic weights elaborates through fine electro-physiologic experiences and psychological tests. But, perception by the HSV don't systematically active all these process. For temporal, energetical, and intellectual economy they reduced the feed-back, in culling for primal visual experiences, and conditioned reflex. This work purpose some holistic visual strategy, induced by the retinal functional vision for the fast perception of coloured objects, and of their spatial structures, this after learning stages of writing and elementary geometry. We remind the functional abilities of retinal areas and suggest some subdivisions.

(1) z_f, foveal area, scanning area, angular field Ω_1 $\theta = \pm 2°$, exclusively populate with great acute cones, distributed among three families S.M.L. with spectral sensibilities : $l_b(\lambda), l_g(\lambda), l_r(\lambda)$: l_g... nearby l_r. The foveal cones, have distinct connections with ganglions fibres, but also intra-lateral connections, and extra-lateral cell which permit pre-processing of chromatic contrast then by superposing signals, those of luminance (Walraven model).

(2) z_{S1}, first supervision area, $\{(\theta = \pm 8°) - \Omega_1\} \cup z_{f'}$ with : $z_{f'} : \theta = [-2,-1] \cup [1,2]$, (i.e. $z_{f'} \subset z_f$). $z_{S1} \cup z_{f'}$ participate in the colour vision, associated with those of shape, by means of more dense lateral connections between cones and macular rods linked with horizontal and amacrine cells.

(3) We conjecture that the photoreceptors located in the part of the area z_{s2} (for $\theta \in \pm[8,15]$) and those of the induce sensitive area z_m (for $\theta \in [15°, 30°]$) are dedicated to the shape processing system : detection of the contours ∂U (edge contour, or occluding contour upon a punctual lighting) ; valuation of the convexities inside the view cone, by the use of the luminance singularities (specular over-reflectance, extinction scattering in hollow or textured area) also using the acute abilities of the fovea.

For an object having C^1 border, we suggest a preliminary phase of visual exploration based on a scanning of the shape by spherical wavelet and by the re-construction of the contour as an envelop. This expanding process of covering must be compared with the morpho-mathematical operation (dilatation), but the objectives are inverted. Optically, the front wave family, reflected by the observed object U, carrying the geometrical information upon U, is theoretically obtained by resolving the boundary problem : $\left|\overrightarrow{grad}\psi(M,t)\right| = k_0 r(M)$ $\partial\psi(\partial U) = \psi_0$, or in homogenous medium : $\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 = a^2$. Equation of constant "slope" surfaces, in space *(x, y, z, u)*. If we have working out the complex : $F(x,y,z,u;\lambda,\mu,\nu) = o(I)$ of a complete integral, we deduce the various kind of solutions, by forming implicit equation of envelopes of each subset of surfaces included in (I). We have used envelope method to study mirror caustic. The most classical, the semi-cylindrical mirror lighted with a beam from the infinite in its plan of symmetry, have a caustic : the nephroïd.

This caustic is built with a minimum of three geometrical process : envelop of reflected ray $\{\delta_\alpha\}$ calculated upon a parametrical equation of δ_α; epicycloid with two turning cusp points, by implicit method introducing a Thom's potential of 4th degree, at last as a chryzode.

We propose a mathematical structure which synthesises this quick process of the first chromatic perception from four attractors, Red, Green, Blue, and the White, which correspond to the three retinal categories and to the one of the solar illuminance. In the framework of the bifurcation theory, we can find the simplest potential $F(x,y,u,v,w)$ such as :

(C1) F generate four critical points : $dF(s)=0$
(C2) F is hyperbolic $\left[d^2F(s)\right]$ with a spectrum $\lambda\ (x,y)$ hetero-signed on three points (S_i) in the neighbouring of the fourth (S_0), and mono-signed on this last point (in order to categorize the direction $\overrightarrow{S_iS_j}$ in relation to $\overrightarrow{S_iS_0}$ eigen-vector of $\left[d^2F(s)\right]$
(C3) $\lambda(S_i)$ is equi-valued on the circle centred on S_0 (a kind of propagation of the influence of S_0 on the critical points).

(C1) implicate a polynomial of the 3rd degree :
$f(x,y)=d_{3,0}x^3+d_{0,3}y^3+d_{2,1}x^2y+d_{1,2}xy^2+a_{2,0}x^2+a_{0,2}y^2+a_{1,1}xy+b_{1,0}x+b_{0,1}y$
which can produce, by section of two conics, four critical points.
For $b_{1,0}=b_{0,1}=0$; $S_0=(0,0)$. Then to satisfy (C3), it is necessary that :
$\lambda(x,y)=\phi(x^2+y^2)$ *(C'3)*
The characteristic polynomial can be wrote with the Bocher's formula :
$C_2(\lambda)=\lambda^2 TrA(x,y)\lambda+\frac{1}{2}\left[(TrA(x,y))^2-(TrA^2(x,y))\right]$ with :

$$A=\left[d^2f(x,y)\right]=\begin{bmatrix}a_{2,0}+l(x,y) & a_{1,1}+m(x,y)\\ a_{1,1}+m(x,y) & a_{0,2}+n(x,y)\end{bmatrix}\quad\begin{cases}l(x,y)=3d_{3,0}x+d_{2,1}y\\ m(x,y)=d_{2,1}x+d_{1,2}y\\ n(x,y)=d_{1,2}x+3d_{0,3}y\end{cases}$$

(C'3) implicate that : $TrA(x,y)=\varphi_1(x^2+y^2)\Rightarrow l(x,v)+n(x,y)=0$;
the choice $a_{11}=a_{22}=u; a_{1,2}=0$ conduit à :

$$\det A=\left(2a_{2,0}^2-(l^2+m^2)\right)=\left(a_{2,0}^2+\left[\left(d_{1,2}^2+d_{2,1}^2\right)\left(x^2+y^2\right)\right]\right)=\varphi_2\left(x^2+y^2\right)$$

$f(x,y)=-\frac{d_{1,2}}{3}x^3+d_{2,1}x^2y+d_{1,2}xy^2+d(x^2+y^2)-\frac{d_{2,1}}{3}y$ which verify *(C3)* and have a difficult straight processing.
(C2) can be realized if : $f(x,y)=U\times(x^2+y^2)+H(x,y)$; $H(x,y)$ is an harmonical function, well-known to have harmonical critical points. The subspace $\boldsymbol{P_3}$ of polynomials of 3rd degree, with 2 variables agree the base : $x(x^2-3y^2)$, $y(y^2-3x^2)$.

The final result is $f_{u,v}(x,y) = x^3 - 3xy^2 + u(x^2 + y^2) + vx + wy$. This is one of the seven potential elaborate by René Thom : the elliptic ombilic. This polynomial is generated by a spreading of the singularity $x\left(x^2 - 3y^2\right)$ A discuss on values of control parameter *(u,v)* have permit to identify the stable states near the critical point S_0.

Morever the experimental verifications of diffraction figures, calculated with approximation of phases in intensity integral, by means of seven Thom's singularities (cf. Fig. 5 : elliptical ombilic) confirm the modelling abilities of the potentials in geometry as well as in physics. However the bringing near, between the front wave genesis by envelop and the ondulatory property of radiation have to receive most developments, reading collaboration through mathematicians, physicists and computer graphics scientists. The well leading of this plan conduct us to restructure the analytic process to characterize curves, surfaces defined with implicit equations : research of critical points center, branch or isolated point, determination of connex parts, bounded or not, limit curve ; type of curvature are qualified with their topologic or metric properties. Beside these dense sets, the needs of computer graphics have leaded to search features for finite pixel subset, developing a discrete geometry (Chassery, 1989) and also a pretopology (Emptoz, 1983). Symmetrically, guided by the cognitive part of perception (already evoked in part II) developed by the psychological studies (Moles, A.A., Bonnet, C.). We are constructing a branch table developing the scale [0,8], of 40 morphomeres of Moles, including the oriented arcs of Malik and the 36 "geons" of Biedermann.

Many forms are likewise registered in short-term memory (5), and have limited effect on the behaviour of the subject (saillance). But the situation is quite different when forms carry a biological significance. These forms gives rise very ample reaction in the subject (pregnance). Besides the static suggestion of these forms, we exhibit a dynamical suggestion by an optical illusion of a motion (cf. figures).

The concentric rings family (Fig. 5) induce dynamically a conical depth by angular restrictions on circular explorations, result of anisotropical receiver fields. The pattern of secant circles (Fig. 6) iso-angular on a ring, suggest statically the specular super-reflectance of a smooth torus. The same pattern, but less dense (Fig. 7), with alternate filling, and masking the covering arc, give the suggestion of toboggan in slow motion, induced by ambiguity upon the depth. Reaction tests, with alternate filling, chose in a hue scale or a saturation one, have developed pregnance scales toward more cognitive suggestion of psycho-affective entity by means of logos.

For static shapes perception system, the abilities to extrapolate, are prime externalised with radiance contrast between picture back ground. The first phase of reading text, activate correlatively to this system the memory area of wrote words, developed from the real- life of the reader (culture). The rhythm of this reading, depending on syntactical eases, and edition flow, is often accelerated by a contextual anticipation on the significant, that will may be confirmed with a second reading.

Keywords :
Flow, energy, spectrum, edge, forms, convexity, envelope, retinal specialisation, bases 3D learning frames.

Bibliography :

[1] Balian, R. (1982), *Du microscopique au macroscopique*. Paris : Ellipse.

[2] Baron, T. (1991), Attribut d'image utilisant la réflectance spéculaire. *Proceedings of the RFIA*. Lyon, INSA. Vol. p. 540-546

[3] Bonnet, C. (1989), *Traité de Psychologie Cognitive*. Paris : Dunod.

[4] Buser, P., Imbert, M. (1986), *Vision*. Paris : Hermann.

[5] Chandrasekhar, S. (1950), Radiative transfert. Oxford : Ox. University Press.

[6] Chassery, J.M. (1989), Images, vision, couleurs. *Proceedings of the Université d'été*. Grenoble. IMAG, Laboratoire TIMC.

[7] Chpolski (1977), *Physique*. Moscou : MIR.

[8] Colombo (1976), *Equations aux dérivées partielles*. Paris : Masson.

[9] Cook, Torrance (1982), A reflectance model for computer graphics. *ACM*. January

[10] Dordet, Y. (1990), *Colorimétrie*. Paris : Eyrolles.

[11] Emptoz, H. (1983), *Modèles prétopologiques pour la reconnaissance des formes* Thèse d'Etat : Université Claude Bernard Lyon 1.

[12] Faugeras, O., Vaillant, R. (1990), *Using occluding contour for 3D object*. 1990, INRIA, Sophia Antipolis.

[13] Faugeras, O. (1993), *Three dimensionnal Computer Vision*. New-York : MIT Press. 663 p.

[14] Galifret, Y. (1990), Vision. *In Encyclopedie Universalis*. (Ed.). Paris.

[15] Hata, S. (1992), Shape detection using color stripe lighting. *Proceedings of the IEEE*. p. 554-556.

[16] Healy, C., Binford, T. (1988), A color metric for computer vision. *Proceedings of the IEEE*. p. 10-17.

[17] Henish, B. (1985), *Rayonnement thermique* Thèse d'Université : INSA Lyon.

[18] Kunt, M. (1993), Traitement numérique. Paris : Presse Universitaire Rimande, Dunod.
[19] Monga, O, Benayoun, S. (1992), *Using partial derivates of 3D images.* 1992, INRIA. Sophia Antipolis.
[20] Pelissier, C. (1982), *Aspects Physico-Mathématiques de la vision des contours* Thèse d'Etat : Pierre et Marie Curie.
[21] Perroche, B. (1988), *Synthèse d'Images.* Toulouse : Hermes.
[22] Petitot, J. (1985), *Morphogénèse du sens : pour un schématisme de la structure.* Coll. "Formes sémiotiques". Paris : Puf. 306 p.
[23] Pratt (1978), *Digital image processing.* New-York : Wiley.
[24] Roubine, E. (1986), *Antennes.* Paris : Masson.
[25] Silvy, J. (1961), Diffusions multiples. *Revue d'Optique.* Octobre. p. 495-517.
[26] Thom, R. (1977), *Stabilité structurelle et morphogénèse.* Paris : InterEdition.
[27] Vallée, R. (1979), *Aspects mathématiques des processus de perception-mémorisation.* GRASCE, Univ. Aix en Provence.
[28] Vallée, R. (1995), *Cognition et système.* : Interdisciplinaire.
[29] Vassalo, C. (1980), *Electro-magnétisme dans la matière.* Paris : Dunod.
[30] Vogel, Th. (1956), *Physique mathématique.* Paris : Armand Colin.