Non-locality Property of Neural Systems Based on Incursive Discrete Parabolic Equation

Daniel M. DUBOIS
Centre for Hyperincursion and Anticipation in Ordered Systems,
CHAOS asbl, Institute of Mathematics, B37, University of Liège
Grande Traverse 12, B–4000 LIEGE 1, Belgium
Fax: + 32 4 366 94 89 – Email: Daniel.Dubois@ulg.ac.be
http://www.ulg.ac.be/mathgen/CHAOS/CASYS.html

Abstract

This paper shows that non-locality property occurs in simple diffusion neural equation: space local incursive discrete equation system transforms to a space non-local recursive equation system. The cable equation used for modelling the potential in neural membrane is similar to the Schrödinger quantum equation with a complex diffusion coefficient.

Keywords : Neural Systems, non-locality, parabolic equation, anticipation, incursive computing

1 Introduction

In a recent paper, a differential equation of membrane neural potential was used as a model of a brain, the incursive computation of which giving rise to non-locality effects (Dubois, 1999). The model that will be considered in this paper is a very simple model of a neural system. This model will not present threshold effects of neurons, as in the visual system. For other neurons with threshold, this model would represent space-time fluctuations around steady states of potential, that is sub-threshold behaviour.

The one dimension cable equation which is classically considered for describing membrane neural potential (action potential along axons and dendrites)

$$\partial V(x,t)/\partial t = a\ V(x,t) + D\ \partial^2 V(x,t)/\partial x^2 \qquad (1)$$

where $V(x,t)$ is a potential depending on space x and time t. For a passive cable, the parameters $D > 0$ and $a < 0$ are constants. For axons and dendrites, a $V(x,t)$ is replaced by a non-linear function $F[V(x,t)]$ of ions exchanges in membranes. The cable equation is a parabolic equation similar to a reaction-diffusion equation.

In this section, I would like to demonstrate the basic mechanics of non-locality from anticipation from an incursive discrete parabolic equation. For this purpose, I will consider the following simplest model of parabolic equation

$$\partial V(x,t)/\partial t = D\ \partial^2 V(x,t)/\partial x^2 \qquad (1a)$$

To compute numerically such an eq. 1a, a discretization may be made in using either the forward time derivative

$$\partial V(x,t)/\partial t \approx [V(x,t+\Delta t) - V(x,t)]/\Delta t \tag{2a}$$

either the backward time derivative

$$\partial V(x,t)/\partial t \approx [V(x,t) - V(x,t-\Delta t)]/\Delta t \tag{2b}$$

The backward equation tends to the forward equation when Δt is small (all the powers of $(\Delta t)^2$, $(\Delta t)^3$, ... become negligible for continuous derivable functions). Both descriptions tend to the differential equation when Δt tends to zero: $\Delta t \to 0$.
The forward potential discrete equation is then given by the following recursive discrete equation, in considering the forward time discrete derivative 2a,

$$V(x,t+\Delta t) = V(x,t) + \Delta t\, D\, [V(x+\Delta x,t)-2.V(x,t)+V(x-\Delta x,t)]/\Delta x^2 \tag{3a}$$

The potential $V(x,t+\Delta t)$, at node x and time step $t+\Delta t$, is a function of $V(x,t)$, at the preceding time t, and of $V(x \pm \Delta x,t)$, at the preceding time t at the two adjacent nodes $x \pm \Delta x$. This is a recursive spatially local discrete system: the potential V does not depend of potentials at high distance.
The backward potential discrete equation is then given by the following incursive, implicit, discrete equation, in considering the backward time discrete derivative 2b,

$$V(x,t+\Delta t) = V(x,t) + \Delta t\, D\, [V(x+\Delta x,t+ \Delta t)-2.V(x,t+\Delta t)+V(x- \Delta x,t+ \Delta t)]/\Delta x^2 \tag{3b}$$

The potential $V(x,t+\Delta t)$, at node x and time step $t+\Delta t$, is a function of $V(x,t)$, at the preceding time t, and of $V(x,t+\Delta t)$, $V(x \pm \Delta x,t+\Delta t)$, at the time $t+\Delta t$, at the node x and at the two adjacent nodes $x \pm \Delta x$. This is a incursive spatially local discrete system: the potential V does not depend of potentials at high distance, but depends on the potentials at future time. This incursive discrete description of the potential equation shows many similarities with the flip-flop hyperincursive memory (see Dubois, 1999).
Let us show now that non-locality emerges from the implicit nature of the discrete incursive equation 3b.

2 Non-Locality in Incursive Space-Time Discrete Parabolic Equations

In defining the dimensionless diffusion $d = (\Delta t/\Delta x^2)\, D$, and in taking the time variable in Δt unit and the space variable in Δx unit, eq. 3b is written as follows:

$$[1+2.d].V(1,t+1)-d.[V(0,t+1)+V(2,t+1)] =V(1,t)$$

$$[1+2.d].V(2,t+1)-d.[V(1,t+1)+V(3,t+1)] = V(2,t)$$

$$[1+2.d].V(3,t+1)-d.[V(2,t+1)+V(4,t+1)] = V(3,t)$$

...

$$[1+2.d].V(N,t+1)-d.[V(N-1,t+1)+V(N+1,t+1)]= V(N,t) \tag{4}$$

which can be described in the following matrix form

$$A\ V(x,t+1) = V(x,t) \tag{4a}$$

In inverting the matrix A, we obtain a recursive system

$$V(x,t+1) = A^{-1}\ V(x,t) \tag{4b}$$

2.1 Non-locality with Periodic Boundary Conditions

In taking periodic boundary conditions $V(0,t+1)=V(N,t+1)$ and $V(N+1,t+1)=V(1,t+1)$, the system defines itself the values of its boundaries.
The following table gives the matrix A(N,N) and its inverse $A^{-1}(N,N)$, for N = 3, 4, 5.

TABLE 1: Matrices A and A^{-1} for N = 3, 4, 5.

A(3,3)	**A^{-1}(3,3)**
$\begin{pmatrix} 1+2d & -d & -d \\ -d & 1+2d & -d \\ -d & -d & 1+2d \end{pmatrix}$	$\begin{pmatrix} \frac{1+d}{1+3d} & \frac{d}{1+3d} & \frac{d}{1+3d} \\ \frac{d}{1+3d} & \frac{1+d}{1+3d} & \frac{d}{1+3d} \\ \frac{d}{1+3d} & \frac{d}{1+3d} & \frac{1+d}{1+3d} \end{pmatrix}$
A(4,4)	**A^{-1}(4,4)**
$\begin{pmatrix} 1+2d & -d & 0 & -d \\ -d & 1+2d & -d & 0 \\ 0 & -d & 1+2d & -d \\ -d & 0 & -d & 1+2d \end{pmatrix}$	$\begin{pmatrix} \frac{1+4d+2d^2}{1+6d+8d^2} & \frac{d}{1+4d} & \frac{2d^2}{1+6d+8d^2} & \frac{d}{1+4d} \\ \frac{d}{1+4d} & \frac{1+4d+2d^2}{1+6d+8d^2} & \frac{d}{1+4d} & \frac{2d^2}{1+6d+8d^2} \\ \frac{2d^2}{1+6d+8d^2} & \frac{d}{1+4d} & \frac{1+4d+2d^2}{1+6d+8d^2} & \frac{d}{1+4d} \\ \frac{d}{1+4d} & \frac{2d^2}{1+6d+8d^2} & \frac{d}{1+4d} & \frac{1+4d+2d^2}{1+6d+8d^2} \end{pmatrix}$
A(5,5)	**A^{-1}(5,5)**
$\begin{pmatrix} 1+2d & -d & 0 & 0 & -d \\ -d & 1+2d & -d & 0 & 0 \\ 0 & -d & 1+2d & -d & 0 \\ 0 & 0 & -d & 1+2d & -d \\ -d & 0 & 0 & -d & 1+2d \end{pmatrix}$	$\begin{pmatrix} \frac{1+3d+d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} \\ \frac{d(1+d)}{1+5d+5d^2} & \frac{1+3d+d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} \\ \frac{d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{1+3d+d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} \\ \frac{d^2}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{1+3d+d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} \\ \frac{d(1+d)}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{1+3d+d^2}{1+5d+5d^2} \end{pmatrix}$

The remarkable property of the inverse matrices A^{-1} is the fact that all their elements are non null, at the contrary to the matrices A.

This means that the space local incursive system transform to a space non-local system: the potential at any point depends of all the potentials at all distances.

The following table gives these matrices for d = 1.

TABLE 2: Matrices A and A^{-1} for N = 3, 4, 5, with d = 1.

A(3,3)	A^{-1}(3,3)
$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$
A(4,4)	**A^{-1}(4,4)**
$\begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{7}{15} & \frac{1}{5} & \frac{2}{15} & \frac{1}{5} \\ \frac{1}{5} & \frac{7}{15} & \frac{1}{5} & \frac{2}{15} \\ \frac{2}{15} & \frac{1}{5} & \frac{7}{15} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{15} & \frac{1}{5} & \frac{7}{15} \end{pmatrix}$
A(5,5)	**A^{-1}(5,5)**
$\begin{pmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{5}{11} & \frac{2}{11} & \frac{1}{11} & \frac{1}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{5}{11} & \frac{2}{11} & \frac{1}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{2}{11} & \frac{5}{11} & \frac{2}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{1}{11} & \frac{2}{11} & \frac{5}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{1}{11} & \frac{1}{11} & \frac{2}{11} & \frac{5}{11} \end{pmatrix}$

So, for example, with N = 4, eq. 4b is given by

$$V(1,t+1) = a\ V(1,t) + b\ V(2,t) + c\ V(3,t) + b\ V(4,t)$$

$$V(2,t+1) = a\ V(2,t) + b\ V(3,t) + c\ V(4,t) + b\ V(1,t)$$

$$V(3,t+1) = a\ V(3,t) + b\ V(4,t) + c\ V(1,t) + b\ V(2,t)$$

$$V(4,t+1) = a\ V(4,t) + b\ V(3,t) + c\ V(1,t) + b\ V(2,t) \qquad (5)$$

where a = 7/15, b = 3/15, c = 2/15 for which a + b + c + d = 1.

The remarkable effect is the fact that the potential V(x,t+1) at the next time step depends on the whole spatial values of the potential. Contrary to the recursive forward discrete equation system, there is no more a local space interaction but an infinite spatial interaction. The weights of the potentials a, b, c decreases with distance.
The analytical parameters a, b, c are given by

$$a = 1 - 2[d/(1 + 4d)][1 + d/(1 + 2d)] \qquad (6a)$$

$$b = [d/(1 + 4d)] \qquad (6b)$$

$$c = [d/(1 + 4d)][2d/(1 + 2d)] \qquad (6c)$$

The incursive equation tends to the recursive equation for small d, that is to say to suppress the powers d^2. Let us first suppress the terms containing d^2, in $A(4,4)^{-1}$. Eq. 4b becomes

$$V(1,t+1) = V(1,t) + [d/(1 + 4d)] [+ V(4,t) - 2V(1,t) + V(2,t)]$$

$$V(2,t+1) = V(2,t) + [d/(1 + 4d)] [+ V(1,t) - 2V(2,t) + V(3,t)]$$

$$V(3,t+1) = V(3,t) + [d/(1 + 4d)] [+ V(2,t) - 2V(3,t) + V(4,t)]$$

$$V(4,t+1) = V(4,t) + [d/(1 + 4d)] [+ V(3,t) - 2V(4,t) + V(1,t)] \quad (7)$$

which is the recursive system with d replaced by $d/(1 + 4d)$.
Second, for $4d \ll 1$, $d/(1 + 4d) = d - 4d^2 + 16d^3 - \ldots$, the powers of d can be suppressed and the recursive system

$$V(1,t+1) = V(1,t) + d [+ V(4,t) - 2V(1,t) + V(2,t)]$$

$$V(2,t+1) = V(2,t) + d [+ V(1,t) - 2V(2,t) + V(3,t)]$$

$$V(3,t+1) = V(3,t) + d [+ V(2,t) - 2V(3,t) + V(4,t)]$$

$$V(4,t+1) = V(4,t) + d [+ V(3,t) - 2V(4,t) + V(1,t)] \quad (8)$$

is obtained. The following table gives the matrices for d = 0.01.

TABLE 3: Matrices A(4,4) and $A^{-1}(4,4)$ for d = 0.01.

A(4,4)				$A^{-1}(4,4)$			
1.02	-0.01	0	-0.01	0.980581	0.00961538	0.000188537	0.00961538
-0.01	1.02	-0.01	0	0.00961538	0.980581	0.00961538	0.000188537
0	-0.01	1.02	-0.01	0.000188537	0.00961538	0.980581	0.00961538
-0.01	0	-0.01	1.02	0.00961538	0.000188537	0.00961538	0.980581

Another very interesting result is the fact that the incursive algorithm is numerically stable for any values of d. For $d \gg 1$, and even for an infinite value of d, the incursive algorithm remains stable. Indeed, when d tends to infinity, for N = 4, eqs. 6abc become $a = b = c = 1/4$, and eqs. 5 become

$$V(1,t+1) = [1/4] [+ V(4,t) + V(1,t) + V(2,t) + V(3,t)]$$

$$V(2,t+1) = [1/4] [+ V(1,t) + V(2,t) + V(3,t) + V(4,t)]$$

$$V(3,t+1) = [1/4] [+ V(2,t) + V(3,t) + V(4,t) + V(1,t)]$$

$$V(4,t+1) = [1/4] [+ V(3,t) + V(4,t) + V(1,t) + V(2,t)] \quad (9)$$

The following table gives these matrices for d = 100.

TABLE 4: Matrices A(4,4) and $A^{-1}(4,4)$ for d = 100.

A(4,4)	$A^{-1}(4,4)$
$\begin{pmatrix} 201. & -100. & 0 & -100. \\ -100. & 201. & -100. & 0 \\ 0 & -100. & 201. & -100. \\ -100. & 0 & -100. & 201. \end{pmatrix}$	$\begin{pmatrix} 0.253111 & 0.249377 & 0.248136 & 0.249377 \\ 0.249377 & 0.253111 & 0.249377 & 0.248136 \\ 0.248136 & 0.249377 & 0.253111 & 0.249377 \\ 0.249377 & 0.248136 & 0.249377 & 0.253111 \end{pmatrix}$

The general result is the following:

- When d tends to zero, the incursive system tends to the recursive system, that is to say a local interaction of potential.

- When d tends to infinity, all the weights of the potential tend to 1/N, that is to say an infinite range of interaction of potential.

To confirm these results, let us show that the non-locality is independent of the periodicity of the boundary conditions in looking at the incursive diffusive system with reflecting boundary conditions.

2.2 Non-locality with Reflecting Boundary Conditions

In taking reflecting boundary conditions V(0,t+1)=V(1,t+1) and V(N+1,t+1)=V(N,t+1), the system also defines itself the values of its boundaries.
The following table gives the matrix A(N,N) and its inverse $A^{-1}(N,N)$, for N = 3, 4, 5.

TABLE 5: Matrices A(N,N) and $A^{-1}(N,N)$ for N = 3, 4, 5.

A(3,3)	$A^{-1}(3,3)$
$\begin{pmatrix} 1+d & -d & 0 \\ -d & 1+2d & -d \\ 0 & -d & 1+d \end{pmatrix}$	$\begin{pmatrix} \frac{1+3d+d^2}{1+4d+3d^2} & \frac{d}{1+3d} & \frac{d^2}{1+4d+3d^2} \\ \frac{d}{1+3d} & \frac{1+d}{1+3d} & \frac{d}{1+3d} \\ \frac{d^2}{1+4d+3d^2} & \frac{d}{1+3d} & \frac{1+3d+d^2}{1+4d+3d^2} \end{pmatrix}$
A(4,4)	**$A^{-1}(4,4)$**
$\begin{pmatrix} 1+d & -d & 0 & 0 \\ -d & 1+2d & -d & 0 \\ 0 & -d & 1+2d & -d \\ 0 & 0 & -d & 1+d \end{pmatrix}$	$\begin{pmatrix} \frac{1+5d+6d^2+d^3}{1+6d+10d^2+4d^3} & \frac{d(1+3d+d^2)}{1+6d+10d^2+4d^3} & \frac{d^2(1+d)}{1+6d+10d^2+4d^3} & \frac{d^3}{1+6d+10d^2+4d^3} \\ \frac{d(1+3d+d^2)}{1+6d+10d^2+4d^3} & \frac{1+4d+4d^2+d^3}{1+6d+10d^2+4d^3} & \frac{d(1+d)^2}{1+6d+10d^2+4d^3} & \frac{d^2(1+d)}{1+6d+10d^2+4d^3} \\ \frac{d^2(1+d)}{1+6d+10d^2+4d^3} & \frac{d(1+d)^2}{1+6d+10d^2+4d^3} & \frac{1+4d+4d^2+d^3}{1+6d+10d^2+4d^3} & \frac{d(1+3d+d^2)}{1+6d+10d^2+4d^3} \\ \frac{d^3}{1+6d+10d^2+4d^3} & \frac{d^2(1+d)}{1+6d+10d^2+4d^3} & \frac{d(1+3d+d^2)}{1+6d+10d^2+4d^3} & \frac{1+5d+6d^2+d^3}{1+6d+10d^2+4d^3} \end{pmatrix}$

A(5,5)
$\begin{pmatrix} 1+d & -d & 0 & 0 & 0 \\ -d & 1+2d & -d & 0 & 0 \\ 0 & -d & 1+2d & -d & 0 \\ 0 & 0 & -d & 1+2d & -d \\ 0 & 0 & 0 & -d & 1+d \end{pmatrix}$
A^{-1}(5,5)
$\begin{pmatrix} \frac{1+7d+15d^2+10d^3+d^4}{1+8d+21d^2+20d^3+5d^4} & \frac{d(1+5d+6d^2+d^3)}{1+8d+21d^2+20d^3+5d^4} & \frac{d^2}{1+5d+5d^2} & \frac{d^3(1+d)}{1+8d+21d^2+20d^3+5d^4} & \frac{d^4}{1+8d+21d^2+20d^3+5d^4} \\ \frac{d(1+5d+6d^2+d^3)}{1+8d+21d^2+20d^3+5d^4} & \frac{1+6d+11d^2+7d^3+d^4}{1+8d+21d^2+20d^3+5d^4} & \frac{d(1+d)}{1+5d+5d^2} & \frac{d^2(1+d)^2}{1+8d+21d^2+20d^3+5d^4} & \frac{d^3(1+d)}{1+8d+21d^2+20d^3+5d^4} \\ \frac{d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{1+3d+d^2}{1+5d+5d^2} & \frac{d(1+d)}{1+5d+5d^2} & \frac{d^2}{1+5d+5d^2} \\ \frac{d^3(1+d)}{1+8d+21d^2+20d^3+5d^4} & \frac{d^2(1+d)^2}{1+8d+21d^2+20d^3+5d^4} & \frac{d(1+d)}{1+5d+5d^2} & \frac{1+6d+11d^2+7d^3+d^4}{1+8d+21d^2+20d^3+5d^4} & \frac{d(1+5d+6d^2+d^3)}{1+8d+21d^2+20d^3+5d^4} \\ \frac{d^4}{1+8d+21d^2+20d^3+5d^4} & \frac{d^3(1+d)}{1+8d+21d^2+20d^3+5d^4} & \frac{d^2}{1+5d+5d^2} & \frac{d(1+5d+6d^2+d^3)}{1+8d+21d^2+20d^3+5d^4} & \frac{1+7d+15d^2+10d^3+d^4}{1+8d+21d^2+20d^3+5d^4} \end{pmatrix}$

TABLE 6: Matrices A and A^{-1} for d = 1.

A(3,3)	A^{-1}(3,3)
$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{5}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \end{pmatrix}$
A(4,4)	**A^{-1}(4,4)**
$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{13}{21} & \frac{5}{21} & \frac{2}{21} & \frac{1}{21} \\ \frac{5}{21} & \frac{10}{21} & \frac{4}{21} & \frac{2}{21} \\ \frac{2}{21} & \frac{4}{21} & \frac{10}{21} & \frac{5}{21} \\ \frac{1}{21} & \frac{2}{21} & \frac{5}{21} & \frac{13}{21} \end{pmatrix}$
A(5,5)	**A^{-1}(5,5)**
$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{34}{55} & \frac{13}{55} & \frac{1}{11} & \frac{2}{55} & \frac{1}{55} \\ \frac{13}{55} & \frac{26}{55} & \frac{2}{11} & \frac{4}{55} & \frac{2}{55} \\ \frac{1}{11} & \frac{2}{11} & \frac{5}{11} & \frac{2}{11} & \frac{1}{11} \\ \frac{2}{55} & \frac{4}{55} & \frac{2}{11} & \frac{26}{55} & \frac{13}{55} \\ \frac{1}{55} & \frac{2}{55} & \frac{1}{11} & \frac{13}{55} & \frac{34}{55} \end{pmatrix}$

For example, for N = 4 with d = 1, eqs. 4b are given by

$V(1,t+1) = a_0\ V(1,t) + b_0\ V(2,t) + c_0\ V(3,t) + d_0\ V(4,t)$

$V(2,t+1) = a_1\ V(2,t) + b_1\ V(3,t) + c_1\ V(4,t) + d_1\ V(1,t)$

$V(3,t+1) = a_2\ V(3,t) + b_2\ V(4,t) + c_2\ V(1,t) + d_2\ V(2,t)$

$$V(4,t+1) = a_3\ V(4,t) + b_3\ V(3,t) + c_3\ V(1,t) + d_3\ V(2,t) \qquad (10)$$

where $a_0 = 13/21$, $a_1 = 5/21$, $a_2 = 2/21$, $a_3 = 1/21$, $b_0 = 5/21$, $b_1 = 10/21$, $b_2 = 4/21$, $b_3 = 2/21$, $c_0 = 2/21$, $c_1 = 4/21$, $c_2 = 10/21$, $c_3 = 5/21$, $d_0 = 1/21$, $d_1 = 2/21$, $d_2 = 5/21$, $d_3 =13/21$. The non-locality is thus confirmed.
The following tables give the matrices for d = 0.01 and d = 100.

TABLE 7: Matrices A(4,4) and $A^{-1}(4,4)$ for d = 0.01.

A(4,4)

1.01	-0.01	0	0
-0.01	1.02	-0.01	0
0	-0.01	1.02	-0.01
0	0	-0.01	1.01

$A^{-1}(4,4)$

0.990195	0.00970873	0.0000951929	9.42504×10^{-7}
0.00970873	0.980582	0.00961448	0.0000951929
0.0000951929	0.00961448	0.980582	0.00970873
9.42504×10^{-7}	0.0000951929	0.00970873	0.990195

TABLE 8: Matrices A(4,4) and $A^{-1}(4,4)$ for d = 100.

A(4,4)				$A^{-1}(4,4)$			
101.	-100.	0	0	0.258621	0.251207	0.246305	0.243867
-100.	201.	-100.	0	0.251207	0.253719	0.248768	0.246305
0	-100.	201.	-100.	0.246305	0.248768	0.253719	0.251207
0	0	-100.	101.	0.243867	0.246305	0.251207	0.258621

2.3 Non-locality with Boundary Conditions with External Inputs

With boundary conditions as external inputs I(0,t+1) and I(4,t+1), the eq. 4b becomes, for x = 0,1,2,3,4, and d = 1,

$V(1,t+1)=[8[V(1,t)+I(0,t+1)]+3V(2,t)+[V(3,t)+I(4,t+1)]]/21$

$V(2,t+1)=[3V(2,t)+9[V(3,t)+I(4,t+1)]+3[V(1,t)+I(0,t+1)]]/21$

$$V(3,t+1)=[[V(3,t)+I(4,t+1)]+3[V(1,t)+I(0,t+1)]+8V(2,t)]/21 \qquad (11)$$

We remark that the inputs are present in the three equations, that is in all the space nodes at the same time step. It means that the inputs at the boundaries are logically transmitted instantaneously, that is to say during the time duration Δt, to each node.

3 Infinite Speed of Propagation in Diffusion Equation

It is well-known that the parabolic equation 1a predicts an infinite speed of propagation for disturbances (Maxwell (1867), Cattanbo (1958), Green and Laws (1972), Vernotte (1958), Chester (1963), Kranys (1966), Müller (1967,1971), Lambermont and Lebon (1973)).

With the initial conditions $V(0,0) = \delta_0$ (Dirac's function) and, $V(x,0) = 0$ for all x different of 0, $\forall\ |x| > 0$, the solution of eq. 1a is

$$V(x,t) = [1/2\sqrt{(\pi t)}] \exp[-x^2/(4t)] \quad (12)$$

which shows clearly that the potential spreads the whole space instantaneously. This corresponds to the paradox the infinite speed of propagation of disturbances in parabolic equations.

So the incursive discrete equation is the best algorithm to simulate the parabolic equation because this gives rise to the non-locality property of the differential parabolic equation.

Indeed, the recursive discrete equation is an algorithm which suppresses the non-locality property and thus does no more represent the original differential parabolic equation.

With an initial condition $V(4, 0) = 1$, the spatial evolution of $V(x, t)$ in function of time gives the spread of the potential with time as shown in the following figure.

x =	0	1	2	3	4	5	6	7	8
t = 0	0	0	0	0	1	0	0	0	0
t = 1	0	0	0	1/4	1/2	1/4	0	0	0
t = 2	0	0	1/16	1/4	3/8	1/4	1/16	0	0
t = 3	0	1/64	3/32	15/64	5/16	15/64	3/32	1/64	0

Figure 1: simulation of the recursive discrete eq. 3a, with $d = (\Delta t/\Delta x^2)\ D = 1/4$. The potential propagates with a velocity $v = \Delta x/\Delta t$.

With such a recursive algorithm, no non-locality occurs. The potential spreads in space domain at the velocity of $\Delta x/\Delta t$.

4 Non-locality in Quantum Mechanics

The non-locality property of such an incursive discrete parabolic system could be similar to the non-locality in Quantum Systems by micro-causal loops, where the duration Δt would be related to the Planck time constant t_p.

After John von Neumann, the Schrödinger equation is similar to the diffusion equation in taking an imaginary diffusion constant $D = i\,\hbar/2m$, so eq. 1a becomes

$$\partial V(x,t)/\partial t = i\,\hbar/2m\ \partial^2 V(x,t)/\partial x^2 \tag{13}$$

and in multiplying both members by $i\,\hbar$, the Schrödinger equation for a free particle is found

$$i\,\hbar\partial\Phi(x,t)/\partial t = -\,\hbar^2/2m\ \partial^2\Phi(x,t)/\partial x^2 \tag{13b}$$

in defining by Φ the complex wave function.

The matrices A and A^{-1} in Tables 1 and 5, are still available for the incursive discretization of eq. 13b, in taking an imaginary diffusion d.

Holistic properties of systems by non-locality could be generated by temporal anticipation as micro-causal loops. Why this phenomenon?
As the spatially local incursive equation system is defined with future time duration by the backward derivative, the inversion of the matrix A has the effect of mixing all the space nodes together at the present time t. The inversion of the matrix A folds each space nodes to the other ones from the future time $t+\Delta t$ to the present time t. A micro-causal loop means that during the duration Δt, small duration anticipation could occur. This is in contradiction with Einstein causality principle but not with the Heisenberg principle of uncertainty $\Delta t\ \Delta E \geq \hbar$. During the interval of time, Δt, we cannot know if there is an arrow of time or a micro causal loop.
Let us remark that in a pure anticipatory duration, the time duration would be negative, $\Delta t < 0$, and thus the interval of energy would be also negative, $\Delta E < 0$. G. Nimtz (1998) obtained experimentally a signal propagating with a velocity higher than the light velocity with negative energy.

5 Conclusion

More and more authors believe that the neural brain could be explained as a quantum neural system. One characteristic of quantum systems is non-locality, well confirmed by entanglement experiments.
This paper is an attempt to show that non-locality can emerge from an incursive discrete description of the cable parabolic equation used for modeling the potential in neural membranes.

Acknowledgements

I would like to thank Francine Monjoie-Garnir for her help in using Mathematica for matrix inversion.

References

Cattaneo, C. (1958), Sur une forme de l'équation éliminant le paradoxe d'une propagation instantanée." C.R. Acad. Sc. Vol. T 247: pp. 431-433.

Chester, M. (1963). Second Sound in Solids." Phys. Rev. Vol. **131**: pp. 2013-2015.

Dubois D. M. (1996). "A Semantic Logic for CAST related to Zuse, Deutsch and McCulloch and Pitts Computing Principles". In F. Pichler, R. Moreno Diaz, R. Albrecht (Eds.): *Computer Aided Systems Theory - EUROCAST'95*. Lecture Notes in Computer Science, 1030, Springer-Verlag Berlin Heidelberg, pp. 494-510.

Dubois D. M. (1996). "Introduction of the Aristotle's Final Causation in CAST: Concept and Method of Incursion and Hyperincursion". In F. Pichler, R. Moreno Diaz, R. Albrecht (Eds.): *Computer Aided Systems Theory - EUROCAST'95*. Lecture Notes in Computer Science, 1030, Springer-Verlag Berlin Heidelberg, pp. 477-493.

Dubois D. M. (1997). Generation of fractals from incursive automata, digital diffusion and wave equation systems, *Biosystems* 43, pp. 97-114.

Dubois D. M. (1998) *Boolean Soft Computing by Non-linear Neural Networks with Hyperincursive Stack Memory*, Invited Paper, in Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications, Edited by Okyay Kaynak, Lotfi A. Zadeh, Burhan Türksen, Imre J. Rudas, NATO ASI Series, Series F: Computer and Systems Sciences, vol. 162, Springer-Verlag, pp. 333-351.

Dubois D. M. (1999) *Hyperincursive McCulloch and Pitts Neurons for Designing a Computing Flip-Flop Memory*, Conference Proceedings CP 465, Computing Anticipatory Systems: CASYS'98 – Second International Conference, Edited by Daniel M. Dubois, Published by The American Institute of Physics, pp. 3-21.

Dubois Daniel M. (1998), "Computing Anticipatory Systems with Incursion and Hyperincursion". Computing Anticipatory Systems: CASYS - First International Conference. Edited by Daniel M. Dubois. Published by The American Institute of Physics, AIP Conference Proceedings 437, pp. 3-29.

Dubois Daniel M. (2000) Review of Incursive, Hyperincursive and Anticipatory Systems - Foundation of Anticipation in Electromagnetism. Computing Anticipatory Systems: CASYS'99 - Third International Conference. Edited by Daniel M. Dubois, Published by The American Institute of Physics, AIP Conference Proceedings 517, pp. 3-30.

Dubois D., and G. Resconi. (1992). *Hyperincursivity: a new mathematical theory*, Presses Universitaires de Liège, 260 p..

Green, A. E., Laws, N. (1972). On the Entropy Production Inequality." Arch. Rat. Mech. Anal. Vol. **45**: pp. 47-53.

Kranys, M. (1966). The Postulate of the Relaxation of Non-Stationary Processes Caused by the Finite Transport Velocity of Interaction in the Matter." Phys. Letters, Vol. **22**: pp. 285-286.

Lambermont, J., Lebon, G. (1973). On a Generalization of the Gibbs Equation for Heat Conduction." Phys. Lett. Vol. **42A**: pp. 499-500.

Maxwell, J. C. (1867). On the Dynamical Theory of Gases." Phil. Trans. Roy. Soc. Vol. **157**: pp. 49-88.

Müller, I. (1967). Zum Paradoxen der Wärmeleitungstheorie." Zeitschrift für Phys. Vol. **198**: pp. 329-344.

Müller, I. (1971). The Coldness, a Universal Function in Thermoelastic Bodies." Arch. Tat. Mech. Anal. Vol. **41**: pp. 319-322.

Nimtz G. (1998). Superluminal signal velocity. Ann. Phys. (Leipzig) 77 - 8, 618 - 624.

Vernotte, P. (1958). Les paradoxes de la théorie continue de l'équation de la chaleur." C.R. Acad. Sc. Vol. **T 247**: pp. 3154-3155.