The Dual Pairing Principle of Quantum State Engineering: Cooper Pairs and Bose–Einstein Condensates

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Abstract. The non–invasive diagnostic protocols of magnetic resonance tomography and spectroscopy have been made effective for the clinical routine by means of the chirality reversing concepts of spin echo and gradient echo. Because Cooper pairs in a spin singlet are formed from time–reversed quantum states, the coadjoint orbit picture of the unitary dual $\hat{N}$ of the real Heisenberg step 2 nilpotent Lie group N gives rise to the symmetry group $\mathbf{SU}(2, \mathbf{C}) \cong \mathbf{Spin}(\mathbf{R}^3)$. Due to the Hopf fibration of the unit sphere $\mathbf{S}_3 \hookrightarrow \mathbf{C} \oplus \mathbf{C}$ over the Bloch sphere $\mathbf{S}_2 \hookrightarrow \mathbf{R} \oplus \mathbf{C}$ with fiber $\mathbf{S}_1 \hookrightarrow \mathbf{C}$, the compact Lie group $\mathbf{SU}(2,\mathbf{C})$ acts via $\mathbf{U}(1,\mathbf{C})$ gauge transformations on the complex line bundle associated with N. The metaplectic symmetries of the symplectic spinor bundle configuration, conjugated by indistinguishable pairs of contragredient flat double–layers occurring in the foliation of $\hat{N}$, permit a natural approach to the quantum information transfer within pairs of Bose–Einstein condensates.

Keywords. Quantum state engineering of complex line bundles; $\mathbf{U}(1,\mathbf{C})$ gauge symmetry; Bose–Einstein condensation of ultracold quantum gases; atom laser beams; Cooper pairs of electrons

1. Introduction

When the pioneer of quantum state engineering, Heike Kamerling Onnes, received the Nobel award in 1913 for his liquefaction of He and the discovery of the phenomenon of superconductivity, he concluded his acceptance speech by expressing the hope that advances in low temperature quantum physics would *contribute towards lifting the veil which thermal motion at normal temperatures spreads over the inner world of atoms and electrons.* Speaking long before the advent of industrial quantum state engineering and the non–invasive diagnostic technique of clinical magnetic resonance tomography and spectroscopy, Onnes could not have guessed how prophetic his words would prove. Ever since their original discovery nearly 100 years ago, superfluidity and superconductivity have led to an incredible number of unexpected and surprising physical phenomena. For the spectrum of novelties revealed by the quest for low and even ultralow temperatures, by far the most dramatic have been the physical phenomena that occur in a degenerate system of Bose–Einstein particles or bosons for short, as Bose–Einstein condensation,

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or in a system of degenerate Fermi–Dirac particles or fermions for short, as Cooper pairing. Even today Bose–Einstein condensation and Cooper pairing continue to offer new and arcane physical phenomena. Superconductivity, superfluidity, and Bose–Einstein condensation remain among the most fascinating phenomena in nature. Their strange and often surprising properties are direct consequences of quantum field theory and offer quantum state engineering applications.

Clinical magnetic resonance imaging scanners are the result of three converging quantum state engineering methodologies carried to a remarkable level of technological sophistication. The large magnetic field density in which the patient is placed is generated by coil configurations cooled to liquid He temperature and exhibiting superconductivity, a phenomenon of fundamentally quantum nature. Only such assemblies of coils can produce without heating effect the large magnetic field densities required for high contrast–resolution magnetic resonance tomography. The choreography of the spin ensembles observed by clinical magnetic resonance tomography is also ruled by the laws of quantum physics ([5]). Finally, the signals picked up in the detection coils are transformed into high contrast–resolution images by powerful computers which also exploit quantum effects within their semiconductor circuits. In this way, quantum state engineering can be directly witnessed for the benefit of the patients ([9], [10]).

The physical phenomenon known as Bose–Einstein condensation was predicted by Albert Einstein in 1924 on the basis of a statistical argument of Satyendra Nath Bose to derive the black–body photon spectrum of cavity radiation: In a system of *indistinguishable* particles obeying Bose statistics and whose total number is conserved, there should be a well–defined, critical temperature below which a finite fraction of all the particles *condense* into the same one–particle state. Einstein's original prediction was for a noninteracting gas, a system felt by some of his contemporaries to be perhaps pathological, but shortly after the observation of the phenomenon of superfluidity in liquid ^{4}He below the temperatue of 2.17 K, Fritz London and Laszlo Tisza hypothesized in 1938 that despite the strong interatomic interactions Bose–Einstein condensation was occurring in this system and was responsible for the appearance of the superfluid properties. Their work was the first to bring out the idea of Bose–Einstein condensation displaying quantum behaviour on a *macroscopic* size scale, the primary reason for much of its current attraction. Their visionary suggestions have stood the test of time and is the basis of the modern understanding of the properties of the superfluid phase.

Forming a Bose–Einstein condensate is simple in principle: One has to make an atomic gas extremely cold until the atomic wave packets start to overlap. Over the last two decades a brilliant program of research in the field of quantum state engineering was established by the success in using optical lasers and evaporative techniques to cool dilute clouds of monatomic gases into the nano K temperature range ([6]). Engineering cooling techniques need an open system which allows entropy to be removed from the system, in laser–cooling in the form of scattered photons which carry away more energy than has been absorbed by the atoms, hence resulting in net cooling, in evaporative cooling in the form of discarded atoms while the remainder rethermalize at steadily lower temperatures, resulting again in a cooling advance. Laser–cooling is applied to precool the atoms so that the atoms are cold enough to be confined in a magnetic trap.

After magnetically trapping the atoms, forced evaporative cooling is applied as the second stage of ultracooling. The sophisticated experimental techniques of combining engineering cooling schemes and trapping methods by magnetic fields culminated in the attainment of Bose–Einstein condensation in such systems in the summer of 1995. More recently the experimental progress has permitted investigation not only of Cooper pairing but of the *crossover* between this phenomenon and Bose–Einstein condensation proper.

Collisions between ultracold atoms have remarkable properties. They induce reproducible and controllable phase shifts on atomic quantum states. These collisional phase shifts permit to construct quantum gates and to engineer the *non–local* concept of quantum entanglement in condensates trapped in a potential well. In this respect, cold atom collisions can be compared to the Rydberg atom collisions mediated by a cavity. In both cases, a process which is commonly considered as generating randomness is made phase coherent and controllable by imposing boundary constraints. However, cold collisions in optical lattices L have an advantage over cavity mediated collisions. They can be immediately generalized to entangle in a single operation a large ensemble of qubits and to build entanglement factories for ensemble of cold atoms which allow the study of the Mott phase transition in a lattice L.

Returning to photons, the successful observation of ultraslow light propagating at group velocities more than seven orders of magnitude below its vacuum speed, and the subsequent stopping and finally storing of light pulses in atomic media has demonstrated a quantum physical protocol to accomplish the challenge of studying methods to transfer quantum information between atoms and photons by means of the modern ultralow temperature technologies of condensed–matter quantum physics. The technique relies on a symplectic spinor bundle configuration which uses a coupling laser light field to coherently control the propagation of a pulse of probe laser light. The probe or signal pulse coherently imprints its amplitude and phase on the quantum coherence between two stable internal states of the atoms. Switching the coupling field off stops the probe pulse and ramps its intensity to zero, freezing the probe's coherent information of intensity and phase into the atomic media, where it can be stored for a controllable time period. Switching the coupling field back on at a later time writes the information back onto a quantum holographically reproduced probe pulse, which then propagates out of the atomic cloud and can be detected. The reproduced output light pulses were indistinguishable in width and amplitude from non–stored ultraslow light pulses, indicating that the switching process preserves the quantum optical information in the atomic medium during the storage time with a high degree of fidelity.

In the present paper, the metaplectic symmetry of the symplectic spinor bundle configuration is studied by means of harmonic analysis on the double $2\,N$ of the three–dimensional real Heisenberg step 2 nilpotent Lie group N ([5]) via a series of stepwise extension techniques. These group extensions have their own mathematically interesting feature because they culminate finally in the coadjoint orbit covering theorem.

2. Symplectic Vector Spaces

Let W denote a finite–dimensional vector space over $\mathbf{R}$, endowed with a symplectic form $< \cdot, \cdot >$. Thus $< \cdot, \cdot >$ is antisymmetric and nondegenerate. The nondegeneracy is equivalent to the condition that the mapping $W \longrightarrow W^*$ from the symplectic vector space W to its dual vector space W^* over $\mathbf{R}$, given by the assignment $w \rightsquigarrow (w' \rightsquigarrow < w, w' >)$, be a real isomorphism. Let the outer tensor product $W_{\mathbf{C}} \cong W \otimes \mathbf{C}$ be the complexification of W. It forms the vector space $W \oplus i.W$ over $\mathbf{C}$ and represents the ambient qubit space in the sense of quantum computing. Indeed, projectivization leads to $\mathbf{P}(W_{\mathbf{C}}) \cong \mathbf{P}_1(\mathbf{C})$ so that bipolar stereographic projection provides the Hopf fibration $\mathbf{S}_1 \hookrightarrow \mathbf{S}_3 \xrightarrow{\eta} \mathbf{S}_2$ which forms a realization of the qubit concept of quantum information science. Notice that the Lie group $\mathbf{SO}(3,\mathbf{R})$ of orientation preserving rotations of the Euclidean vector space $\mathbf{R}^3$ is diffeomorphic to the real projective space $\mathbf{P}_3(\mathbf{R})$ and therefore compact and connected. Its two–fold covering group is the compact, connected, and simply connected Lie group $\mathbf{S}_3 \cong \mathbf{SL}(1,\mathbf{H}) \cong \mathbf{SU}(2,\mathbf{C}) \cong \mathbf{Spin}(\mathbf{R}^3)$ which consists of the complex matrices

$$\left\{ \begin{pmatrix} w & -\bar{w}' \\ w' & \bar{w} \end{pmatrix} \mid (w, w') \in \mathbf{C} \times \mathbf{C}, |w|^2 + |w'|^2 = 1 \right\},$$

and has as its center a subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z}$. The three–dimensional real Lie algebra

$$\mathrm{Lie}(\mathbf{SU}(2,\mathbf{C})) = \left\{ \begin{pmatrix} iz & -\bar{w} \\ w & -iz \end{pmatrix} \mid z \in \mathbf{R}, w \in \mathbf{C} \right\}$$

of the quantum mechanical rotation group $\mathbf{SU}(2,\mathbf{C})$ is spanned over $\mathbf{R}$ by the traceless Pauli spin matrices

$$\left\{ \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$$

and is therefore isomorphic to the real Lie algebra of pure quaternions $\Im\mathbf{H}$. In this context $\Im\mathbf{H} \cong \mathbf{R}^3$ denotes the real hyperplane in $\mathbf{H} \cong \mathbf{C} \oplus \mathbf{C}$ with normal $1 \in \mathbf{H}$. The Pauli spin matrices are in correspondence to the canonical basis of the real vector space $\mathbf{R}^3 \cong \mathrm{Lie}(\mathbf{SO}(3,\mathbf{R}))$. The real Lie algebra $\mathbf{R}^3$ under the vector product $\times$ as its Lie–Jacobi bracket comes equipped with the Hopf fibered spin group $\mathbf{SU}(2,\mathbf{C}) \cong \mathbf{Spin}(\mathbf{R}^3)$ as an additional, only at the first glance rather *unexpected* gauge symmetry group of the three–dimensional real Heisenberg step 2 nilpotent Lie group N. The gauge symmetry is generated by the natural duality between symplectic groups and orthogonal groups.

The symplectic form $< \cdot, \cdot >$ on W extends by $\mathbf{C}$–bilinearity to a symplectic form, still denoted $< \cdot, \cdot >$, on $W_{\mathbf{C}}$. Let $\mathbf{Sp}(W)$ and $\mathbf{Sp}(W_{\mathbf{C}})$ be the symplectic groups of W and $W_{\mathbf{C}}$, respectively. The groups of isometries $\mathbf{Sp}(W)$ and $\mathbf{Sp}(W_{\mathbf{C}})$ of the symplectic forms $< \cdot, \cdot >$ on W and $W_{\mathbf{C}}$, respectively, consist of $\mathbf{R}$–linear and $\mathbf{C}$–linear endomorphisms of W and $W_{\mathbf{C}}$, respectively. An element $g \in \mathbf{Sp}(W)$ may be extended to a $\mathbf{C}$–linear endomorphism, still denoted $g \in \mathbf{Sp}(W_{\mathbf{C}})$, of $W_{\mathbf{C}}$ such that

$$\mathbf{Sp}(W) = \{ g \in \mathbf{GL}(W) \mid < g(w), g(w') > = < w, w' >, (w, w') \in W \times W \},$$

and

$$\mathbf{Sp}(W_{\mathbf{C}}) = \{g \in \mathbf{GL}(W_{\mathbf{C}}) | < g(w), g(w') >=< w, w' >, (w, w') \in W_{\mathbf{C}} \times W_{\mathbf{C}}\}.$$

The extension process yields an injection of symplectic groups $\mathbf{Sp}(W) \hookrightarrow \mathbf{Sp}(W_{\mathbf{C}})$. Complex conjugation on $\mathbf{C} \cong \mathbf{R} \oplus \mathbf{R}$ associated with the orthogonal reflection matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{O}(2, \mathbf{R})$$

with respect to the real axis induces a complex antilinear involutive endomorphism of $W_{\mathbf{C}}$ by the assignment $w \otimes \zeta \rightsquigarrow w \otimes \bar{\zeta}$, where $w \in W$, and $\zeta \in \mathbf{C}$.

A complex structure on W is formed by an operator $I \in \mathrm{End}(W)$ such that $I^2 = -1$. The complex structure is said to be compatible with $< \cdot, \cdot >$ if $I \in \mathbf{Sp}(W)$. It can have eigenvalues equal only to $\{\pm i\}$, and these eigenvalues must occur with equal multiplicity, which will be $\frac{1}{2} \dim_{\mathbf{R}} W$ for each. Let $W_I^+ \subseteq W_{\mathbf{C}}$ be the $(+i)$–eigenspace of the complex structure I, and let $W_I^- \subseteq W_{\mathbf{C}}$ be the corresponding $(-i)$–eigenspace. Then $W_I^- = \overline{W_I^+}$ where the overline indicates complex conjugation, and the decomposition

$$\boxed{W_{\mathbf{C}} = W_I^+ \oplus W_I^-}$$

of conjugate vector subspaces holds. The $\mathbf{R}$–linear mapping $p_I^+ = \frac{1}{2}(1 - iI)$ projects $W_{\mathbf{C}}$ onto W_I^+ with kernel W_I^-, and the complementary $\mathbf{R}$–linear operator $p_I^- = \overline{p_I^+} = \frac{1}{2}(1 + iI)$ projects $W_{\mathbf{C}}$ onto W_I^- with kernel W_I^+. Notice that the projector p_I^+ is a $\mathbf{R}$–linear isomorphism from W onto W_I^+, and the projector p_I^- is the corresponding $\mathbf{R}$–linear isomorphism from W onto W_I^-. Specifically, $p_I^+(Iw) = ip_I^+(w), p_I^-(Iw) = -ip_I^-(w)$ for $w \in W$. It follows that the assignment $I \rightsquigarrow W_I^+$ is a bijection from the set of complex structures on W onto the set of totally complex vector subspaces of $W_{\mathbf{C}}$ of dimension $\frac{1}{2} \dim_{\mathbf{R}} W$. The assignment $(w, w') \rightsquigarrow i < p_I^+(w), p_I^-(w') >= \frac{1}{2}(< Iw, w' > + i < w, w' >)$ establishes that if W is endowed with the complex structure I performing the multiplication by the imaginary unit i by an application of the mapping I, then a symmetric Hermitian–bilinear form arises when the real symplectic plane W is considered as a vector space over $\mathbf{C}$. The action of $\mathbf{Sp}$(W) on $W_{\mathbf{C}}$ preserves the symmetric Hermitian–bilinear form. Its positive definiteness is equivalent to the positive definiteness of its real part, which is the symmetric bilinear form $< I\cdot, \cdot >$ associated with W. There exists a symplectic frame in W such that $< I\cdot, \cdot >$ is a positive definite symmetric bilinear form.

3. Nilpotent Harmonic Analysis

For the real symplectic plane $W = \mathbf{R} \oplus \mathbf{R}^*$ let

$$I_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbf{SL}(2, \mathbf{R})$$

act as a multiplication–with–i transformation with $I_0^2 = -1$ on W, the first coordinate of $w \in W$ being assigned to the transversal isotropic line $\mathbf{R}$, and the second coordinate being assigned to the transversal isotropic dual line $\mathbf{R} \cong \mathbf{R}^*$. Notice that the symplectic matrix I_0, well–known from the hypoelliptic Cauchy–Riemann differential equations of holomorphic functions, is independent of the basis used to define it. The differential equations characterizing the momentum mapping for the action of a Lie group on a symplectic manifold going into the real dual vector space of the associated Lie algebra can be considered as a generalization of the Cauchy–Riemann equations. The fundamental idea, however, to endow the planetary orbit plane with its natural symplectic stucture is due to Johannes Keppler (1571 to 1630) who invented the symplectic sampling method as a high precision numerical technique to *predict* the spatio–temporal positions of the planets from the astronomical data available ([4]). The idea to equip each coadjoint orbit with its natural symplectic structure can be traced back to Sophus Lie's monumental theory of transformation groups (1890). Lie also had many of the ideas of momentum mappings going into the dual of the Lie algebra so that he should be considered as the legitimate successor of Keppler ([8]). For many years the original work of Lie appears to have been forgotten, although it leads to the insight that the symplectic leaves of the Lie–Poisson foliation on the dual $\text{Lie}(N)^*$ of the Heisenberg Lie algebra $\text{Lie}(N)$ coincide with the coadjoint orbits, and finally to the coadjoint orbit covering theorem.

Since I_0 is the infinitesimal generator of the maximal compact subgroup $\mathbf{SO}(2,\mathbf{R}) \cong \mathbf{R}/\mathbf{Z}$ of $\mathbf{SL}(2,\mathbf{R})$, in a unitary linear representation of $\mathbf{SL}(2,\mathbf{R})$ the symplectic matrix I_0 must act diagonally with eigenvalues in the set $i\mathbf{Z}$; in other words, iI_0 must act diagonally with eigenvalues in $\mathbf{Z}$. The standard symplectic form $< \cdot, \cdot >$ on the real plane W is defined by the recipe $< (x,\lambda),(x',\lambda') >= \lambda'(x) - \lambda(x')$ for $(x,\lambda) = w = (x,y)$, where $x \in \mathbf{R}$ and $\lambda = y \in \mathbf{R}^*$ such that

$$< w, w' >= \det \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}$$

holds. The standard symplectic form on the tangent space of the real dual plane W^* is defined by the d–closed exterior differential two–form $\omega = \mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2i}\,\mathrm{d}w \wedge \mathrm{d}\bar{w}$ in the Graßmann power $\bigwedge^2 W^*$. Notice that ω is actually at the heart of the coadjoint orbit method in representation theory. The affine connection one–form on the cotangent space of the real dual plane W^* associated with the standard Kähler form ω on $\mathbf{C} \cong \mathbf{R}_+ \times \mathbf{SO}(2,\mathbf{R})$ reads $\theta = \frac{1}{2}\,(x.\mathrm{d}y - y.\mathrm{d}x) = \pi\, r^2 \mathrm{d}t$, so that the identity $\omega = \mathrm{d}\theta = 2\pi\, r.\, \mathrm{d}r \wedge \mathrm{d}t$ holds. The standard symplectic structure I_0 defines on W a geometry of *signed* areas in the sense of the Kepplerian second law of planetary motion which leads to the concept of self–orthogonality on W: $< w, w >= 0\,, w \in W$. Its group of isometries is given by

$$\boxed{\mathbf{Sp}(W) \cong \mathbf{SL}(2,\mathbf{R})}$$

so that the complex structure I_0 is compatible with the symplectic form $< \cdot, \cdot >$. It follows that the pair $(W^+_{I_0}, W^-_{I_0})$ forms a complete polarization in the qubit space $W_{\mathbf{C}}$.

Define N, the simply connected three–dimensional Heisenberg step 2 nilpotent Lie group attached to $\mathbf{R}$, by $\text{Heis}(W) = W \oplus \mathbf{R}$ as a set that admits the group law

$$(w,z)(w',z') = (w + w', z + z' + \frac{1}{2} < w, w' >)\,.$$

Thus $N = \text{Heis}(W)$ forms a central extension of the transversal symplectic plane $(W, <\cdot,\cdot>)$ by the longitudinal quantization axis $\mathbf{R} \hookrightarrow N \longrightarrow W$. Therefore N forms the universal covering of the central extension

$$\boxed{\mathbf{T} \hookrightarrow N \longrightarrow W}$$

where the one–dimensional compact torus group $\mathbf{T} \cong \mathbf{R}/\mathbf{Z} \cong \mathbf{S}_1 \cong \mathbf{Spin}(\mathbf{R}^2)$ represents the central phase circle action of phase–coherent quantum information. It follows also that the roto–translation group of planar Euclidean rigid motions under its sub–Riemannian metric and its natural contact geometry is *locally* modeled by the Heisenberg Lie group N.

It is traditional to realize the line bundle N as the Lie group of all unipotent real matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbf{R} \oplus \mathbf{R}, z \in \mathbf{R} \right\}.$$

Obviously its background manifold is $\mathbf{R}^3$. The subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid z \in \mathbf{R} \right\}$$

is isomorphic to the real line $\mathbf{R}$ and forms the *center* of the Heisenberg nilpotent Lie group N. The subgroups

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid (x, z) \in \mathbf{R} \oplus \mathbf{R} \right\},$$

and

$$\left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid (y, z) \in \mathbf{R} \oplus \mathbf{R} \right\}$$

are isomorphic to the real plane $\mathbf{R}^2$ and form *normal* subgroups of the Heisenberg nilpotent Lie group N. Less obvious is the isomorphy of N in its unipotent real matrix model to the first order jet space

$$\mathcal{K}^1_{\mathbf{R}}(\mathbf{R}) = \left\{ (f'(t), t, f(t)) \mid f \in \mathcal{C}^1_{\mathbf{R}}(\mathbf{R}), t \in \mathbf{R} \right\}$$

under its sub–Riemannian structure which derives from the contact one–form $\mathrm{d}f = f'(t)\,\mathrm{d}t$ of local coordinates $\mathrm{d}z - x\,\mathrm{d}y$ in $\mathbf{R}^3$. It follows the contact identity

$$\left(\mathrm{d}z - x\,\mathrm{d}y\right) \wedge \mathrm{d}\left(\mathrm{d}z - x\,\mathrm{d}y\right) = -\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,.$$

The real Lie algebra $\text{Lie}(N) = W \oplus \mathbf{R}$ obtains by making the real line $\mathbf{R}$ central, and for any pair $(w, w') \in W \times W$ one defines the Lie–Jacobi bracket $[w, w'] = < w, w' > z$ with a fixed basal element located on the longitudinal quantization axis $z \in \mathbf{R}$. It is easy to see that Lie(N) is isomorphic to the Heisenberg Lie algebra over $\mathbf{R}$ of nilpotent matrices

$$\left\{ \begin{pmatrix} 0 & w & iz \\ 0 & 0 & \bar{w} \\ 0 & 0 & 0 \end{pmatrix} | w \in \mathbf{C}, z \in \mathbf{R} \right\}$$

where $w = x + iy, \bar{w} = x - iy$. While complex notation is used, Lie(N) is still a real Lie algebra with three real dimensions under the matrix commutator as its Lie–Jacobi bracket. Its one–dimensional center, the longitudinal quantization axis of Lie(N), is defined by $w = 0$, so that

$$\left\{ \begin{pmatrix} 0 & 0 & iz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | z \in \mathbf{R} \right\} \cong \mathrm{Lie}(\mathbf{T}),$$

and Lie(N)/center is isomorphic to the qubit space $W_{\mathbf{C}}$. In terms of the prequantization procedure, the central extension N defines by means of the unitary character of W which represents the phase factor χ_0 of the symplectic Fourier transform $\mathcal{F}_W : \mathcal{S}(W) \longrightarrow \mathcal{S}(W)$ of order 2, a smooth complex line bundle over the symplectic frames at the points of W, still denoted N, with closed two–form ω of curvature and connection one–form θ.

The Rabi frequency ν on the central quantization axis of the real dual Lie(N)* can become comparable to the laser light frequency allowing the population to flip coherently from ground state to the excited quantum state and backwards during a laser pulse of only a few cycles of light. For the concept of carrier–wave Rabi flopping, let $\chi : t \rightsquigarrow e^{2\pi\nu it} \in \mathbf{T}$ denote a non–trivial character of $\mathbf{R}$, and particularly $\chi_0 : t \rightsquigarrow e^{2\pi it} \in \mathbf{T}$ the unitary character of normalized Rabi frequency label $\nu = 1$. Then $\chi_0(w, w') = \chi_0(\frac{1}{2} < w, w' >)$ for all pairs $(w, w') \in W \times W$ defines a two–cocycle on W. In terms of the central character $\chi \in \hat{\mathbf{R}}$ the phase–coherent action of the generic element

$$\begin{pmatrix} 1 & 0 & z' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (z' \in \mathbf{R})$$

belonging to the central quantization axis $x = y = 0$ on the trivialization $N \times \mathbf{C}$ of the complex vector bundle N over W with fiber isomorphic with $\mathbf{C}$ reads as follows:

$$\boxed{\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, v \right) \rightsquigarrow \left(\begin{pmatrix} 1 & x & z + z' \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \chi(z')v \right)}$$

Therefore the phase–coherent action takes place on the complex line $\mathbf{C}$ through the element

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in W,$$

for all numbers $v \in \mathbf{C}$.

In broad terms, representation theory of the Heisenberg step 2 nilpotent Lie group

N and the double Heisenberg group $2\,N$ is the study of the symmetries of quantum fields. The unitary dual $\hat{N}$ is determined by the fundamental

Strong Stone–von Neumann Theorem. There exists one and up to equivalence only one irreducible unitary linear representation ρ_χ of N with central character $\chi \in \hat{\mathbf{R}}$ which is square integrable modulo the central quantization axis.

The essentially *unique* representation ρ_χ of N admits a useful alternate realization on a complex vector space of holomorphic functions, the Fock representation of quantum state engineering. Recall that during the early stages of the development of quantum mechanics a central mathematical question was finding a representation for the canonical commutation relations. In terms of the twisted differentiation ∂_w at the point $w \in W$, and the Dirac measure $\varepsilon_0 \in \mathcal{S}'(\mathbf{R})$ located at the origin of the real symplectic plane W, these commutation relations admit the infinitesimal form $[\,\partial_w, \partial_{w'}] = 2\pi i < w, w' > \varepsilon_0$, which corresponds to the representational form

$$\boxed{[\,\rho_{\chi_0}(w), \rho_{\chi_0}(w')] = 2\pi i < w, w' > \mathrm{id}_{\mathcal{S}'(\mathbf{R})}}$$

for all pairs $(w, w') \in W \times W$ and the identity operator $\mathrm{id}_{\mathcal{S}'(\mathbf{R})}$ acting on the complex vector space of tempered distributions $\mathcal{S}'(\mathbf{R})$. The problem was solved by the Stone–von Neumann theorem. Although the irreducible unitary linear representation ρ_{χ_0} of the Heisenberg step 2 nilpotent Lie group N is unique up to equivalence, it admits different concrete realizations such as the Schrödinger model, the Bargmann–Segal model, or the Fock model of quantum state engineering mentioned above. The square integrability modulo the central quantization axis allows the application of Schur's lemma.

The closed exterior differential two–form $\omega = \mathrm{d}\theta \in \bigwedge^2 W^*$ is the natural symplectic form associated with the coadjoint orbit picture of N in the real dual vector space $\mathrm{Lie}(N)^*$ of the Heisenberg Lie algebra $\mathrm{Lie}(N)$. Therefore it explains the fact that the complex line bundle N over W represents the mathematical structure responsible for the laser–cooling techniques which actually depend on the mechanical properties of light momenta modelled by the momentum mapping of the Hamiltonian N–homogeneous manifold W^*.

4. Metaplectic Harmonic Analysis

The $\mathbf{R}$–linear action $g(w, z) = (g(w), z)$ of elements $g \in \mathbf{Sp}(W)$ on $(w, z) \in \mathrm{Lie}(N)$ embeds the symplectic group $\mathbf{Sp}(W) \cong \mathbf{SL}(2, \mathbf{R})$ into the automorphism group of $\mathrm{Lie}(N)$ and hence of N. In both cases, the action of $\mathbf{Sp}(W)$ on the one–dimensional center is trivial. The trivial action on the longitudinal quantization axis implements Emmy Noether's theorem on the conservation law for the angular momentum ([7]). The metaplectic group $\mathbf{Mp}(W)$ forms a two–fold cover of $\mathbf{Sp}(W)$. Thus there is the central short exact sequence

$$\{1\} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{Mp}(W) \longrightarrow \mathbf{Sp}(W) \longrightarrow \{1\}\,.$$

The Lie group $\mathbf{Mp}(W)$ exists and is unique up to isomorphy because a maximal compact subgroup of $\mathbf{Sp}(2,\mathbf{R})$ is isomorphic to the circle group $\mathbf{U}(1, \mathbf{C}) \cong \mathbf{Spin}(\mathbf{R}^2) \cong \mathbf{SO}(2, \mathbf{R})$, so that the fundamental group of $\mathbf{SL}(2,\mathbf{R})$ is isomorphic to the additive group of integers $\mathbf{Z} \cong \hat{\mathbf{T}}$. Thus $\mathbf{Mp}(W)$ gives rise to the short exact sequence

$$\{1\} \longrightarrow \mathbf{O}(2,\mathbf{R}) \longrightarrow \mathbf{Mp}(W) \longrightarrow \mathbf{Sp}(W) \longrightarrow \{1\},$$

which induces the aforementioned short exact sequence

$$\{1\} \longrightarrow \mathbf{Z}/2\,\mathbf{Z} \longrightarrow \mathbf{Mp}(W) \longrightarrow \mathbf{Sp}(W) \longrightarrow \{1\}.$$

The computational approach from inside demonstrates that $\mathbf{Mp}(W)$ is the closure of the normalized spin–oscillator semigroup. Its physical meaning is at the origin of the Jaynes–Cummings model of quantum optical resonance: A fermionic system consisting of a two–level atom interacts with a bosonic system consisting of a quantized single–mode field. This fermion–boson coupling leads to quantum entanglement of the atom and the mode field. The linear interaction model also predicts that, at *precisely* half of the revival time, the fermionic and bosonic system become disentangled.

Let $\tilde{g} \longrightarrow g$ denote the covering projector of $\mathbf{Mp}(W)$ onto $\mathbf{Sp}(W)$. A consequence of the Stone–von Neumann Theorem is the

Shale–Weil Theorem. There exists a unitary linear representation σ_χ of the metaplectic group $\mathbf{Mp}(W)$, unique up to equivalence, operating on the space of the irreducible unitary linear representation ρ_χ of N such that the covariance identity

$$\boxed{\sigma_\chi(\tilde{g})\rho_\chi(h)\sigma_\chi(\tilde{g}^{-1}) = \rho_\chi(g(h))}$$

holds for elements $\tilde{g} \in \mathbf{Mp}(W)$ and $h \in N$ in the complex Hilbert space $L^2(\mathbf{R})$ to be found among the sections of $\mathrm{Hilb}(\mathcal{S}'(\mathbf{R}))$ in the line bundle N over W.

In particular, the intertwining operators $\sigma_{\chi_0}(\tilde{I}_0) = \mathcal{F}_\mathbf{R}\,, \sigma_{\bar{\chi}_0}(\tilde{I}_0) = \bar{\mathcal{F}}_\mathbf{R}$ are nothing else than the Fourier transform and Fourier cotransform filters, respectively, acting on the complex Schwartz space $\mathcal{S}(\mathbf{R})$ of smooth vectors of the representations ρ_{χ_0} and σ_{χ_0}, and its complex dual vector space of tempered distributions $\mathcal{S}'(\mathbf{R})$. In the Schwartz isomorphism theorem, the Fourier cotransform $\bar{\mathcal{F}}_\mathbf{R}$ operates as a wave packet transform on the vector space $\mathcal{S}'(\mathbf{R})$.

Notice that the metaplectic representation σ_χ may be regarded as a projective unitary representation of $\mathbf{Sp}(W)$, and also as a unitary representation of the semi–direct product $\mathbf{Mp}(W) \times N$, or the semi–direct sum $\mathrm{Lie}(\mathbf{SL}(2,\mathbf{R})) \oplus \mathrm{Lie}(N)$ on $\mathcal{S}'(\mathbf{R})$. Notice also that σ_χ is faithful, but *not* irreducible, and that iI_0 acts diagonally on the vector subspace $\mathcal{S}(\mathbf{R})$ of $L^2(\mathbf{R})$ with eigenvalues in the set $\frac{1}{2}\mathbf{Z}$. The reducibility of σ_{χ_0} follows from the $\mathrm{Lie}(\mathbf{SL}(2,\mathbf{R}))$ *lowest* weight module decomposition

$$\boxed{\mathcal{S}(\mathbf{R}) = \mathcal{M}_{\frac{1}{2}} \oplus \mathcal{M}_{\frac{3}{2}}}$$

where $\mathcal{M}_{\frac{1}{2}}$ is spanned by the even quantum harmonic oscillator wave functions, and $\mathcal{M}_{\frac{3}{2}}$ is spanned by the odd quantum harmonic oscillator wave functions in the Schwartz space $\mathcal{S}(\mathbf{R})$ of complex–valued smooth functions, rapidly decaying at infinity. Therefore σ_{χ_0} is the direct sum of two irreducible unitary linear representations. It implies, together with the Lie($\mathbf{SL}(2,\mathbf{R})$) highest weight module decomposition associated with the action of the contragredient metaplectic representation $\check{\sigma}_{\chi_0} = \sigma_{\bar{\chi}_0}$ on the complex dual *highest* weight module decomposition

$$\boxed{\mathcal{S}'(\mathbf{R}) = \bar{\mathcal{M}}_{-\frac{1}{2}} \oplus \bar{\mathcal{M}}_{-\frac{3}{2}}}$$

the Kepplerian third law of planetary motion. This derivation of the Kepplerian third law depends on the embedding of the planetary motion into the unitary dual of the first order jet space $\mathcal{K}^1_{\mathbf{R}}(\mathbf{R}) \cong N$, and is independent of Pauli's quantum mechanical adaption of the familiar Runge–Lenz vector construction. It is in disagreement with the convention not to give Johannes Keppler a major position among the founding masters of symmetry and mechanics. It is in agreement with the unexpected feature that there is as yet no empirical evidence that quantum physics has limited applicability: A technique of generating elliptical atomic trajectories in planes perpendicular to the cavity axis is based on the *atom–cavity microscope*. Atoms are captured in a magneto–optical trap and dropped through a high–finesse optical cavity ([2]).

The metaplectic analogue of the internal $\mathbf{SO}(4,\mathbf{R})$ Lie group symmetry of the Coulomb gauge is the decomposition into mutual orthogonal, irreducible $\mathbf{O}(4,\mathbf{R}) \times \mathbf{Mp}(2,\mathbf{R})$–modules of the transcendental theory of spherical harmonics

$$\boxed{L^2(\mathbf{R}^4) = \bigoplus_{m\geq 0} \overline{\mathcal{H}^4_m \otimes \mathcal{M}_{2+m}}}$$

where $\mathcal{H}^4_m$ denotes the real vector space of spherical harmonic polynomials of degree $m \geq 0$ on $\mathbf{R}^4$, damped by the radial Gaussian distribution. The metaplectic representation is unique up to unitary equivalence, and is *genuine* in the sense that it does not factor through the symplectic group $\mathbf{Sp}(2,\mathbf{R})$. It is known under various names, for instance Shale–Weil representation, or spin–oscillator representation. It is a fascinating topic to realize that it is important in quantum state engineering and that its discovery underlies the theory of θ–series, especially as developed by Carl Ludwig Siegel, and gives rise to the natural duality between symplectic groups and orthogonal groups as expressed by the dual reductive pair $(\mathbf{Sp}(2,\mathbf{R}), \mathbf{O}(2,\mathbf{R}))$ in $\mathbf{Sp}(4,\mathbf{R})$. In particular, the symplectic group $\mathbf{Sp}(2,\mathbf{R})$ and the orthogonal group consisting of the two components of proper and improper orthogonal transformations

$$\mathbf{O}(2,\mathbf{R}) \cong \mathbf{SO}(2,\mathbf{R}) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{SO}(2,\mathbf{R})$$

are each other's centralizers inside $\mathbf{Sp}(4,\mathbf{R})$. Note that the compact manifold $\mathbf{O}(2,\mathbf{R})$ of all isotropic vector subspaces of the symplectic plane W gives rise to a principal covering circle bundle of symplectic spinors over the qubit space $W_{\mathbf{C}}$. Due to the fact that laser light pulses are modelled by tempered distributions which form the complex dual vector space $\mathcal{S}'(\mathbf{R})$ of the Schwartz space of smooth vectors $\mathcal{S}(\mathbf{R})$ of the representations ρ_{χ_0} and σ_{χ_0}, the principal covering circle bundle of symplectic spinors over $W_{\mathbf{C}}$ coherently controls the resonant laser field interactions.

The Hilbert subspace $L^2(\mathbf{R})$ of the complex vector space of tempered distributions

$\mathcal{S}'(\mathbf{R})$ admits as its Schwartz kernel the canonical injection K of $\mathcal{S}(\mathbf{R})$ into $\mathcal{S}'(\mathbf{R})$. In terms of the adjoint operator $\mathcal{F}^*_{\mathbf{R}} = {}^t\bar{\mathcal{F}}_{\mathbf{R}} = \bar{\mathcal{F}}_{\mathbf{R}}$ of the Fourier transform filter $\mathcal{F}_{\mathbf{R}}$, the image space $\mathcal{F}_{\mathbf{R}}(L^2(\mathbf{R}))$ admits in $\mathcal{S}'(\mathbf{R})$ the transversal Schwartz kernel of resonance $\mathcal{F}_{\mathbf{R}} \circ K \circ \mathcal{F}^*_{\mathbf{R}} = \mathcal{F}_{\mathbf{R}} \circ \mathcal{F}^*_{\mathbf{R}} = K$ which is the complex Hilbert space $L^2(\mathbf{R}) \in \mathrm{Hilb}(\mathcal{S}'(\mathbf{R}))$ itself ([11], [12]).

5. Linear Symmetries of Indistinguishability

The quantum state engineering implementation of the fundamental linear symmetry between matter and light wave transfer begins with the preparation of *two* isolated sodium Bose–Einstein condensates in a double–well potential formed by combining a harmonic magnetic trap and a repulsive optical dipol barrier. The entire potential is turned off 1 ms before the quantum information transfer begins, whereupon the probe and coupling optical laser beams are introduced. The quantum information transfer starts with the injection of a probe laser pulse into the first Bose–Einstein condensate while the atomic cloud is illuminated by the *counterpropagating* coupling optical laser beam of the same Rabi frequency label ν. The laser light pulse propagates into the condensate under ultraslow light conditions. After the light pulse is spatially compressed within the atomic cloud, the coupling beam is switched off, leaving an imprint of the probe pulse's phase and amplitude in the form of atomic population amplitude. Each atom's component has a momentum corresponding to *two* photon recoils, namely absorption from the probe beam and *stimulated emission of radiation* into the coupling beam, and is ejected towards the second Bose–Einstein condensate. When this messenger atom pulse arrives, the coupling beam is switched back on, and the probe light pulse is quantum holographically reproduced in the second Bose–Einstein condensate. Quantum holographically reproduced laser light pulses are imaged and then detected with a photomultiplier tube.

When laser light pulse storage and quantum holographic retrieval occur in two distinct atomic clouds separated before condensation, each atom's wave function is initially localized to either but not both of the two isolated Bose–Einstein condensates. Therefore, the dark state superposition imprinted during storage exists only for atoms from the first Bose–Einstein condensate. Nevertheless, a phase–coherent light pulse can still be reproduced from the second Bose–Einstein condensate through bosonic matter wave stimulation. In the symplectic spinor bundle configuration of a principal covering circle bundle, the coupling laser light field and the matter field for atoms form a symmetric pair: Bosonic stimulation into the macroscopically occupied photon field of the coupling optical laser drives the phase–coherent dynamics during the initial light pulse injection, whereas stimulation into the macroscopically occupied matter field of the second Bose–Einstein condensate secures phase–coherence during quantum holographic retrieval of the probe light pulse.

Because the symplectic Fourier transform $\mathcal{F}_W : \mathcal{S}(W) \longrightarrow \mathcal{S}(W)$ has order 2, a doubling procedure, analogous to the double symmetry Lie group $\mathbf{SO}(4,\mathbf{R})$ of the interaction in the Coulomb gauge, is needed to perform the order 4 of the Fourier transform filter $\mathcal{F}_{\mathbf{R}}$ and its adjoint filter $\mathcal{F}^*_{\mathbf{R}} = \bar{\mathcal{F}}_{\mathbf{R}}$. The double symplectic embedding procedure provides the Heisenberg group $2\,N = \mathrm{Heis}(W_{\mathbf{C}})$ corresponding to

$\mathbf{SU}(2,\mathbf{C}) \times \mathbf{SU}(2,\mathbf{C}) \hookrightarrow \mathbf{SO}(4,\mathbf{R})$.

Define injections j_1 and j_2 of N into $2\,N$ by the formulae $j_1(w,z) = ((w,0),z)\,, j_2(w,z) = ((0,-w),-z)$ for $w \in W$ and $z \in \mathbf{R}$. Thus $j_1 \times j_2 : N \longrightarrow 2\,N$ is a surjective homomorphism with the diagonal of the centers as kernel. Let $2\,\rho_{\chi_0}$ be the representation of the double Heisenberg group $2\,N$ corresponding to the central character χ_0. Then there is an isomorphism

$$2\,\rho_{\chi_0} \circ (j_1 \times j_2) \cong \rho_{\chi_0} \otimes \rho_{\bar{\chi}_0}$$

onto the outer tensor product of the irreducible unitary linear representation ρ_{χ_0} of N and its contragredient version $\check{\rho}_{\chi_0} = \rho_{\bar{\chi}_0}$. Because χ_0 is a non–trivial central character of N, the irreducible unitary linear representations ρ_{χ_0} and $\rho_{\bar{\chi}_0}$ are inequivalent. The preceding outer tensor product identity establishes that, even as the amplitude and phase of the wave functions representing the Bose–Einstein condensate temporally evolve, the relative phase of the two components continues to be well defined.

Let $\mathbf{Sp}(W_{\mathbf{C}})$ be the symplectic group of the qubit space $W_{\mathbf{C}}$. There are two contragredient embeddings of $\mathbf{Sp}(W)$ into $\mathbf{Sp}(W_{\mathbf{C}})$: $j_1(g)(w_1,w_2) = (g(w_1),w_2)\,, j_2(g)(w_1,w_2) = (w_1, g(w_2))$ for $g \in \mathbf{Sp}(W)$ and pairs $(w_1,w_2) \in W_{\mathbf{C}} = W^+_{I_0} \oplus W^-_{I_0}$. These mappings lift uniquely to maps between the corresponding metaplectic groups. It follows from the outer tensor product identity above the identity

$$2\,\sigma_{\chi_0} \circ (j_1 \times j_2)\tilde{} = \sigma_{\chi_0} \otimes \sigma_{\bar{\chi}_0}$$

where $\check{\sigma}_{\chi_0} = \sigma_{\bar{\chi}_0}$ denotes the contragredient metaplectic representation of $\mathbf{Mp}(W)$. Switching the circularly polarized field of the coupling optical laser back on at a later time writes the information back onto a quantum holographically reproduced probe pulse which then propagates out of the atomic cloud. Due to the isotropic quantum harmonic oscillator model of tightly confined Bose–Einstein condensates, the metaplectic symmetry of the symplectic spinor bundle configuration is represented by the outer tensor product

$$\boxed{2\,\sigma_{\chi_0} \circ (j_1 \times j_2)\tilde{}(iI_0) = (\sigma_{\chi_0} \otimes \sigma_{\bar{\chi}_0})(i\tilde{I}_0)}$$

Due to the Schwartz kernel theorem, this linear symmetry of the Heisenberg picture $2\,\rho_{\chi_0} \circ (j_1 \times j_2)$ associates with $L^2(\mathbf{R})$ among the sections of $\mathrm{Hilb}(\mathcal{S}'(\mathbf{R}))$ the *positive* Schwartz kernel or transversal reproducing kernel $K : \mathcal{S}(\mathbf{R}) \hookrightarrow \mathcal{S}'(\mathbf{R})$ of laser light pulses in $L^2(\mathbf{R})$. It coherently controls the resonant fields of laser light pulses interacting with ultracold, dense atomic clouds and turns the symplectic spinor bundle configuration into a particularly rich photonic system of quantum optical information retrieval. As coherently controlling the retrieval time to within tens of microseconds controls the propagation depth of the messenger atom pulse to micrometer precision, the light pulse can be reproduced at metaplectic frame positions where the phase patterns of the messenger and second Bose–Einstein condensate match. The norm identity $||K||_{L^2(W)} = 1$ establishes the experimental result that the revived light pulses have the same shape as the incoming light pulse and that there are no detectable losses from the storage and quantum holographic retrieval processes ([1], [8]). At a fundamental level, the observation of retrievals reveals optical laser field entanglement. Information about the atomic state is phase–coherently imprinted onto the laser light field. Due to indistinguishability combined with Schur's lemma for square integrable representations, the induced retrieval results from the quantum erasure of the photonic imprint onto the matter field, and from the unitary destruction of the laser light field entanglement. The *collapse* phenomenon and the associated *revival* are manifestations of the quantum physical complementary ([2]).

It follows from the involutive action $\sigma_\chi \rightsquigarrow \sigma_{\bar{\chi}}$ on metaplectic representations of $\mathbf{Mp}(W)$ that the quantum holographically reproduced output pulses were indistinguishable in width and amplitude from non–stored ultraslow light pulses, indicating that the switching process preserves the quantum optical information in the atomic medium during the storage time with a high degree of fidelity. Thus the phase–coherent storage of photon states in matter reveals to be a reversible linear photonic process.

6. Conclusions and Perspectives

The advent of optical lasers stemmed directly from fundamental research, in that it was a discovery which owed nothing to any expectation of practical usage. In fact, it was the outcome of pure curiosity research. Albert Einstein's theory of stimulated emission of radiation which was concerned with the thermodynamic study of the interaction of an ensemble of atoms with electromagnetic radiation was developed at the dawn of the quantum era. It slept peacefully in the archives of science until the physicist Charles Townes demonstrated in 1956 that a microwave resonator could be suitable to realize an ammonia maser operating at 23 GHz. Translating the resonator concept to the window of optical frequencies, Thomas H. Maiman succeeded in 1960 in constructing by means of a Fabry–Perot resonator a coherent quantum optical source, the ruby laser producing quantum optical signals well suited to the transfer of information.

Combining monochromaticity and high intensity has proved crucial to cool and trap atoms to extremely low temperatures. In this way, the laser permits to tame radiation by exploiting the properties of atomic stimulated emission. The recent developments of atom optics to display the wavelike properties of matter on a macroscopic size scale of quantum state engineering, and the methodology of quantum information processing have radically changed quantum physics and represents a source of new mathematical models and questions. In this mathematical models, the Heisenberg nilpotent Lie group N and the Hopf fibered symmetry group $\mathbf{SU}(2, \mathbf{C})$ as well as the metaplectic group $\mathbf{Mp}(2, \mathbf{R})$ of automorphisms of N play a dominant role.

An important aspect of quantum state engineering is the elliptic non–Euclidean geometry of the projectivization of the qubit space $\mathbf{P}(W_\mathbf{C})$. It leads to the concepts of Clifford translations of the first and second kind. These are bijections of $\mathbf{P}(W_\mathbf{C})$ forming a group isomorphic to $\mathbf{SO}(3,\mathbf{R})$. The spin group $\mathbf{SU}(2, \mathbf{C}) \cong \mathbf{Spin}(\mathbf{R}^3) \cong \mathbf{S}_3$ is a non–trivial covering group of the rotation group $\mathbf{SO}(3,\mathbf{R})$ and gives rise to the compact Heisenberg nilmanifold which forms a principal circle bundle over the two–dimensional compact torus $\mathbf{T}^2 \cong \mathbf{S}_1 \times \mathbf{S}_1$ with the niltheta functions as special cases of its automorphic forms. The associated three–dimensional lattice L is given by the $\mathbf{Z}$–module

$L \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. The geometry of the Heisenberg nilmanifold underlies the magnetic trap as well the quantum Hall effect in a double–layer two–dimensional electron system in a strong perpendicular magnetic field. It is important to emphasize that the small value of the counterflow Hall resistance *does not* result from a cancellation of opposite sign quantum Hall effects in two layers but from spontaneous interlayer phase–coherence. This state exhibits the quantum Hall effect when equal electrical currents flow in parallel through the two layers. In contrast, if the currents in the two layers are equal, but oppositely directed, both the longitudinal and Hall resistances of each layer vanish in the low–temperature limit. The experimental counterflow setup ensures only that the total current flowing in the two layers are globally equal and oppositely directed. It does not eliminate the possibility of local regions where the two currents are not precisely equal. The interlayer collective phase may be considered in several equivalent ways, including as a Bose–Einstein condensate of interlayer excitons, or a pseudospin ferromagnet.

A Bose–Einstein condensate superimposed on the optical lattice L permits to drive the system into a Mott insulating phase. The double–layer configurations occurring in the foliated unitary dual $\hat{N}$ contain a unique homogeneous real plane associated with the one–dimensional unitary linear representations and hence corresponding to the characters of N. Although often neglected, this *singular* plane governs the quantum collapse and revival phenomena of the matter wave field of a Bose–Einstein condensate as well as the transition from a superfluid to a Mott insulator in a gas of ultracold atoms. The associated line bundle plays a major role in the series of classical experiments of stellar interferometry performed by the asrophysicist and astronomer Robert Hanbury Brown in collaboration with the mathematician Richard Q. Twiss, as well in their modern atomic optics versions which establish the *bunching* process of photonics as a resonance phenomenon. The bunching phenomenon of the atomic analog of the Hanbury Brown–Twiss effect below the Bose–Einstein condensation transition temperature can be visualized by a position sensitive microchannel plate detector placed below the center of the magnetic trap. Position sensitvity is achieved by means of a delay line anode placed at the rear side of the microchannel plate detector.

On the other hand, photons in a laser were *not* bunched. The double–layer configuration, augmented with the *annihilating* singular plane Lie(center)° of quantum collapses, visualizes geometrically the wave field–particle duality inside $\mathrm{Lie}(N)^*$. Either bunching or quantum coherence occurs at low temperatures above and below the Bose–Einstein condensation threshold, respectively, but *not* both phenomena simultaneously.

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