Conceptual Representation of Particles, Waves, and Heisenberg's Uncertainty Relation

Karl Erich WOLFF[1]

University of Applied Sciences Darmstadt, Department of Mathematics
Schöfferstr. 3, D-64295 Darmstadt, Germany
ERNSTSCHRÖDERCENTER FOR CONCEPTUAL KNOWLEDGE PROCESSING
Research Group Concept Analysis at Darmstadt University of Technology
Fax: +49 6257-69361, E-mail: karl.erich.wolff@t-online.de

Abstract

This paper introduces a mathematical representation of the fundamental physical notions *particles*, *waves*, and *Heisenberg's Uncertainty Relation* in such a general way that neither Hilbert spaces nor the real or complex numbers are used. This new approach is based on the mathematical notion of a *lattice* as defined by Birkhoff [2] (1940) who introduced lattices as a generalization of hierarchies in geometry, logic and algebra. In 1982 lattice theory has been connected by Wille [22] (1982) with the philosophical construct of a *concept* using a mathematical definition of *formal concepts* and *concept lattices*.

Formal Concept Analysis (FCA), the mathematical theory of concept lattices, was then used by the author to introduce *Temporal Concept Analysis* [30] which is based on *Conceptual Time Systems* where the notion of a *state* is introduced as a formal concept. The conceptual definition of *life tracks of objects* led to a generalization of the formal representation of objects in *Conceptual Semantic Systems* [29] where *distributed objects* yield a clear mathematical representation of the idea of a *wave packet* together with a definition of *particles* and *waves*.

In this paper the author's previous definitions of particles and waves are extended, the notion of *measurement* is introduced and combined with the notion of a *view* and a (distributed) object to represent "how distributed" that object is represented by the measurement in the chosen view. That leads to a conceptual analogue of the notion of *"simultaneously measurable"* in Quantum Theory and to a conceptual analogue of Heisenberg's Uncertainty Relation.

Keywords: particles, waves, states, Heisenberg's Uncertainty Relation, concept lattices, conceptual semantic systems

---

[1] This work is supported by DFG project COMO, GZ: 436 RUS 113/829/0-1.

1 Introduction

The purpose of this paper is to present, for physicists, some new and general ideas concerning the notions of *particles*, *waves*, and *Heisenberg's Uncertainty Relation*. The central background for these ideas is a mathematical theory called *Formal Concept Analysis* [10] (*FCA*) which is based on a mathematization of *concept* and *concept hierarchy* as introduced by Wille (1982) [22].

What is the purpose for introducing a new mathematical theory for physicists who seem to be quite satisfied with classical mathematics [3,6,8,9,11,12,19,21] based on real and complex numbers, on statistics, vector spaces, metrics, tensors and manifolds? A short and coarse answer to this question is that the development of discrete mathematics during the last fifty years led to new conceptual tools which are powerful also in the foundations of physics. That yields a clear understanding of many problems around uncertainty and precision, particles and waves, space and time, state and situation, measurement and probability, and the identity of objects [5,6]. Some first results will be described in this paper.

One of the main advantages of FCA is its ability to handle notions around *granularity* in a simple and effective way. That covers the granularity of more or less coarse measurement values as well as the granularity of organizational concepts like university, department, student, of spatial concepts like point, place, country, or the continuously fine granularity of the real numbers. Such conceptual structures can be represented by *concept lattices* in FCA. The great success of the lattice of real numbers and continuous functions in applications supported the belief that "Natura non facit saltus" which became problematically when Max Planck introduced his quantum of action.

1.1 Planck's Quantum of Action

When Max Planck investigated the black-body radiation in 1900 [18] he could derive his radiation law only under the assumption that the energy is not a continuous, infinitely divisible quantity, but a discrete quantity composed of an integral number of finite equal parts. In his paper [18] Max Planck wrote:

> *It is now a matter of finding the probability W so that the N resonators together possess the vibrational energy U_N. Moreover, it is necessary to interpret U_N not as a continuous, infinitely divisible quantity, but as a discrete quantity composed of an integral number of finite equal parts. Let us call each such part the energy element ε; consequently we must set*
>
> (4) $U_N = P\,\varepsilon$
>
> *where P represents a large integer generally, while the value of ε is yet uncertain.*

Based on the energy element ε Max Planck introduced and determined numerically his famous quantum of action h by

$$\varepsilon = h \nu$$

which led to Planck's energy distribution law as a continuous function of temperature and frequency.

Since granular structures like quantized actions and quantized energies did not fit with his well-established continuous world view Planck called his introduction of the quantum of action an "act of desperation".

1.2 Einstein's Light-Quanta, his Granularity Remark, and Physical Objects

While Planck's introduction of the quantum of action was first understood by him "as a formal assumption" [7], p. 49 Einstein's hypothesis of light-quanta [8], introduced to explain the photo-electric effect, was considered by Bohr as a very important contribution [1], p. 25. While Einstein started to investigate the wavelike phenomena of light by the notion of light-quanta de Broglie proposed in 1926 a converse connection between particles and waves, namely [1], p. 26

that the matter, and particularly the electrons, should be considered as wavelike.

Auletta continues [1], p. 26

> *The situation was now very difficult because there was evidence at the same time of a corpuscular nature and of a wavelike nature of microphysical entities (constituents of both radiation or matter). This is the problem of Wave/Particle Dualism, the heart and the only mystery of QM [Feynman et al. 1965, 1-1].*

Connected to the problem of Wave/Particle Dualism are several problems related to granularity in physics. Some of these problems are mentioned by Einstein [9] in his footnote on page 893 (translated by the author):

> *The inaccuracy which lies in the concept of simultaneity of two events at (about) the same place and which has to be bridged also by an abstraction, shall not be discussed here.*

In this paper we shall discuss this inaccuracy by a formal treatment of granularity using conceptual semantics. That will also lead to a clear mathematical treatment of the problems around "physical objects" as discussed in the fine collection of important papers on that subject by Elena Castellani [6]. We quote from the beginning of her introduction:

Whichever are the particular descriptions employed, having mass, being located in space and time, and persisting through time seem to constitute the fundamental features required for something to qualify as a "physical object".

Many of the main problems around “identity”, “individuality”, “part/whole relation”, “genidentity”, “space-time location” can be investigated using conceptual semantics. Besides the problems around granularity of objects, space, and time also the dependencies between uncertainties of different observables had been investigated. Using Planck's quantum of action Heisenberg [13] (1927) introduced his famous Uncertainty Relation

$$\Delta q \, \Delta p \geq h/4\pi$$

where Δq and Δp are standard deviations of the Gaussian distributions for the position q and the momentum p of a quantum particle in one spatial dimension [1], p. 118. For more details the reader is referred to [31].

In the next sections we shall introduce a conceptual analogue of Heisenberg's Uncertainty Relation described in the framework of Conceptual Semantic Systems [27,28,29,31].

2 Conceptual Semantics

A central motivation for logicians like E. Schröder and C. S. Peirce was to investigate reality by suitable formal representations which can be used as tools for human communication. The basic tools in human communication are statements or judgments which are built from words denoting concepts. Usually these concepts are accepted in the community as elements of often-used semantics. In the following we represent concepts, their semantics and statements about these concepts in a mathematical structure, called a *Conceptual Semantic System*.

In physics the most accepted formal semantics are the semantics of numbers, mainly those of the real and complex numbers. It is well-known that *Newton* and *Leibniz* introduced the infinitesimal calculus long before *Dedekind* (1872) found a mathematically clear construction of the set of real numbers from the ordered set $(\mathbf{Q}, \leq)$ of rational numbers using so-called *Dedekind cuts*. That famous construction can be understood in Formal Concept Analysis in such a way that each Dedekind cut is a *formal concept* in the sense of FCA, namely an element of a *concept lattice* constructed from the ordered set $(\mathbf{Q}, \leq)$ of rational numbers as described precisely in the following sections.

Slightly more general semantics than those for numbers are the semantics for tuples of numbers, mainly used as coordinate tuples. In the standard case of n-tuples of real numbers for a given integer n the n-dimensional Euclidean vector space is the classical semantics for these tuples.

From data tables with rows representing tuples of numbers it is a straightforward step to data tables whose values are arbitrary values, not necessarily numbers. In data base theory the only generally used semantics for the values is the distinction whether they are equal or not. As usual we call that a nominal semantics. In the following section we introduce much more general conceptual semantics. They are described as *formal contexts* in Formal Concept Analysis (FCA) as explained in the next section.

3 Formal Concept Analysis

FCA has its mathematical roots in the theory of ordered sets, in particular in the theory of lattices as introduced by *Birkhoff* [2] (1940). The origins of lattice theory lie in the hierarchies of structures arising from *algebra*, *geometry*, and *mathematical logic*. Examples of lattices are the Boolean lattices in logic, the lattice of all subspaces of a vector space, and as a well-known example from physics the lattice of all closed subspaces of a Hilbert space which is the basic structure in Quantum Logic. These lattices can be represented in FCA as concept lattices which are used as the main tools for a conceptual description of semantics.

By definition, a lattice is an ordered set $(L, \leq)$ (i.e. $\leq$ is a reflexive, antisymmetric, and transitive relation on L) such that for any elements $x, y \in L$ there exist the infimum inf(x,y):= the greatest lower bound of x, y and the supremum sup(x,y):= the least upper bound of x, y.

These abstract lattices have been connected by Wille (1982) [22] with the philosophical construct of a *concept*. In traditional philosophy a concept is understood as a unit of thought consisting of two parts, the extension and the intension where the extension consists of all objects belonging to the concept and the intension comprises all attributes valid for all those objects.

To have a clear mathematical definition of a concept Wille introduced first the notion of a *formal context* (G, M, I) where G is a set whose elements are called (*formal*) *objects* (*Gegenstände* in German), M is a set of elements called (*formal*) *attributes* (*Merkmale* in German) and I is a binary relation between G and M, i.e. $I \subseteq G \times M$. If a formal object g and a formal attribute m are related by I, that is $(g,m) \in I$, we say that *g has the attribute m* or *m is valid for g*, denoted by *g I m*.

To define the notion of a *formal concept* of a formal context (G,M,I) we employ the upper and lower derivations $\uparrow$, $\downarrow$ where for any $X \subseteq G$ the upper derivation of X is the set of common attributes of X, denoted by:

$$X^{\uparrow} := \{m \in M \mid \forall g \in X \;\; g\, I\, m\},$$

and for any subset $Y \subseteq M$ the lower derivation of Y is the set of all objects which have all attributes of Y, denoted by:

$$Y^{\downarrow} := \{g \in G \mid \forall m \in Y \;\; g\, I\, m\}.$$

The following definition of a *formal concept* is the basic definition in FCA:

Definition: A *formal concept* of a formal context **K** := (G, M, I) is a pair (A,B) where

$$A \subseteq G,\ B \subseteq M \text{ and } A^{\uparrow} = B \text{ and } B^{\downarrow} = A.$$

For any formal concept (A,B) the set A is called the *extent*, the set B the *intent* of (A,B). The set of all formal concepts of **K** is denoted by **B(K)**.

The conceptual hierarchy among concepts is defined by set inclusion:
For $(A_1, B_1), (A_2, B_2) \in \mathbf{B(K)}$ let
$(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$).

An important role is played by the *object concepts* $\gamma(g) := (\{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow})$ for $g \in G$ and dually the *attribute concepts* $\mu(m) := (\{m\}^{\downarrow}, \{m\}^{\downarrow\uparrow})$ for $m \in M$.
The ordered set $(\mathbf{B(K)}, \leq)$ is a complete lattice, called the *concept lattice of* ***K***. By definition, a complete lattice is an ordered set such that any subset has an infimum and a supremum [2]. In $(\mathbf{B(K)}, \leq)$ the infimum of a subset $S \subseteq \mathbf{B(K)}$ is the formal concept whose extent is the intersection of the extents of the formal concepts in S; the supremum of S is the formal concept whose intent is the intersection of the intents of the formal concepts in S. It is shown in the Basic Theorem of FCA [10] that any complete lattice is isomorphic to a concept lattice.

For example, the complete lattice $(\mathbf{R} \cup \{\infty, -\infty\}, \leq)$ consisting of the ordered set of real numbers together with ∞ and $-\infty$ is isomorphic to the concept lattice of the context $(\mathbf{Q}, \mathbf{Q}, \leq_Q)$ where $(\mathbf{Q}, \leq_Q)$ is the rational order. For further details the reader is referred to the textbook on FCA [10].

In the following section we use formal contexts to describe hierarchies which are used as semantics for knowledge domains.

4 Conceptual Semantic Systems

We now recall the basic definitions for Conceptual Semantic Systems as introduced by the author [27,28,29].

4.1 Basic Definitions

Definition: "Conceptual Semantic System"

Let M be a set and, for each $m \in M$, let $S_m := (G_m, N_m, I_m)$ be a formal context and $\underline{\mathbf{B}}(S_m) := (\mathbf{B}(S_m), \leq_m)$ its concept lattice; let G be a set and

$$\kappa: G \times M \rightarrow \bigcup_{m \in M} \mathbf{B}(S_m)$$

be a mapping such that $\kappa(g,m) \in \mathbf{B}(S_m)$.
Then the quadruple $\mathbf{C} := (G, M, (\underline{\mathbf{B}}(S_m))_{m \in M}, \kappa)$ is called a *Conceptual Semantic System (CSS) with semantic scales* S_m ($m \in M$). The elements of M are called many-valued attributes; the elements of G are called *instances*. We write $m(g) := \kappa(g,m)$ and $m(G) := \{m(g) | g \in G\}$ and mention that κ can be represented by a data table whose values m(g) "in column m" are formal concepts of the given semantic scale S_m.

We interpret the concepts of the semantic scales as "types" and the concept lattice of a semantic scale as a "type hierarchy". The equation "m(g) = **c**" is interpreted as "instance g tells something about the concept $\mathbf{c} \in \mathbf{B}(S_m)$". For any instance g the tuple $(m(g) \mid m \in M)$ is interpreted as a short description of a statement connecting the concepts m(g) where $m \in M$. A concept m(g) may denote for example the grammatical subject, or the grammatical object, or the grammatical predicate of a statement. That allows for the representation of arbitrary, not only binary, relations. A special example is the parametric representation of the unit circle by triples (t, cos(t), sin(t)) where $0 \leq t \leq 2\pi$. In that sense a CSS is a parametric representation of relational conceptual knowledge.

One of the central points in our intended interpretation of Conceptual Semantic Systems is that the instances are not interpreted as concepts, as for example objects like persons or particles. That differs strongly from the usual interpretation of many-valued contexts [10] where the formal objects are used to represent objects like persons or other entities which then have to form a key in the data table of the many-valued context. In Conceptual Semantic Systems we are much more free in practical applications since we do not need an object domain whose objects form a key in the data table. That freedom allows for the representation of "distributed concepts" which will be introduced in the following. But first of all we represent the information given in a data table of κ by a single formal context, called the *semantically derived context*.

Definition: "Semantically Derived Context"

Let $\mathbf{C} := (G, M, (\underline{\mathbf{B}}(S_m))_{m \in M}, \kappa)$ be a Conceptual Semantic System with semantic scales $S_m := (G_m, N_m, I_m)$ and let int(**c**) denote the intent of a concept **c**. Then the formal context $\mathbf{K} := (G, N, J)$ where $N := \{(m,n) \mid m \in M, n \in N_m\}$ and

$$g\ J\ (m,n) :\Leftrightarrow n \in \mathrm{int}(m(g))$$

is called the *semantically derived context of* **C**.

To express this definition in a short tabular language we can say that we construct a data table of the formal context **K** from a data table of κ by the rule: "Replace each $m \in M$ (in the head of each column) by the set $\{(m,n) \mid n \in N_m\}$ and replace each concept m(g) by its (characteristic function of the) intent". Therefore the CSS $(G, M, (\underline{\mathbf{B}}(S_m))_{m \in M}, \kappa)$ can be reconstructed from **K** and $(S_m)_{m \in M}$.

It is easy to see that the semantically derived context of a CSS can be obtained also by plain scaling as the usual derived context [8]. Therefore we write in the following only "derived context" instead of "semantically derived context".

4.2 Occuring Concepts and Realizations

Definition: "occuring scale concepts"

Let $\mathbf{C} := (G, M, (\underline{\mathbf{B}}(S_m))_{m \in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} = (G, N, J)$, and for $m \in M$ let γ_m be the object concept mapping of

$$\mathbf{K}_m := (G, \{m\} \times N_m, J \cap (G \times (\{m\} \times N_m))), \text{ the m-part of } \mathbf{K}.$$

For $m \in M$ and a concept $\mathbf{c} = (A_c, B_c) \in \mathbf{B}(S_m)$ and $g \in G$ we say that

- *c occurs at instance g in* $\mathbf{K}_m$ $:\Leftrightarrow$ $\mathrm{int}(\gamma_m(g)) = \{m\} \times B_c$

- c *co-occurs at instance g in* $\mathbf{K}_m$ $:\Leftrightarrow$ $\mathrm{int}(\gamma_m(g)) \supseteq \{m\}\times B_c$.

For $m \in M$ and a concept $\mathbf{c} = (A_c, B_c) \in \mathbf{B}(S_m)$ we say that

- c *occurs in* $\mathbf{K}_m$ $:\Leftrightarrow$ $\mathrm{int}(\gamma_m(g)) = \{m\}\times B_c$ for some instance g
- c *co-occurs in* $\mathbf{K}_m$ $:\Leftrightarrow$ $\mathrm{int}(\gamma_m(g)) \supseteq \{m\}\times B_c$ for some instance g.

By definition of the derived context $m(g) = \mathbf{c} \Leftrightarrow \mathrm{int}(\gamma_m(g)) = \{m\}\times B_c$. Hence

$$m(g) = m(h) \Leftrightarrow \gamma_m(g) = \gamma_m(h),$$
$$m(g) \leq m(h) \Leftrightarrow \gamma_m(g) \leq \gamma_m(h)$$
$$\text{for all } m \in M,\ g, h \in G.$$

Clearly, $m(g) \leq \mathbf{c} \Leftrightarrow \mathrm{int}(\gamma_m(g)) \supseteq \{m\}\times B_c$.

It is obvious that these notions are very useful for a conceptual investigation of *reasoning with granularity*, for example to conclude from "My father took a flight from Frankfurt to London" that the statement "A man travelled from Germany to Great Britain" also holds.

The formal concepts of the semantic scales yield "realized concepts" in the derived context $\mathbf{K} = (G, N, J)$ as introduced in the following definition.

Definition: "Realization of a concept of a semantic scale"

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} = (G, N, J)$. Then for $m \in M$ the following mapping

$$r_m\colon\ \mathbf{B}(S_m) \to \mathbf{B}(\mathbf{K})$$

which maps $\mathbf{c} = (A_c, B_c) \in \mathbf{B}(S_m)$ to

$$r_m(\mathbf{c}) := ((\{m\}\times B_c)^{\downarrow}, (\{m\}\times B_c)^{\downarrow\uparrow})$$

is called the *m-realization of c*. Let $R_m := \{r_m(\mathbf{c}) \mid \mathbf{c} \in \mathbf{B}(S_m)\}$ be the set of all m-realizations and $R := \{r_m(\mathbf{c}) \mid m\in M, \mathbf{c} \in \mathbf{B}(S_m)\}$ the set of all realizations. We call $\mathbf{L}(R) := \{\bigwedge D \mid D \subseteq R\}$ *the lattice of realizations*.

The m-realization of $\mathbf{c}$ is the greatest concept in $\mathbf{B}(\mathbf{K})$ containing $\{m\}\times B_c$ in its intent. We emphasize to use the m-realization of a concept $\mathbf{c}$ in $\mathbf{B}(S_m)$ as a formal description of a "really observed concept", as for example a "really observed whale" as opposed to the semantic concept "whale"; the "really observed whale" is connected with the instances which "tell something about this whale" since these instances form the extent of the realization of the semantic concept.

Lemma 1:

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K}$. Let $m \in M$, $\mathbf{c}_1 = (A_1, B_1)$, $\mathbf{c}_2 = (A_2, B_2) \in \mathbf{B}(S_m)$. Then

$$\mathbf{c}_1 \leq \mathbf{c}_2 \Leftrightarrow r_m(\mathbf{c}_1) \leq r_m(\mathbf{c}_2).$$

Proof: $\mathbf{c}_1 \leq \mathbf{c}_2 \Leftrightarrow \{m\}\times B_1 \supseteq \{m\}\times B_2 \Leftrightarrow (\{m\}\times B_1)^{\downarrow} \subseteq (\{m\}\times B_2)^{\downarrow} \Leftrightarrow r_m(\mathbf{c}_1) \leq r_m(\mathbf{c}_2)$ where the second equivalence holds since $\{m\}\times B_1$ and $\{m\}\times B_2$ are intents of concepts in $\mathbf{K}_m$ and for $i\in \{1,2\}$ the $\mathbf{K}_m$-extent $(\{m\}\times B_i)^{\downarrow m}$ equals the $\mathbf{K}$-extent $(\{m\}\times B_i)^{\downarrow}$.

The instances in the extent of the realization of a concept $\mathbf{c} \in \mathbf{B}(S_m)$ are characterized in the following Lemma 2 which connects occuring concepts and their realizations.

Lemma 2:

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K}$. Let $g \in G$, $m \in M$, $\mathbf{c} = (A_c, B_c) \in \mathbf{B}(S_m)$, γ the object concept mapping of $\mathbf{K}$, and $^\downarrow$ the lower derivation in $\mathbf{K}$. Then

$$m(g) \leq \mathbf{c} \Leftrightarrow g \in (\{m\}\times B_c)^\downarrow \Leftrightarrow \gamma(g) \leq r_m(\mathbf{c}).$$

Proof: The first equivalence holds by definition of the derived context. The second equivalence holds since

$g \in (\{m\}\times B_c)^\downarrow \Leftrightarrow \forall_{h\in G} (g^\uparrow \subseteq h^\uparrow \rightarrow h \in (\{m\}\times B_c)^\downarrow) \Leftrightarrow ext(\gamma(g)) \subseteq ext(r_m(\mathbf{c}))$

where $^\uparrow$ denotes the upper derivation in $\mathbf{K}$.

In spite of the equivalence in Lemma 2 it is not true that $(m(g) = \mathbf{c} \Rightarrow \gamma(g) = r_m(\mathbf{c}))$ since there are examples with $m(g) = \mathbf{c}$ and $\gamma(g) < r_m(\mathbf{c})$. Also the converse implication $(\gamma(g) = r_m(\mathbf{c}) \Rightarrow m(g) = \mathbf{c})$ does not hold, which can be easily shown by counterexamples where $\gamma(g) = r_m(\mathbf{c})$ and $m(g) < \mathbf{c}$ and $\mathbf{c}$ does not occur in $\mathbf{K}_m$. But if $\mathbf{c}$ occurs at some instance h in $\mathbf{K}_m$ and $m(g) < m(h) = \mathbf{c}$ then $\gamma(g) < r_m(\mathbf{c})$, since $m(g) < m(h)$ implies that there exists an $n \in N_m$ such that $(m,n) \in g^\uparrow \setminus h^\uparrow$, hence $(m,n) \in int(\gamma(g))$ and $(m,n) \notin int(r_m(\mathbf{c}))$, hence $\gamma(g) \neq r_m(\mathbf{c})$, but $\gamma(g) \leq r_m(\mathbf{c})$ by Lemma 2, therefore $\gamma(g) < r_m(\mathbf{c})$.

4.3 Locations

The purpose for the introduction of the following notations is to define in Conceptual Semantic Systems the notion of "particles" in the physical sense that "each particle is at each moment at exactly one place" and to distinguish these "particles" from "distributed objects" like waves or wave packets which "occupy at each moment the whole space or some part of the space".

In Conceptual Semantic Systems the notions of "moment" and "place" can be defined with respect to a suitable granularity as formal concepts of semantic scales of many-valued attributes for "time" and "space".

As opposed to classical space-time theories we also represent objects, as for example "particles", "waves", "systems", "diseases" and so on, as formal concepts of semantic scales which yields a simple representation of sub-systems or of parts of objects.

A basic example for the combination of only three kinds of things, namely "objects", "moments", and "places" is the formal representation of movements in spatio-temporal systems. For that purpose the author has introduced spatio-temporal Conceptual Semantic Systems where three many-valued attributes P, T, L for "persons" (or "objects"), for "time" and for "locations" are specified (cf. Wolff [27]).

In the special case that the set {P,T} forms a key, which means that the mapping P×T which maps each instance $g \in G$ to the pair $(P(g), T(g))$ is injective, the spatio-temporal Conceptual Semantic Systems have been investigated by the author using the name "Conceptual Time Systems with actual Objects and Time relation (CTSOT)". In a CTSOT each object p is at each moment t (time granule) at exactly one place, namely at the object concept of the formal object (p,t) in the space part. For applications of CTSOTs the reader is referred to [30].

A crucial step on the way to a formal definition of "particles" and "waves" was the introduction of Conceptual Semantic Systems where we do not assume that a subset of the set of many-valued attributes forms a key. Clearly the singleton set $\{G\}$ consisting of the set G of instances of a CSS is used as an "artificial key".

The main consequence of the introduction of the key of instances is that an instance g can tell that an object p was at time t at place x and another instance h can tell that the same object p was at the same time t at another place y. That is very unusual if we interpret objects as it is usually done in physics; but it is very common if we say for example that a person p was during the last year at place x and at place y. In that sense a person may be understood as a "distributed object". To introduce the definition of a "particle" we need some notion for the set of places where an object was during some time. For that purpuse we use Conceptual Semantic Systems with three many-valued attributes P, T, L where we interpret the tuple $(P(g), T(g), L(g))$ as the statement that the "person" $P(g)$ *was* at "time granule" $T(g)$ at "place" $L(g)$ without a formal representation of the relation "*was*". Clearly we could represented the relation "was" as a formal concept of some semantic scale for relations.

Definition: "L-location, occuring places, occuring time granules"

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K}$. Let P, T, L are elements of M, $\mathbf{p} \in \mathbf{B}(S_P)$, $\mathbf{t} \in \mathbf{B}(S_T)$, and γ_L (resp. γ_T) the object concept mapping of $\mathbf{K}_L$ (resp. $\mathbf{K}_T$) the L-part (T-part) of the derived context $\mathbf{K}$. Then

$$\lambda_L(\mathbf{p},\mathbf{t}) := \{\gamma_L(g) | P(g) = \mathbf{p}, T(g) = \mathbf{t}\}$$

is called *the L-location of the actual object* $(\mathbf{p},\mathbf{t})$. The set $\gamma_L(G) := \{\gamma_L(g) \mid g \in G\}$ is called the set of *occuring places* in $\mathbf{K}_L$, the set $\gamma_T(G) := \{\gamma_T(g) \mid g \in G\}$ is called the set of *occuring time granules* in $\mathbf{K}_T$.

The L-location $\lambda_L(\mathbf{p},\mathbf{t})$ is defined as the set of places in $\mathbf{K}_L$ which occur at instances g where also $\mathbf{p}$ and $\mathbf{t}$ occur. We remark that the occuring place $\gamma_L(g)$ has as intent the set $\{L\} \times B_c$ where $B_c = \text{int}(L(g))$, hence $\gamma_L(g)$ determines the "place" $L(g)$ in the concept lattice of the semantic scale S_L since $\gamma_L(g) = \gamma_L(h) \Leftrightarrow L(g) = L(h)$.

In the following we generalize the notion of an L-location slightly to a *location with respect to a view Q* where $Q \subseteq N$ and N is the set of attributes of the derived context $\mathbf{K} = (G,N,J)$. We use the concept lattice $\mathbf{B}(\mathbf{K}_Q)$ of the Q-part $\mathbf{K}_Q := (G, Q, J \cap (G \times Q))$ of $\mathbf{K}$ like a map in which the *Q-locations* of actual objects $(\mathbf{p},\mathbf{t})$ or more generally of tuples of scale concepts are represented. Tuples of scale concepts can be used to describe "concatenations of concepts" in colloquial speech.

Definition: "Q-location of a tuple of scale concepts"

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} = (G,N,J)$. For $M^* \subseteq M$ we call $\tau(M^*) := \times(\mathbf{B}(S_m) \mid m\in M^*)$ *the set of tuples of concepts over* M^*, hence a tuple $(\mathbf{c}_m \mid m\in M^*) \in \tau(M^*)$ iff $\forall_{m\in M^*}\ \mathbf{c}_m \in \mathbf{B}(S_m)$.
If $M^* = \{m\}$ we replace the 1-tuple $(\mathbf{c}_m)$ by the single concept $\mathbf{c}_m$.
Let $Q \subseteq N$ and γ_Q the object concept mapping of the Q-part $\mathbf{K}_Q := (G, Q, J \cap (G \times Q))$ of $\mathbf{K}$. Then

$$\lambda_Q(\mathbf{c}_m \mid m\in M^*) := \gamma_Q(\{g \in G \mid \forall_{m\in M^*}\ m(g) = \mathbf{c}_m\})$$

is called *the Q-location of the tuple* $(\mathbf{c}_m \mid m\in M^*)$ of scale concepts.

4.4 Aspects of Concepts and Instance Selections

Closely related to the Q-locations are the *aspects of concepts with respect to a view* which can be defined in an arbitrary formal context.

Definition: "Q-aspect of a concept"

Let $\mathbf{K} := (G, N, J)$ be a formal context, $Q \subseteq N$, and γ_Q the object concept mapping of the Q-part $\mathbf{K}_Q := (G, Q, J \cap (G \times Q))$ of $\mathbf{K}$. For any $\mathbf{d} = (A_\mathbf{d}, B_\mathbf{d}) \in \mathbf{B}(\mathbf{K})$ the set

$$\alpha_Q(\mathbf{d}) := \{\gamma_Q(g) \mid g \in A_\mathbf{d}\} \quad (= \gamma_Q(ext(\mathbf{d})))$$

is called the *aspect of the concept* $\mathbf{d}$ *with respect to the view* Q or *the Q-aspect of* $\mathbf{d}$.

In this paper we use the notion of a "Q-aspect of a concept" only for the concepts of the derived context $\mathbf{K} := (G, N, J)$ of a CSS. Then the extent $A_\mathbf{d}$ of a concept $\mathbf{d} \in \mathbf{B}(\mathbf{K})$ is a subset of the set G of instances. For any *view* $Q \subseteq N$ this extent $A_\mathbf{d}$ is mapped by the object concept mapping γ_Q onto the Q-aspect of $\mathbf{d}$. The introduction of views allows not only for combining different parts of the derived context, but also for many kinds of "factorizations" by selecting special attributes of the derived context.

The following Lemma 3 relates the Q-aspects and the Q-locations of a tuple of concepts.

Lemma 3:

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} := (G, N, J)$. Let $M^* \subseteq M$, and for each $m\in M^*$ let $\mathbf{c}_m \in \mathbf{B}(S_m)$. Let $Q \subseteq N$ and γ_Q the object concept mapping of the Q-part $\mathbf{K}_Q$ of $\mathbf{K}$. Then

(1) $\alpha_Q(\bigwedge\{r_m(\mathbf{c}_m) \mid m\in M^*\}) = \{\gamma_Q(g) \mid \forall_{m\in M^*}\ m(g) \leq \mathbf{c}_m\} \supseteq \lambda_Q(\mathbf{c}_m \mid m\in M^*)$,

and equality holds in the inclusion in (1) if

(2) $\forall_{m\in M^*}\ \forall_{g\in G}\ (m(g) \leq \mathbf{c}_m \rightarrow m(g) = \mathbf{c}_m)$.

Proof: Since $ext(\bigwedge\{r_m(\mathbf{c}_m) \mid m\in M^*\}) = \bigcap\{ext(r_m(\mathbf{c}_m)) \mid m\in M^*\}$ (by the Basic Theorem of FCA, cf. [10], p. 20) we obtain from Lemma 2
$\alpha_Q(\bigwedge\{r_m(\mathbf{c}_m) \mid m\in M^*\}) = \{\gamma_Q(g) \mid \forall_{m\in M^*}\ g \in ext(r_m(\mathbf{c}_m))\} =$

$\{\gamma_Q(g) \mid \forall_{m\in M^*}\ g \in (\{m\}\times \mathrm{int}(\mathbf{c}_m))^{\downarrow}\} = \{\gamma_Q(g) \mid \forall_{m\in M^*}\ m(g) \leq \mathbf{c}_m\} \supseteq$
$\{\gamma_Q(g) \mid \forall_{m\in M^*}\ m(g) = \mathbf{c}_m\} = \lambda_Q(\mathbf{c}_m \mid m\in M^*)$.

Remark 1: Condition (2) in Lemma 3 is not necessary for the equality in the inclusion in (1), since γ_L may be not injective.
Remark 2: Condition (2) holds if for all $m\in M^*$ the semantic scale S_m is a nominal scale and $\mathbf{c}_m$ is an object concept and only object concepts of S_m occur in $\mathbf{K}_m$.

Corollary:

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System. Let P, T, L are elements of M, $\mathbf{p} \in \mathbf{B}(S_P)$, $\mathbf{t} \in \mathbf{B}(S_T)$, and let α_L denote the aspect function with respect to the view $\{(L,n) \mid n \in N_L\}$ where N_L is the set of attributes of the scale S_L. Then

(C1) $\alpha_L(r_P(\mathbf{p}) \wedge r_T(\mathbf{t})) = \{\gamma_L(g) \mid P(g) \leq \mathbf{p}, T(g) \leq \mathbf{t}\} \supseteq \{\gamma_L(g) \mid P(g) = \mathbf{p}, T(g) = \mathbf{t}\}$
and equality holds in the inclusion in (1) if
(C2) $\forall_{g\in G}\ (P(g) \leq \mathbf{p} \rightarrow P(g) = \mathbf{p})$ and $\forall_{g\in G}\ (T(g) \leq \mathbf{t} \rightarrow T(g) = \mathbf{t})$.

Remark 3: That the aspect $\alpha_L(r_P(\mathbf{p}) \wedge r_T(\mathbf{t}))$ may be a proper superset of the *location* $\lambda_L(\mathbf{p},\mathbf{t}) := \{\gamma_L(g) \mid P(g) = \mathbf{p}, T(g) = \mathbf{t}\}$ of the "actual person $(\mathbf{p}, \mathbf{t})$" had not been made explicit in previous publications of the author. But the distinction between $\alpha_L(r_P(\mathbf{p}) \wedge r_T(\mathbf{t}))$ and $\lambda_L(\mathbf{p},\mathbf{t})$ led the author to the following definition of an *instance selection*.

Definition: "instance selection"

Let $\mathbf{C} := (G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System. For $M^* \subseteq M$ any mapping $\sigma: \tau(M^*) \rightarrow \mathbf{P}(G) := \{X \mid X \subseteq G\}$ is called an *instance selection on* $\tau(M^*)$ and for $\mathbf{c} \in \tau(M^*)$ the set $\sigma(\mathbf{c}) \subseteq G$ is called the *instance selection of* $\mathbf{c}$ or the *selection of* $\mathbf{c}$.

The following two examples of instance selections σ_α and σ_λ which are defined for $\mathbf{c} := (\mathbf{c}_m \mid m\in M^*) \in \tau(M^*)$ by

1. $\sigma_\alpha(\mathbf{c}) := \{g \in G \mid \forall_{m\in M^*}\ m(g) \leq \mathbf{c}_m\}$
2. $\sigma_\lambda(\mathbf{c}) := \{g \in G \mid \forall_{m\in M^*}\ m(g) = \mathbf{c}_m\}$

yield for the Q-aspect $\alpha_Q(\bigwedge\{r_m(\mathbf{c}_m) \mid m\in M^*\}) = \gamma_Q(\sigma_\alpha(\mathbf{c}))$ and for the Q-location $\lambda_Q(\mathbf{c}) = \gamma_Q(\sigma_\lambda(\mathbf{c}))$. Clearly, σ_λ is the usual selection in data base theory.

4.5 Precise and Distributed Tuples of Concepts

As a preparation for the definition of particles and waves in Conceptual Semantic Systems we now introduce the notions that a tuple $\mathbf{c}$ is represented *precisely* respectively *distributed* in some view Q. The main example is that a particle $\mathbf{p}$ is at each time granule $\mathbf{t}$ at exactly one place which will be formally expressed as "the tuple $(\mathbf{p},\mathbf{t})$

is precise in $\mathbf{B}(\mathbf{K}_Q)$ with respect to the instance selection σ".

Definition: "σ-precise, σ-distributed, fully σ-distributed tuples of scale concepts"

Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} = (G,N,J)$, $M^* \subseteq M$ and $\sigma\colon \tau(M^*) \to \mathbf{P}(G)$ an instance selection on $\tau(M^*)$.
Let $Q \subseteq N$, then a tuple $\mathbf{c} \in \tau(M^*)$ is called

- *σ-precise in* $\mathbf{B}(\mathbf{K}_Q)$ if $|\gamma_Q(\sigma(\mathbf{c}))| = 1$;
- *σ-distributed in* $\mathbf{B}(\mathbf{K}_Q)$ if $|\gamma_Q(\sigma(\mathbf{c}))| \geq 2$;
- *fully σ-distributed in* $\mathbf{B}(\mathbf{K}_Q)$ if $\gamma_Q(\sigma(\mathbf{c})) = \gamma_Q(G)$.

This definition generalizes a definition introduced in [31] by the author:

A tuple $\mathbf{c} \in \tau(M^*)$ is σ_α-precise, σ_α-distributed, fully σ_α-distributed in $\mathbf{B}(\mathbf{K}_Q)$ iff the concept $\mathbf{d} := \bigwedge\{r_m(\mathbf{c}_m) | m\in M^*\}$ is precise, distributed, fully distributed in $\mathbf{B}(\mathbf{K}_Q)$.

The following Lemma 4 relates the instance selections σ_α and σ_λ.

Lemma 4:
Let $(G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} := (G, N, J)$. Let $M^* \subseteq M$, $\mathbf{c} \in \tau(M^*)$, and $Q \subseteq N$. Then

(4.1) $\mathbf{c}$ is σ_α-precise in $\mathbf{B}(\mathbf{K}_Q)$ $\Rightarrow$ $\mathbf{c}$ is σ_λ-precise in $\mathbf{B}(\mathbf{K}_Q)$,

(4.2) $\mathbf{c}$ is σ_λ-distributed in $\mathbf{B}(\mathbf{K}_Q)$ $\Rightarrow$ $\mathbf{c}$ is σ_α-distributed in $\mathbf{B}(\mathbf{K}_Q)$,

(4.3) $\mathbf{c}$ is fully σ_λ-distributed in $\mathbf{B}(\mathbf{K}_Q)$ $\Rightarrow$ $\mathbf{c}$ is fully σ_α-distributed in $\mathbf{B}(\mathbf{K}_Q)$.

Proof: Lemma 4 follows from Lemma 3.

4.6 Particles and Waves in Conceptual Semantic Systems

In [27] the definition of particles and waves in spatio-temporal Conceptual Semantic Systems (where three many-valued attributes P, T, L are specified for the description of objects, time and space) was given in terms of a certain location of an actual object. We now define that in a more general setting where only "the time attribute T" is specified, and "objects" are represented by tuples, and instance selections are used.

Definition: "particles and waves"
Let $\mathbf{C} = (G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be as CSS with derived context $\mathbf{K} = (G, N, J)$.
Let $T \in M$; we call T the *time attribute* of the *time-CSS* $(\mathbf{C},T)$. Let $Q \subseteq N$, and $\mathbf{K}_Q$ the Q-part of $\mathbf{K}$. Let $M^* \subseteq M \setminus \{T\}$, and let σ be an instance selection on $\tau(M^*\cup\{T\})$; for each tuple $\mathbf{c} \in \tau(M^*)$ and each $\mathbf{t} \in \mathbf{B}(S_T)$ the set $\sigma(\mathbf{c},\mathbf{t})$ is called the *selection of (c,t)*. With respect to this instance selection σ we call a tuple $\mathbf{c} \in \tau(M^*)$ a

- *σ-particle in* $\mathbf{B}(\mathbf{K}_Q)$ if $|\gamma_Q(\sigma(\mathbf{c},\mathbf{t}))| \leq 1$ for all $\mathbf{t} \in \mathbf{B}(S_T)$ occuring in $\mathbf{K}_T$;
- *σ-wave in* $\mathbf{B}(\mathbf{K}_Q)$ if $|\gamma_Q(\sigma(\mathbf{c},\mathbf{t}))| \geq 2$ for all $\mathbf{t} \in \mathbf{B}(S_T)$ occuring in $\mathbf{K}_T$;
- *full σ-wave in* $\mathbf{B}(\mathbf{K}_Q)$ if $\mathbf{c}$ is a σ-wave in $\mathbf{B}(\mathbf{K}_Q)$ and $\gamma_Q(\sigma(\mathbf{c},\mathbf{t})) = \gamma_Q(G)$ for all $\mathbf{t} \in \mathbf{B}(S_T)$ occuring in $\mathbf{K}_T$.

The inclusion in (1) in Lemma 3 shows that each σ_α-particle in **B**(**K**$_Q$) is a σ_λ-particle in **B**(**K**$_Q$), and each σ_λ-wave in **B**(**K**$_Q$) is an σ_α-wave in **B**(**K**$_Q$), and each full σ_λ-wave in **B**(**K**$_Q$) is a full σ_α-wave in **B**(**K**$_Q$). In the following example we show how to apply the given definitions.

5 A Particle-Wave Example: Surfer on a Wave

The purpose of the following small example is to show how moving objects, for example particles and waves in the sense of physics, can be represented using Conceptual Semantic Systems. We choose a very small discrete representation of the movement of a surfer from the crest of a propagating wave to the trough. The surfer as well as the wave will be represented as formal concepts in a small semantics for "objects". Then we shall see that – with respect to some view Q – the surfer is a σ_λ-particle, and the wave is a σ_λ-wave in the sense of the given definitions.

We start by introducing all necessary concepts as formal concepts of suitable semantics. To make explicit that the formal representation of the surfer and the wave only uses that the surfer and the wave are two distinct concepts such that the surfer is not a wave and the wave is not a surfer we employ the following nominal semantics S_O for the objects

=	Surfer	Wave
Surfer	×	
Wave		×

Table 1: Nominal semantics S_O for Surfer and Wave

The concept lattice **B̲**(S_O) is ure 1.

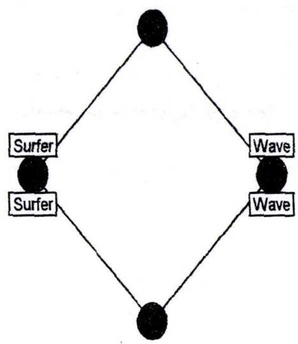

Figure 1: The concept lattice of the nominal semantics for the objects

We denote the object concept of the formal object Surfer by the (bold-written) word **Surfer** which will be used as a value in the data table of the following CSS. Similarly

Wave denotes the object concept of the formal object Wave.

For the representation of time we employ the following ordinal semantics $S_T := (\{0, 1, 2\}, \{0, 1, 2\}, \geq)$:

≥	0	1	2
0	×		
1	×	×	
2	×	×	×

Table 2: Ordinal semantics S_T for time

The concept lattice $\underline{\mathbf{B}}(S_T)$ is represented in the coarse structure of the nested diagram in Figure 2. The object concepts of 1, 2, 3 are denoted by **1**, **2** , **3**.

For a simple representation of the space in which the surfer and the wave move we employ a small discrete plane with only five points on the x-axis and three points on the z-axis as shown in the inner diagram of Figure 2. For the conceptual representation of the x-axis we use the ordinal semantics

$$S_x := (\{0, 1, 2, 3, 4\}, \{0, 1, 2, 3, 4\}, \geq)$$

yielding a concept lattice $\underline{\mathbf{B}}(S_x)$ which is a chain with five concepts, called **0**, **1**, **2**, **3**, **4**,

instances	objects	time	x	z
1	**Surfer**	**0**	**0**	**1**
2	**Surfer**	**1**	**2**	**0**
3	**Surfer**	**2**	**4**	**-1**
4	**Wave**	**0**	**0**	**1**
5	**Wave**	**0**	**1**	**0**
6	**Wave**	**0**	**2**	**-1**
7	**Wave**	**0**	**3**	**0**
8	**Wave**	**0**	**4**	**1**
9	**Wave**	**1**	**0**	**0**
10	**Wave**	**1**	**1**	**1**
11	**Wave**	**1**	**2**	**0**
12	**Wave**	**1**	**3**	**-1**
13	**Wave**	**1**	**4**	**0**
14	**Wave**	**2**	**0**	**-1**
15	**Wave**	**2**	**1**	**0**
16	**Wave**	**2**	**2**	**1**
17	**Wave**	**2**	**3**	**0**
18	**Wave**	**2**	**4**	**-1**

Table 3: Data table for CSS $\mathbf{C_1}$: "Surfer on a Wave"

and for the z-axis we employ the ordinal semantics

$$S_z := (\{-1, 0, 1\}, \{-1, 0, 1\}, \leq)$$

yielding a concept lattice $(\mathbf{B}(S_Z), \leq)$ which is a chain with three concepts $\mathbf{-1} \leq \mathbf{0} \leq \mathbf{1}$.

Now we construct a CSS $\mathbf{C_1} := (G, M, (\underline{\mathbf{B}}(S_m))_{m \in M}, \kappa)$ where $G := \{1,...,18\}$, $M :=$ {objects, time, x, z}, $S_{objects} := S_O$, $S_{time} := S_T$, S_x and S_z as defined above, and κ is given as in Table 3. Clearly, T = time is chosen as the specified time attribute.

We interpret instance 1 as the statement that the **Surfer** was at time **0** at the place described by the x-coordinate **0** and the z-coordinate **1**. Instance 4 is interpreted as the statement that the **Wave** has at time **0** at the x-coordinate **0** the amplitude **1**.

The derived context of the CSS $\mathbf{C_1}$ is given in the following Table 4:

K	objects		time			x					z		
instances	Surfer	Wave	0	1	2	0	1	2	3	4	-1	0	1
1	×		×			×							×
2	×		×	×		×	×	×				×	×
3	×		×	×	×	×	×	×	×	×	×	×	×
4		×	×			×							×
5		×	×			×	×					×	×
6		×	×			×	×	×			×	×	×
7		×	×			×	×	×	×			×	×
8		×	×			×	×	×	×	×			×
9		×	×	×		×						×	×
10		×	×	×		×	×						×
11		×	×	×		×	×	×				×	×
12		×	×	×		×	×	×	×		×	×	×
13		×	×	×		×	×	×	×	×		×	×
14		×	×	×	×	×					×	×	×
15		×	×	×	×	×	×					×	×
16		×	×	×	×	×	×	×					×
17		×	×	×	×	×	×	×	×			×	×
18		×	×	×	×	×	×	×	×	×	×	×	×

Table 4: The derived context **K** of the CSS $\mathbf{C_1}$

Reading example: Instance 2 has in the time part the attributes (time,0) and (time,1) since the intent of the formal concept time(2) = $\mathbf{1} \in \mathbf{B}(S_T)$ is int($\mathbf{1}$) = {0,1}. That shows how the concepts of the semantics are represented in the derived context by their intents.

Now we select a view, namely the spatio-temporal view
Q := {(time,0), (time,1), (time,2),(x,0),(x,1),(x,2),(x,3),(x,4),(z,-1),(z,0),(z,1)}
consisting of the attributes in the (last 11) "spatio-temporal" columns in Table 4 and draw the concept lattice of $\mathbf{K}_Q$ in form of a nested line diagram (in Figure 2) where the

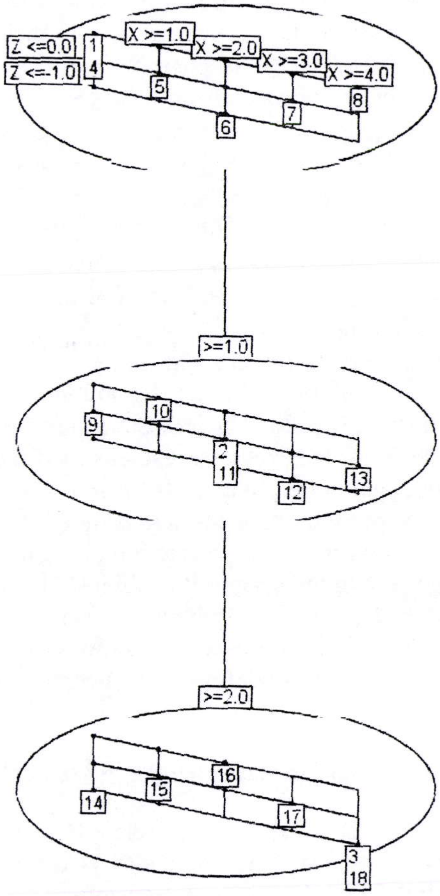

Figure 2: Surfer on a Wave

coarse diagram represents the ordinal time semantics S_T.

As an example we consider the semantic concept $\mathbf{c} := \mathbf{Surfer} \in \mathbf{B}(S_O)$ and its realization $r_{obj}(\mathbf{Surfer}) = (\{1,2,3\}, \{1,2,3\}^{\uparrow})$ where $\{1,2,3\}^{\uparrow} =$ $\{(\text{objects,Surfer}), (\text{time},0), (x,0), (z,1)\}$ is the upper derivation of $\{1,2,3\}$ in $\mathbf{K}$. We now show that the "actual object Surfer at time 0", formally described by the tuple $(\mathbf{Surfer},\mathbf{0})$ is σ_α-distributed in $\mathbf{B}(\mathbf{K}_Q)$, but σ_λ-precise in $\mathbf{B}(\mathbf{K}_Q)$. For that purpose we observe that $\gamma_Q(3) < \gamma_Q(2) < \gamma_Q(1)$, hence by Lemma 3 $\gamma_Q(\sigma_\alpha(\mathbf{Surfer},\mathbf{0})) =$ $\gamma_Q(\{g\in G \mid \text{objects}(g) \leq \mathbf{Surfer}, \text{time}(g) \leq \mathbf{0}\}) = \alpha_Q(r_{obj}(\mathbf{Surfer}) \wedge r_T(\mathbf{0})) = \gamma_Q(\{1,2,3\})$ since $\text{time}(1) = \mathbf{0}$ and $\mathbf{0}$ co-occurs at instances 2 and 3 in $\mathbf{K}_T$. Therefore the tuple $(\mathbf{Surfer},\mathbf{0})$ is σ_α-distributed in $\mathbf{B}(\mathbf{K}_Q)$, but clearly σ_λ-precise in $\mathbf{B}(\mathbf{K}_Q)$, since $\gamma_Q(\sigma_\lambda(\mathbf{Surfer}, \mathbf{0}))= \{\gamma_Q(1)\}$. Obviously the concept **Surfer** is a σ_λ-particle in $\mathbf{B}(\mathbf{K}_Q)$, but not an σ_α-particle in $\mathbf{B}(\mathbf{K}_Q)$.

We just mention, that in the modified CSS $\mathbf{C_{1_n}}$ which is obtained from $\mathbf{C_1}$ by replacing the ordinal time semantics by a *nominal* time semantics the concept **Surfer** is not only a σ_λ-particle, but also an σ_α-particle in the concept lattice of the Q-part.

Focussing now in $\mathbf{C_1}$ on the formal concept **Wave** we first mention that $\gamma_Q(\sigma_\lambda(\mathbf{Wave,0})) = \gamma_Q(\{4,5,6,7,8\})$ has 5 concepts, hence $(\mathbf{Wave,0})$ is σ_λ-distributed in $\mathbf{B(K_Q)}$; indeed, the concept **Wave** is a full σ_λ-wave in $\mathbf{B(K_Q)}$ since $|\gamma_Q(\sigma_\lambda(\mathbf{Wave,t})| = 5 \geq 2$ for all $\mathbf{t} \in \mathbf{B}(S_T)$ occuring in $\mathbf{K_T}$, and $\gamma_Q(G) = \gamma_Q(G\backslash\{1,2,3\}) = \gamma_Q(\sigma_\lambda(\mathbf{Wave}))$, which can be seen in Figure 2 since $\gamma_Q(1) = \gamma_Q(4)$, $\gamma_Q(2) = \gamma_Q(11)$, $\gamma_Q(3) = \gamma_Q(18)$. These three places are the places where "the **Surfer** is on the **Wave**".

We now interpret the very coarse visualization of the **Surfer** on the **Wave** in Fig. 2. At time = **0** the distribution of the five object concepts in $\gamma_Q(\{4,5,6,7,8\})$ can be understood as a discrete form of the cos-function whose first crest is the place of the **Surfer** at x = **0** and z = **1**. At time = **1** the distribution of the five object concepts in $\gamma_Q(\{9,10,11,12,13\})$ can be understood as a discrete sin-function which is interpreted as the result of a "right-shift" of the previous discrete cos-function. At time = **2** we see a discrete (–cos)-function, again interpreted as the result of a "right-shift" of the previous discrete sin-function. The place of its last trough is the place of the **Surfer** at instance 3.

The CCS $\mathbf{C_1}$ is clearly a very coarse description of the movement of a surfer from the crest of a propagating wave to the trough. It is obvious that temporal phenomena can be represented with this method in any granularity, even in the granularity of the real numbers. The purpose of this example was to show how particles and waves can be concretely represented and generally defined in Conceptual Semantic Systems.

6 A Conceptual Analogue of Heisenberg's Uncertainty Relation

In this section some first steps are done to relate Heisenberg's Uncertainty Relation to Conceptual Semantic Systems. The main idea is to use the generality of concept lattices and the flexibility of Conceptual Semantic Systems to make explicit the granularity notions in the ideas around Heisenberg's Uncertainty Relation.

There are *three main steps* which lead from Heisenberg's Uncertainty Relation to the conceptual analogue which will be presented in this paper.

The first step is to replace the very regular and therefore useful and simple structure of a vector space by the structure of a concept lattice. Both structures are used as tools for the representation of knowledge, the vector spaces with its nice algebraic structure, usually together with a metric, serve as the standard frame for the representation of several ideas of "spaces"; concept lattices can be used to represent the ordinal structure of multi-dimensional real spaces; but concept lattices can also be used to represent semantic structures as for example tree structures or nominal scales.

The second step is to replace the well-established and for many purposes very useful idea of functional dependencies by the more general idea of relational dependencies. A very simple example is the unit circle in the real plane which can not be described by a function which maps the x-coordinate to the y-coordinate, but by the relation $\{(x,y)|\ x^2 + y^2 = 1\}$ – which can be expressed again by a function, namely by the function which maps each "parameter" $t \in [0, 2\pi]$ to (t, cos(t), sin(t)). In Conceptual Semantic Systems the instances play the role of such a parameter which connects meaningful values, or concepts, in a relational way.

The third step is the introduction of distributed objects, or more generally of distributed concepts in Conceptual Semantic Systems. It seems to be very fruitful to make explicit how the mathematical representation of distributed concepts can be used to describe the ideas around Heisenberg's Uncertainty Relation in a conceptual framework.

In this paper we do not yet try to connect the mathematical quantum-theoretical formalism of Hilbert spaces and the lattice of closed sub-spaces of a Hilbert space with Conceptual Semantic Systems.

6.1 Uncertainty, Distributions, Variances

The term "Uncertainty" in "Heisenberg's Uncertainty Relation" is a translation of the German word "Unschärfe" which refers to "not sharp" or "not precise". "Uncertainty" is usually associated with a "broad" distribution of measurement values, for example a continuous distribution like a Gaussian distribution on the real axis. Whether a distribution is "broad" or "narrow" is usually measured by a single number, namely its variance. The variance is used in formal representations of Heisenberg's Uncertainty Relation. Clearly, the definition of the variance employs addition, subtraction, the square, the square-root, and in higher dimensions much more classical algebraical, metrical and analytical tools which are not available in more general knowledge representations as for example in concept lattices.

Though variances of distributions can not be defined in arbitrary concept lattices, the distributions arising from arbitrary data are well represented, for example as the distribution of formal objects (instances in CSSs) on the set of object concepts of the derived context.

6.2 Conceptual Precision, Locations, Aspects

As opposed to "uncertainty" in the sense of the German word "Unschärfe", the term "precision" refers to some kind of exactness. A typical example of precision is the assignment of a single number to some "measured" object, or slightly more general, the assignment of a single concept as an element of a semantic scale to a concept of another semantic scale. Clearly, if the chosen semantic scale is coarse, the measurement value may be precise in the sense of being just a single concept in a semantic scale, but it may be not very informative since some subconcept would be more appropriate. Therefore it seems to be suitable to formalize the term "precision" with respect to a certain granularity notion, for which we employ semantic scales.

The main idea in our formal representation of "precision" is that a formal concept $\mathbf{c} \in m(G)$ is *precisely* represented in the concept lattice of some view Q if the instances where $\mathbf{c}$ occurs in $\mathbf{K}_m$ have the same object concept in $\mathbf{B}(\mathbf{K}_Q)$

$$m(g) = \mathbf{c},\ m(h) = \mathbf{c} \Rightarrow \gamma_Q(g) = \gamma_Q(h)$$

which is equivalent to $|\gamma_Q(\sigma_\lambda(\mathbf{c}))| = 1$ (for each concept $\mathbf{c} \in m(G)$).
The stronger condition

$$m(g) \leq \mathbf{c},\ m(h) \leq \mathbf{c} \Rightarrow \gamma_Q(g) = \gamma_Q(h)$$

is equivalent to $|\gamma_Q(\sigma_\alpha(\mathbf{c}))| = 1$ (for each concept $\mathbf{c} \in m(G)$).

In the Surfer-Wave example the tuple **(Surfer,0)** is σ_λ-precise in $\mathbf{B}(\mathbf{K}_Q)$ since $|\gamma_Q(\sigma_\lambda(\mathbf{Surfer,0}))| = 1$ (for the chosen spatio-temporal view Q), but it is not σ_α-precise in $\mathbf{B}(\mathbf{K}_Q)$ since $|\gamma_Q(\sigma_\alpha(\mathbf{Surfer,0}))| = 3$.

6.3 A Conceptual Analogue of Heisenberg's Uncertainty Relation

The following definition of a conceptual analogue of Heisenberg's Uncertainty Relation is based on the idea that a certain system (or subsystem) $\mathbf{c}$ may be "precisely measurable in two spaces" which corresponds to "simultaneously measurable" in Quantum Theory. For that purpose we introduce "measurements" as formal concepts of a semantic scale. Each "system" or "subsystem" is represented as a tuple $\mathbf{c}$ of formal concepts, and for each measurement ξ and each tuple $\mathbf{c}$ we introduce a selection $\sigma(\xi,\mathbf{c})$ *of the measurement* ξ *and the tuple* $\mathbf{c}$ which is interpreted as the set of those instances which "refer" to the measurement ξ and the tuple $\mathbf{c}$ in the given CSS. Clearly, as in practice, a given tuple $\mathbf{c}$ can be measured repeatedly in different measurements. The "space" into which a tuple is measured is represented as a view of the derived context. In the most famous example of Heisenberg's Uncertainty Relation a single particle is measured with respect to two views Q and Q' describing the x-axis and the momentum-axis.

Definition: "precisely measurable, Heisenberg's Uncertainty Relation"

Let $\mathbf{C} := (G, M, (\underline{\mathbf{B}}(S_m))_{m\in M}, \kappa)$ be a Conceptual Semantic System with derived context $\mathbf{K} = (G,N,J)$ and $m_o \in M$. We call the formal concepts of the semantic scale of m_o *measurements*. For $M^* \subseteq M \setminus \{m_o\}$ let σ be an instance selection on $\tau(\{m_o\} \cup M^*)$; for each measurement ξ and each tuple $\mathbf{c} \in \tau(M^*)$ the set $\sigma(\xi,\mathbf{c}) \subseteq G$ is called the *selection of* $(\xi\mathbf{c})$. Let Q, Q' $\subseteq$ N. With respect to σ we define:

- A tuple $\mathbf{c} \in \tau(M^*)$ is called *precisely measurable in* Q *and* Q' if there exists a measurement ξ such that

$$|\gamma_Q(\sigma(\xi,\mathbf{c}))| = 1 = |\gamma_{Q'}(\sigma(\xi,\mathbf{c}))| .$$

- Q is called *c-complementary to* Q' if for all measurements ξ

$$|\gamma_Q(\sigma(\xi,\mathbf{c}))| \cdot |\gamma_{Q'}(\sigma(\xi,\mathbf{c}))| > 1.$$

We call the last inequality *the conceptual analogue of Heisenberg's Uncertainty Relation with respect to the chosen instance selection* σ.

- Q is called *complementary to* Q' *in* M^* if for all $\mathbf{c} \in \tau(M^*)$
  Q is $\mathbf{c}$-complementary to Q'.

Special choices for the instance selection:

Let ξ be a measurement, $\mathbf{c} = (\mathbf{c}_m | m \in M^*) \in \tau(M^*)$:

1. For $\sigma_\lambda(\xi,\mathbf{c}) := \{g \in G \mid m_o(g) = \xi, \forall_{m \in M^*}\, m(g) = \mathbf{c}_m\}$ the set $\gamma_Q(\sigma_\lambda(\xi,\mathbf{c}))$ is called *the Q-location of* **c** *in measurement* ξ.
2. For $\sigma_\alpha(\xi,\mathbf{c}) := \{g \in G \mid m_o(g) \leq \xi, \forall_{m \in M^*}\, m(g) \leq \mathbf{c}_m\}$ the set $\gamma_Q(\sigma_\alpha(\xi,\mathbf{c}))$ is called *the Q-aspect of* **c** *in measurement* ξ.

Clearly many other instance selections can be defined similarly.

6.4 An Example

The following small example serves to connect the conceptual analogue of Heisenberg's Uncertainty Relation with the classical application of Heisenberg's Uncertainty Relation where a particle moves along the x-axis. Instead of taking into account all possible measurements we take only two measurements **A** and **B**. Each measurement yields a distribution in the so-called "phase space" spanned by the x-axis and the momentum axis. In the following CSS $\mathbf{C}_2$ we choose $G = \{1,\ldots,20\}$, $M = \{m_o, P, x, v\}$ where m_o is used as "measurement attribute" which has a nominal semantic scale $(\{A,B\},\{A,B\},=)$ whose two object concepts are interpreted as two measurements, called "**A**" and "**B**". P is an attribute with a semantic scale $(\{c\}, \{c\},\{(c,c)\})$ with only one formal concept **c** denoting "the single moving particle". The attributes x (for the x-coordinates) and v (for the momentum) have the same formal context $(\{0,1,2,3,4,5\}, \{0,1,2,3,4,5\}, \geq)$ as semantic scale. Its six formal concepts are the object concepts **i** of the formal objects i. The mapping κ is given in Table 5.

In Figure 3 a nested line diagram shows the direct product of the concept lattices of the m_o-, x-, and v-scales into which the concept lattice of the "(m_o,x,v)-part" of the derived context of $\mathbf{C}_2$ is represented by the bold points. The inner part of the nested line diagram is the "phase space" which is the direct product of the concept lattices of the x- and v-scales. Each object concept is marked by the labels of those instances which have this concept as its object concept. The bold points without instance marks represent suprema of object concepts.

Now we develop in this example the conceptual analogue of Heisenberg's Uncertainty Relation. We choose $M^* := \{P\}$, then $(\mathbf{c})$ is the only tuple in $\tau(M^*)$, denoted just by "**c**" (without brackets). For each measurement $\xi \in \{\mathbf{A},\mathbf{B}\}$ and **c** we use $\sigma_\lambda(\xi,\mathbf{c}) = \{g \in G \mid m_o(g) = \xi, P(g) = \mathbf{c}\}$ as selection, hence $\sigma_\lambda(\mathbf{A},\mathbf{c}) = \{1,\ldots,10\}$ and $\sigma_\lambda(\mathbf{B},\mathbf{c}) = \{11,\ldots,20\}$. We now choose
$Q := \{(x,i) | i \in \{0,1,2,3,4,5\}\}$ and $Q' := \{(v,i) | i \in \{0,1,2,3,4,5\}\}$ as the sets of attributes of the x-part respectively v-part of the derived context of $\mathbf{C}_2$. In the line diagram, for $i \in \{0,1,2,3,4,5\}$ the attribute (x,i) is denoted by "$x \geq i$", and "$x \geq 0$" is omitted, analogously for the attribute (v,i).

G	m_0	P	x	v
1	**A**	**c**	**2**	**0**
2	**A**	**c**	**2**	**1**
3	**A**	**c**	**2**	**3**
4	**A**	**c**	**2**	**4**
5	**A**	**c**	**3**	**1**
6	**A**	**c**	**3**	**2**
7	**A**	**c**	**3**	**3**
8	**A**	**c**	**3**	**4**
9	**A**	**c**	**3**	**5**
10	**A**	**c**	**3**	**5**
11	**B**	**c**	**1**	**2**
12	**B**	**c**	**2**	**2**
13	**B**	**c**	**3**	**2**
14	**B**	**c**	**5**	**2**
15	**B**	**c**	**0**	**3**
16	**B**	**c**	**1**	**3**
17	**B**	**c**	**2**	**3**
18	**B**	**c**	**3**	**3**
19	**B**	**c**	**5**	**3**
20	**B**	**c**	**5**	**3**

Table 5: The data table of the CSS $\mathbf{C}_2$

The concept lattices of the Q-part and of the Q'-part of the derived context are chains. It can be easily seen, for example from the inner diagram of measurement **A** (by "projecting the object concepts to the Q-border line in the left") that $|\gamma_Q(\sigma_\lambda(\mathbf{A},\mathbf{c}))| = |\gamma_Q(\{1,\ldots,10\})| = 2$. Hence we get that Q is **c**-complementary to Q' since $|\gamma_Q(\sigma_\lambda(\mathbf{A},\mathbf{c}))| \cdot |\gamma_{Q'}(\sigma_\lambda(\mathbf{A},\mathbf{c}))| = 2 \cdot 6 = 12 > 1$ and $|\gamma_Q(\sigma_\lambda(\mathbf{B},\mathbf{c}))| \cdot |\gamma_{Q'}(\sigma_\lambda(\mathbf{B},\mathbf{c}))| = 5 \cdot 2 = 10 > 1$. Since **c** is the only tuple in $\tau(M^*)$ we get that Q is complementary to Q'.

7 Conclusion and Future Research

This paper shows a new mathematical representation of some basic notions in physics concerning particles, waves, measurements, and Heisenberg's Uncertainty Relation. This new approach is based on the mathematical theory of Formal Concept Analysis. As opposed to classical physics which is based on mathematical tools related

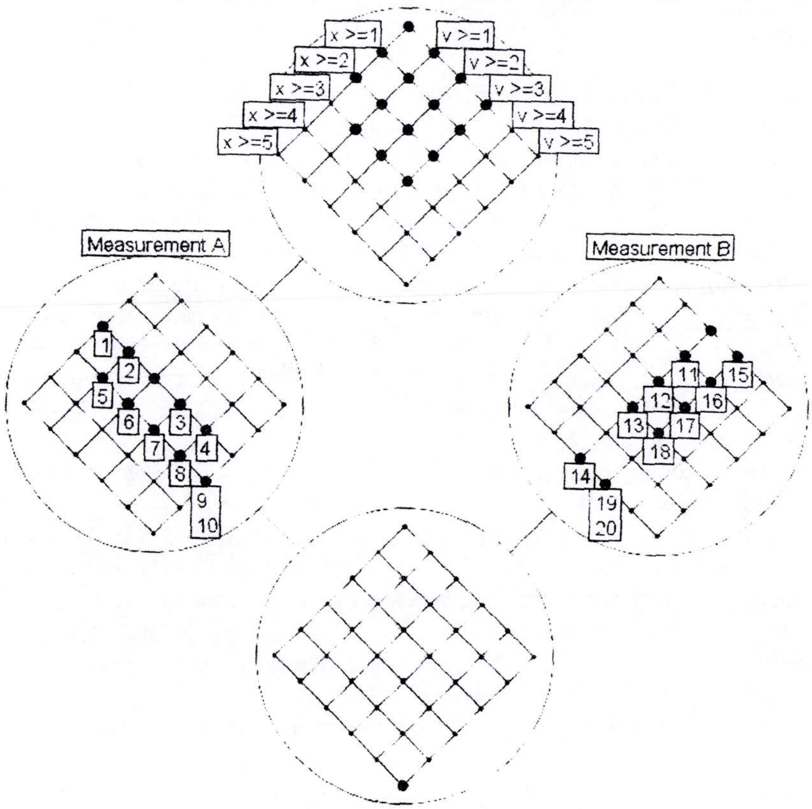

Figure 3: A typical example for Heisenberg's Uncertainty Relation

to real numbers this new approach is based on the general notion of a *concept*.

The main result is that in Conceptual Semantic Systems notions like "object", "particle", "wave", and "measurement" can be represented in a contextual-conceptual way which includes the continuous descriptions in physics, but without using the algebraical and metrical structure of classical spaces. In my actual understanding, the "problem of the Wave/Particle Dualism" as mentioned in section 1.2 disappears in the mathematical framework of Conceptual Semantic Systems since this framework contains an explicit notion of granularity and a relational knowledge representation which is based on formal concepts refering to accepted contextual data.

Hence Planck's desperation concerning discrete quanta in the continuum is based on an obsolete viewpoint, and Einstein's wish for a granularity theory in physics starts to become true.

Future research will focus on the application of conceptual methods to the classical mathematical structures as they are used in physics. For that purpose the connection of conceptual and algebraical structures has to be developed.

References

1. Auletta, G: *Foundations and Interpretations of Quantum Mechanics.* World Scientific Publishing Co. Pte. Ltd., Singapore, 2000.
2. Birkhoff, G.: *Lattice theory*; 3rd ed.; Amer.Math.Soc., Providence. 1967.
3. Born, M.: Die statistische Deutung der Quantenmechanik. Nobelvortrag. In: [3], II, 430-441. English translation: The statistical interpretation of quantum mechanics. Nobel Lecture, December 11, 1954. http://nobelprize.org/physics/laureates/1954/born-lecture.html
4. Born, M.: *Ausgewählte Abhandlungen.* Vandenhoeck and Ruprecht, Göttingen 1963.
5. Butterfield, J. (ed.): *The Arguments of Time.* Oxford University Press, 1999.
6. Castellani, E. (ed.): *Interpreting Bodies: Classical and Quantum Objects in Modern Physics.* Princeton University Press 1998.
7. DPG00: 100 Jahre Quantentheorie. Ed.: Deutsche Physikalische Gesellschaft. Lentz Druck, Berlin, 2000.
8. Einstein, A.: Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. Annalen der Physik **17** (1905): 132-148.
9. Einstein, A.: Zur Elektrodynamik bewegter Körper. Annalen der Physik **17** (1905): 891-921.
10. Ganter, B., R. Wille: *Formal Concept Analysis: mathematical foundations.* Springer-Verlag, Berlin-Heidelberg 1999.
11. Hawking, S.; Penrose, R.; *The Nature of Space and Time.* Princeton University Press, 1996.
12. *Heisenberg: Gesammelte Werke.* Edited by: W. Blum, H.-P. Dürr, H. Rechenberg. Springer-Verlag, Heidelberg 1984, Series B.
13. Heisenberg, W: Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. Zeitschrift für Physik **43** (1927) 172-198.
14. Heisenberg, W: *Physik und Philosophie.* Hirzel-Verlag, Stuttgart 1959.
15. Kalman, R.E., Falb, P.L., Arbib, M.A.: *Topics in Mathematical System Theory.* McGraw-Hill Book Company, New York, 1969.
16. Lin, Y.: *General Systems Theory: A Mathematical Approach.* Kluwer Academic / Plenum Publishers, New York, 1999.
17. Mesarovic, M.D., Takahara, Y.: *General Systems Theory: Mathematical Foundations.* Academic Press, London, 1975.
18. Planck, M.: Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum. Verhandlungen der Deutschen Physikalischen Gesellschaft **2**, 1900; Nr. 17, 237-245.
19. Rollnik, H.: *Quantentheorie.* Vieweg, Braunschweig 1995.
20. von Neumann, J.: *Mathematische Grundlagen der Quantenmechanik.* Berlin, Springer, 1932.
21. von Neumann, J.: *Mathematical Foundations of Quantum Mechanics.* Princeton, University Press, 1955 (engl. translation of von Neumann 1932).

22. Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Ordered Sets (ed. I.Rival). Reidel, Dordrecht-Boston 1982, 445-470.
23. Wille, R.: Introduction to Formal Concept Analysis. In: G. Negrini (ed.): Modelli e modellizzazione. Models and modelling. Consiglio Nazionale delle Ricerche, Instituto di Studi sulli Ricerca e Documentatione Scientifica, Roma 1997, 39-51.
24. Wolff, K.E.: A first course in Formal Concept Analysis - How to understand line diagrams. In: Faulbaum, F. (ed.): SoftStat '93, Advances in Statistical Software **4**, Gustav Fischer Verlag, Stuttgart 1994, 429-438.
25. Wolff, K.E.: Concepts in Fuzzy Scaling Theory: Order and Granularity. 7th European Conference on Intelligent Techniques and Soft Computing, EUFIT'99. FUZZY SETS and SYSTEMS **132**, 2002, 63-75.
26. Wolff, K.E.: Concepts, States and Systems. In: Dubois, D.M. (ed.): Computing Anticipatory Systems CASYS'99, American Institute of Physics Conference Proceedings 2000, vol. **517**, 83-97.
27. Wolff, K.E.: 'Particles' and 'Waves' as Understood by Temporal Concept Analysis. In: K.E. Wolff, H.D. Pfeiffer, H.S. Delugach (eds.): Conceptual Structures at Work. 12th International Conference on Conceptual Structures, ICCS 2004. Huntsville, AL, USA, July 2004. Proceedings. Springer Lecture Notes in Artificial Intelligence, LNAI **3127**, Springer-Verlag, Berlin Heidelberg 2004, 126-141.
28. Wolff, K.E.: Wave Packets, Particles, and Waves – A Granular Conceptual Representation. In: George E. Lasker (ed.): Proceedings of the16th International Conference on Systems Research, Informatics, and Cybernetics. Baden-Baden, 2004.
29. Wolff, K.E.: States of Distributed Objects in Conceptual Semantic Systems. In: F. Dau, M.L.Mugnier, G.Stumme (eds.): Common Semantics for Sharing Knowledge. Springer Lecture Notes in Artificial Intelligence, LNAI **3596**, Springer-Verlag, Berlin Heidelberg 2005, 250-266.
30. Wolff, K.E.: States, Transitions, and Life Tracks in Temporal Concept Analysis. In: B. Ganter, G. Stumme, R. Wille (eds.): Formal Concept Analysis – State of the Art. Springer Lecture Notes in Artificial Intelligence, LNAI **3626**, Springer-Verlag, Berlin Heidelberg 2005, 127-148.
31. Wolff, K.E.: A Conceptual Analogue of Heisenberg's Uncertainty Relation. In: B. Ganter, L. Kwuida: Contributions to ICFCA 2006. 4th International Conference on Formal Concept Analysis, Dresden, Germany, 2006. Verlag Allgemeine Wissenschaft, 2006, 19-30.