# Hyperincursive Simulation of Ecosystems Chaos and Patchiness by Diffusive Chaos

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**Abstract**

In Pearl-Verhulst's finite difference equation, R. May showed that fractal chaos appears for large values of the command parameter. In this paper, it is shown that, surprisingly, chaos emerges for small values of the command parameter when Laplacian spatial diffusion is taken into account. For small diffusion the space pattern is uniform and stable and for large diffusion, discrete space-time structures emerge and then a chaotic patchiness. A mathematical demonstration by incursion shows that the emergence of such structures is due to the space diffusion parameter which gives rise to a bifurcation cascade and chaos. This is a new type of emergence of space-time structures what I suggest to call "diffusive chaos" different from the Turing "morphogenesis by diffusive instability". A gradient spatial transport by advection can also give rise to bifurcations and chaos, what I call "advective chaos" depending of the velocity intensity. A simulation with negative diffusion shows stable fractal periodic patterns.
In Lotka-Volterra's discrete model, numerical instabilities occur. D. Dubois had found a new method for stabilising such instabilities by the concept and method of incursion, an inclusive recursion, where the equations are sequentially computed. With space diffusion such incursive equations show the emergence of a chaotic space-time patchiness which is followed by continuous space patchiness represented by travelling waves. Diffusive chaos could explain space-time structures called patchiness in marine plankton.

**Keywords:** ecological systems, incursion, hyperincursion, diffusive chaos, patchiness.

## 1. Introduction

An important subject in mathematical systems theory is the emergence of space-time structures. Turing (1952) initiated this subject in proposing a chemical basis of morphogenesis. His starting point was to consider space-time differential equations, that is to say differential equations representing the chemical dynamics with a space diffusion given by a Laplacian (second space derivative). His showed that the emergence of space patterns is due to "diffusive instability" in non-linear systems. In ecology, the Volterra predator-prey model with spatial diffusion was firstly studied by Dubois (1975, 1979, 1981) to explain space-time structures called patchiness in marine plankton. Numerical simulations showed prey-predator "travelling waves" in good agreement with experimental data. An alternative model was also proposed from the Turing theory (Dubois, 1977). It must be pointed out that the chemical autocatalysis equation of Lotka (1925) is similar to the Volterra model. This paper deals with the presentation of a completely new mechanism of emergence of space-time structures from fractal chaos in discrete non-linear equations, what I propose to call a "diffusive chaos" (Dubois, 1996a). This is completely different from Turing 's "diffusive instability". Two types of systems will be considered: the Pearl-Verhulst and the Lotka-Volterra discrete models with space diffusion.

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R. May (1976) showed that several solutions exist in the discrete Pearl-Verhulst map: a fixed point for small values of the command parameter, a bifurcation diagram for medium value and chaos for great values. This paper will show that space-time fractal chaos structures emerge from the Pearl-Verhulst model for small values of the command parameter in adding a space diffusion. For small diffusion, the system is space homogeneous and in increasing only the diffusion, chaos emerges giving rise to space-time structures (Dubois, 1996a). This will be mathematically demonstrated and numerical simulations will be presented.

The discrete Lotka-Volterra equations are cellular automata which give rise to numerical instabilities. A new technique to stabilise these instabilities was proposed by the introduction of the concept and method of incursion and hyperincursion (Dubois, 1992, 1995, 1996abcd; Dubois and Resconi, 1992, 1994, 1995). In adding diffusion in the incursive Lotka-Volterra model, it will be showed that diffusion initiates space-time structures by a chaotic behaviour. This chaos will then give rise to space continuous structures like "waves", but with discrete values in time.

## 2. Chaos in Temporal Pearl-Verhulst Equation

Pearl (1924) and Verhulst (1845, 1847) considered the Malthus growth equation with a saturation like the following differential equation:

$$(1) \qquad \frac{dx}{dt} = rx(1-x)$$

The theory of fractal chaos has been introduced in ecology by a discretisation of this equation by R. May (1976). From the definition of the derivative

$$(2) \qquad \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

the following discrete Pearl-Verhulst equation is obtained:

$$(3) \qquad x(t+\Delta t) = x(t) + \Delta t.r.x(t).(1-x(t))$$

It can be considered as an automaton. For $-1<\Delta t.r<0$ there is a fixed point at the origin $x = 0$, for $0<\Delta t.r<2$ the fixed point is $x = 1$ and for $2<\Delta t.r<3$, there are the direct cascade of bifurcations and then chaos. The first bifurcation appears thus for $\Delta t.r = 2$.
A characteristic of chaos is its uncontrollability. Such a chaoic system can be controlled by incursion, what was called "incursive control" (Dubois, 1995; Dubois, and Resconi, 1994).

## 3. Diffusive Chaos in Space-time Pearl-Verhulst Equation

Let us now consider an extension of the Pearl-Verhulst equation in considering a spatial diffusion D(s) (depending eventually on the space dimension s) along a discrete space dimension denoted by s, in taking a discrete time t (Dubois, 1996a):

$$(4) \qquad x(s,t+\Delta t) = x(s,t)+\Delta t.r.x(s,t).(1-x(s,t))+\Delta t.D(s)[x(s-\Delta s,t)-2.x(s,t)+x(s+\Delta s,t)]/\Delta s^2$$

Without lack of generality, we can choose a unit time step $\Delta t = 1$ and a unit space step $\Delta s = 1$, or dimensionless rate constant $\rho = \Delta t.r$ and dimensionless diffusion $d(s) = \Delta t.D(s)/\Delta s^2$ can be defined. So the equation becomes:

(5) $x(s,t+1)=x(s,t)+\rho.x(s,t).(1-x(s,t))+d(s)[x(s-1,t)-2.x(s,t)+x(s+1,t)]$

In taking a value of the parameter $\rho = 1$, that is to say in a case where the solution of the discrete time Pearl-Verhulst equation is a fixed point, surprisingly the solution shows bifurcations and chaos behaviour when spatial diffusion is taken into account.

With an initial homogeneous space distribution $x(s,0)=1$, $s=2$ to $199$, with periodical boundary conditions for $x(1,t)$ and $x(200,t)$), the solution remains space homogeneous. In taking a very small perturbation of the initial condition of one automaton $x(100,0)=1+0.01$, chaos emerges as shown in Figures 1abcd. Figure 1a shows the emergence of a space-time structure at the position of the perturbation after 50 time iterates for $d=0.28$.

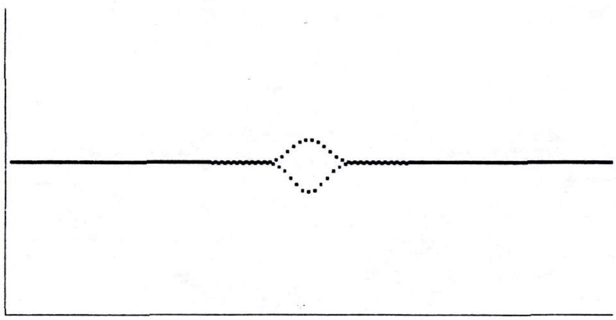

**Figure 1a**

At iterate $t=200$, Fig. 1b shows a spreading bifurcation of period 2.

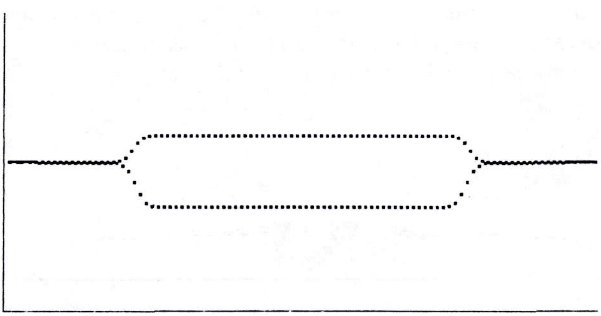

**Figure 1b**

Fig. 1c considers the iterate 150 with a greater diffusion constant d=0.36. Space-time structure appears.

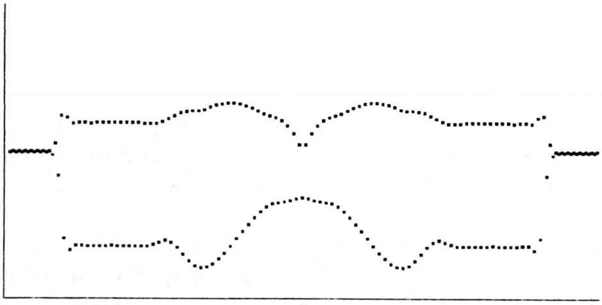

**Figure 1c**

Fig. 1d gives 8 successive iterates from t = 143 to 150 exhibiting "wave" patterns.

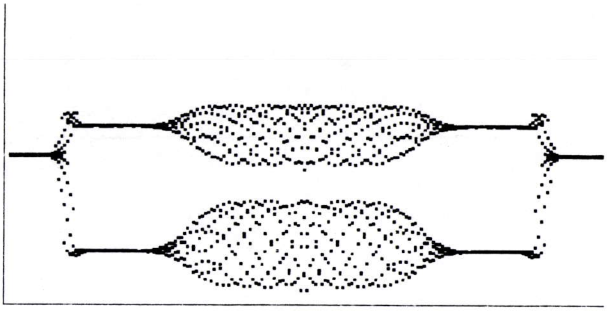

**Figure 1d**

In Figure 1e, a parabolic diffusion was considered: d(s)=0.24+0.24 s.(1-s/200), the maximum value being at the centre of the automata. The initial condition is x(2,0)=1+0.01 and all other automata being equal to 1. Simulation shows that chaos emerges when the diffusion is important. The perturbation was given in the zone of weak diffusion: it is what is called a sensibility to initial conditions and a butterfly effect is well seen.

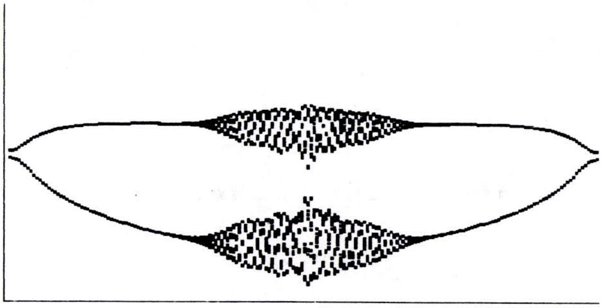

**Figure 1e**

With a negative diffusion constant $d = -0.45$ with $\rho = 2.3$ and $x(2,0)=1+0.01$, Fig. 1f shows 8 successive iterates from $t = 293$ to $300$: a very curious "periodically stable" fractal pattern occurs.

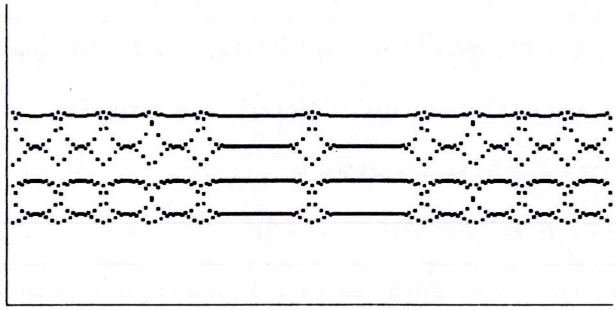

**Figure 1f**

From the Pearl-Verhulst model with space advection (gradient) where v is a velocity (for example a biological population x in a river which flows at velocity $V(s)=v(s).\Delta s/\Delta t$):

(6) $x(s,t+1)=x(s,t)+\rho.x(s,t).(1-x(s,t))+v(s)[x(s-1,t)-x(s,t)]$

asynchronous chaos emerges (Dubois, 1996a).
Fig. 1g gives the 8 successive iterates from $t = 143$ to $150$, with initial conditions $x(s,0)=1$ except $x(50,0)=1+0.01$ and $v=0.652$, $\rho=1$.

Let us establish the mathematical proof of these numerical results.

**Theorem (Dubois, 1996a)**: For the fixed point $x = 1$ solution of the Pearl-Verhulst equation (4), i.e. for $0<\Delta t.r<2$, the first bifurcation appears when a spatial diffusion is taken into account for which the following relation is satisfied:

(7) $\Delta t.r/(1-2.\Delta t.D(s)/\Delta s^2) \geq 2$ or $\rho/(1-2.d(s)) \geq 2$

thus, the first bifurcation appears when

(8) $d(s) \geq (2-\rho)/4$

In the case of the simulations presented here, $\rho = 1$ and thus $d(s) \geq 1/4$.

**Proof**: The first bifurcation is defined by

(9) $x(s,t) = x(s,t+2)$

The simpler non-homogeneous spatial pattern can be defined by

(10) $x(s-1,t) = x(s+1,t)$

Let us assume that the state of x at position s at time t+1 will be equal to the state x at an adjacent position s+1 (or s-1, from eq. (10) at the preceding time t:

(11) $x(s,t+1) = x(s+1,t) = x(s-1,t)$

In replacing eq.(11) in eq. (4) with eq. (10), the following incursive equation is obtained:

(12) $x(s,t+1)=x(s,t)+\rho.x(s,t).(1-x(s,t))+d(s)[x(s,t+1)-2.x(s,t)+x(s,t+1)]$

which can be transformed to the recursive equation:

(12b) $x(s,t+1)=x(s,t)+[\rho/(1-2.d(s))].x(s,t).(1-x(s,t))$

This eq.(12b), defined at position s does no more depend on the adjacent positions s-1 and s+1. It is identical to the Pearl-Verhulst equation without diffusion but the growth rate $\rho$ is now divided by (1-2.d(s)): the effective growth rate increases with the diffusion parameter.
As the first bifurcation appears when the growth rate is equal to 2, the eq. (7) is thus demonstrated.

■

The numerical simulations on computer given in Figs. 1abcdef confirm this theorem. It means that the simpler space-time pattern, for $\rho = 1$ and $d = 1/4$, is given by a period-2 time oscillation coupled with a period-2 space oscillation. When the diffusion d increases, bifurcations appear and then chaos.

For the Pearl-Verhulst equation (6) with advection, a similar incursive demonstration as for diffusive chaos can be made. In such an "advective chaos", the condition to obtain the first bifurcation is given by the following relation, similar to eq. (7):

(13) $\rho/(1-v(s)) \geq 2$

For $\rho = 1$, the first bifurcation appears when $v(s) = 0.5$. Figure 1g shows 8 succesive time iterates from t = 93 to 100 starting with a perturbation at x(50,0)=1+0.01, with v=0.652. The pattern is chaotic.

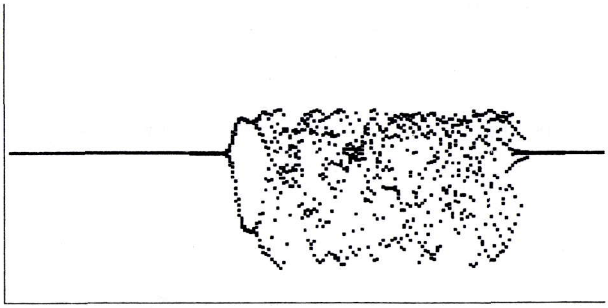

**Figure 1g**

Evidently, diffusion and advection can be considered together for simulating more real systems. Fig. 1h shows 8 successive iterates from 93 to 100 with d=0.28 and v=0.17, the initial perturbation being at x(100,0)=1+0.01.

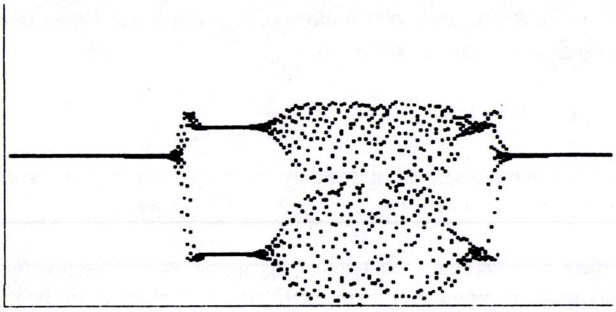

**Figure 1h**

In this case, the same incursive demonstration can be made for calculating the critical values of both diffusion and velocity

(14) $\rho/(1-2d(s)-v(s)) \geq 2$

for obtaining bifurcations and then chaos.

## 4. Hyperincursive Discrete Temporal Lotka-Volterra Equations

This section deals with the discretization of differential equations of the non-linear Lotka-Volterra differential equations model. Recall that these equations are related to the Volterra predator-prey ecological system and to the Lotka auto-catalytic chemical reactions system. The solutions of the original differential equations system are given by periodic oscillations (orbital stability). Analytical solutions exist only for small distances from the steady state, which are identical to the harmonic oscillator. It is well-known that some discretization schemes of these equations gives instabilities. We will show that some incursive schemes of discretization give stationary solutions, but also chaotic behaviours.
Let us first consider the non-linear model given by the discretized Lotka-Volterra equations:

(15a) $X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).Y(t)]$
(15b) $Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.X(t).Y(t)]$

where t is a discrete time with steps $\Delta t$, and a,b,c,d are the parameters.
These equations are related to the Volterra predator-prey ecological system and to the Lotka auto-catalytic chemical reactions system. The solutions of the original differential equations system, at the limit $\Delta t=0$, is given by periodic oscillations (orbital stability). For $\Delta t$ different of zero, the solutions are unstable. Analytical solutions exist only for small oscillations from the steady state $X_0= c/d$ and $Y_0=a/b$, which are identical to the harmonic linear oscillator. In taking $X=X_0 + x$ and $Y=Y_0 + y$, when x and y are small, the linearization of the equations 15a-b gives

(16a) $x(t+\Delta t) = x(t) - \Delta t.(bc/d).y(t)$
(16b) $y(t+\Delta t) = y(t) + \Delta t.(ad/b).x(t)$

and the discrete harmonic oscillator equation is obtained:

(17) $x(t+2\Delta t) - 2x(t+\Delta t) + x(t) = -\Delta t^2.\omega^2.x(t)$

with $\omega^2 = ac$, where $\omega$ is the frequency. The solutions of eqs. 16a-b and 17 are unstable. At the limit $\Delta t=0$, the analytical solution is given by

(18) $x(t) = X_0 + A.SIN(\omega t+\phi)$

where the amplitude A and the phase $\phi$ are defined by the initial conditions at time t=0, where $\omega^2 = ac$. The variable y has a phase delay of $\pi/2$ on the variable x.

We will show that incursive discrete Lotka-Volterra equations give stationary solutions (orbital stability), but also chaotic behaviours (Dubois, 1992, 1995, 1996b; Dubois and Resconi, 1994, 1995). Different incursive discrete equations systems exist to stabilzed numerically these discretized equations (15a-b). A few models are now presented and numerically simulated on computer.

### 4.1. Model I

The iterative values of $X(t+\Delta t)$ of the first equation (15a) can be propagated to the second equation (15b), in an incursive way, as proposed by Dubois (1992, 1993):

(19a) $X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).Y(t)]$
(19b) $Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.\mathbf{X(t+\Delta t)}.Y(t)]$

where we compute the value $Y(t+\Delta t)$ in function of the value of $X(t+\Delta t)$ at the same time step, instead of the value at its preceding step as it is classically done with a recursive parallel way. The incursive discretization corresponds to anticipatory asynchronous iterations. Notice that for small values of the time step $\Delta t$, which correspond to the continuous case, we obtain the same results as the original Lotka-Volterra equations. Indeed, in replacing the variable $X(t+\Delta t)$ in eq. (19b) by the eq. (19a), we obtain:

(19c) $Y(t+\Delta t) = Y(t) - \Delta t.c.Y(t) + d.\Delta t.X(t).Y(t) + d.\Delta t^2.X(t).[a-b.Y(t)].Y(t)$

where the new term is of the second order in time step, $\Delta t^2$, which means that the predator contains an anticipatory model of the prey at the next time step.

We can interpret the incursion in the following way. The first two computing steps of iterations, starting with the initial conditions X(0) and Y(0), can be written as (Dubois 1992, p. 133):

1rst step $X(t+1) = X(t) + \Delta t.[a.X(t) - b.X(t).Y(t)]$
$Y(t+2) = Y(t) + \Delta t.[-c.Y(t) + d.X(t+1).Y(t)]$
2nd step $X(t+3) = X(t+1) + \Delta t.[a.X(t+1) - b.X(t+1).Y(t+2)]$
$Y(t+4) = Y(t+2) + \Delta t.[-c.Y(t+2) + d.X(t+3).Y(t+2)]$
etc...

We see that X and Y are not computed at the time steps: we cannot know the values of X and Y simultaneously at the same time step.

Figures 2ab show an example of the simulation (Dubois, 1992) of the incursive discrete equations (19a-b) in the phase space (X,Y), for successive values of the initial contions. For initial conditions not too far from the steady state, the numerical solutions show stabilized oscillations (orbital stability) by incursion. For medium values of the initial conditions, multiple stable oscillations appear and then, for lager values, again the orbital stability and then the chaos appears. This behaviour of the incursive solutions is due to phase shifts: advanced or delayed phases between X and Y.
Let us notice that for a lot of values of the parameters, the incursive solutions give stable oscillations meanwhile the recursive ones give numerically explosive and very unstable solutions.

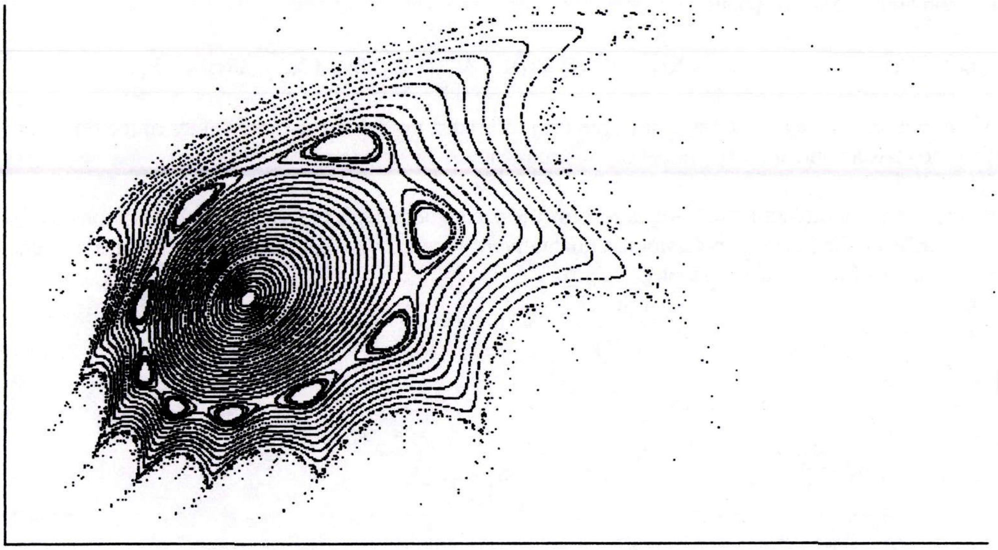

**Figure 2a:** in the phase space, the horizontal axis gives the X values and the vertical axis gives the Y values.

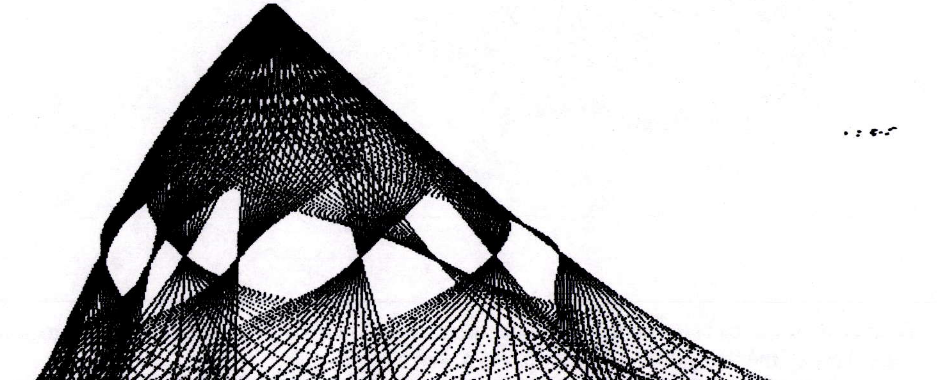

**Figure 2b:** values of X (from left to right) for successive initial conditions (from top to bottom).

When we defined the hyperincursion in the Fractal Machine, we said that several solutions exist at each path, meanwhile for the incursion, only one path. In fact, it exists a second path for the incursive discrete Lotka-Volterra equations system in inversing the order in which the two equations are computed. The second path is given by

$$(20a)\quad Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.X(t).Y(t)]$$
$$(20b)\quad X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).\mathbf{Y(t+\Delta t)}]$$

where now we propagate the value of $Y(t+\Delta t)$ in the equation of $X(t+\Delta t)$. In replacing $Y(t+\Delta t)$ in eq. 20b by eq. 20a, we obtain

$$(20c)\quad X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).Y(t)] + \Delta t^2 .b.X(t).[c - d.X(t)].Y(t)$$

where we recognize a factor similar to the Pearl-Verhulst model, for discrete values of the time step. The prey has an anticipatory model of the predator.

Figure 2c gives the numerical simulations of these equations with the same initial conditions as for the Figure 1a. The general behaviour is qualitatively identical, but not quantatively for medium and large values of the initial conditions.

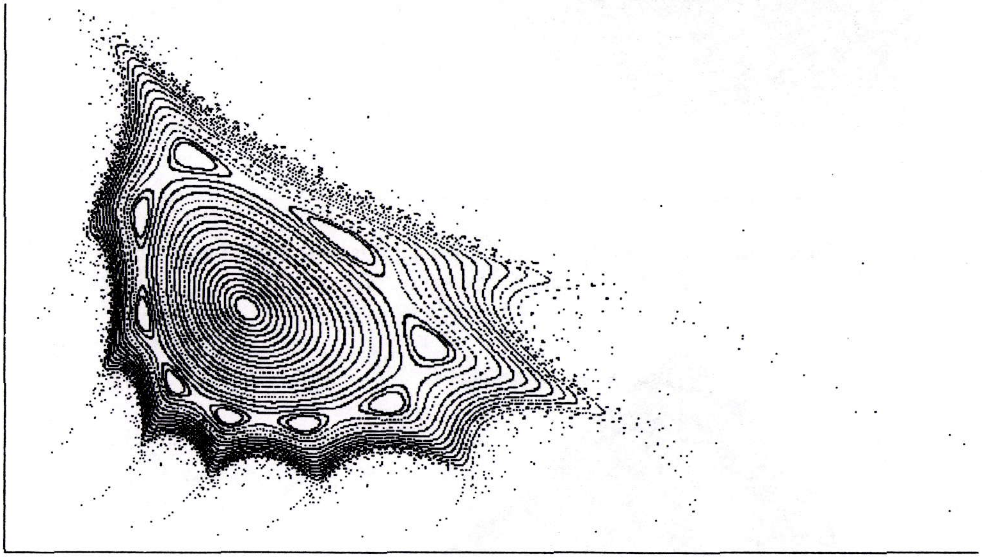

**Figure 2c**

The difference of values between the two hyperincursive solutions given in Figs 2a and 2c measures the uncertainty of the real values of the variables X and Y.

It is only when the frequency (or the time step $\Delta t$) of the oscillations around the steady state is very weak that the two hyperincursive solutions become identical to the solution of the original Lotka-Volterra differential equations.

It would be interesting to simulate the intermediate equations between the two above hyperincursive discrete Lotka-Volterra equations. What we mean is, for example, to consider the following discrete equations

(21a) $X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).\mathbf{Y(t+\Delta t/2)}]$
(21b) $Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.\mathbf{X(t+\Delta t/2)}.Y(t)]$

in view of obtaining the more predictable values of the two variables at the same time steps $\Delta t/2$. To do this, it is necessary to know 4 initial conditions, $X(0)$, $X(\Delta t/2)$, $Y(0)$ and $Y(\Delta t/2)$, instead of 2 ($X(0)$ and $Y(0)$).
Figures 3abcd give the numerical solutions for 4 successive initial conditions for which $X(\Delta t/2)=X(0)$ and $Y(\Delta t/2)=Y(0)$.

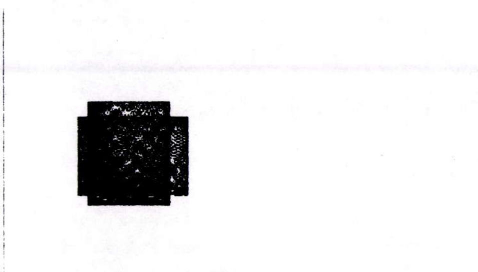

**Figure 3a**

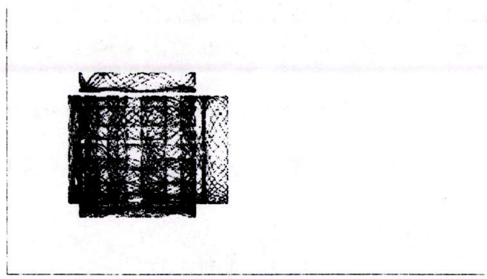

**Figure 3b**

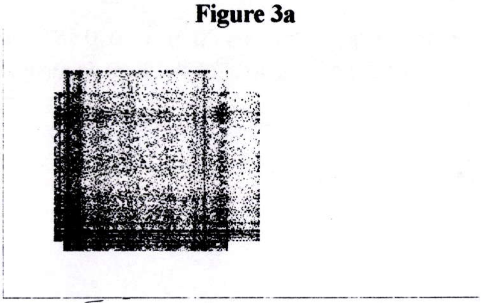

**Figure 3c**

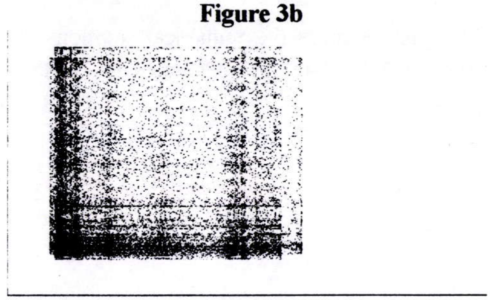

**Figure 3d**

Finally, in Dubois & Resconi (1992), the following hyperincursive Lotka-Volterra discrete equations were considered:

(22a) $X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.\mathbf{X(t+\Delta t).Y(t+\Delta t)}]$
(22b) $Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.\mathbf{X(t+\Delta t).Y(t+\Delta t)}]$

It was demonstrated that it is not possible to transform these hyperincursive equations (with one path of computation, i. e. a parallel computation), to simple recursive equations. Moreover, at each time step $\Delta t$, 2 values of each of the variables X and Y exist, so that an exponential number of values must be computed.
Let us notice that we can choose, at each step, only one particular value for each variable, we select a particular solution. If one considers the time reverse equations (22a-b), the successive values of the two variables are those chosen in the direct time direction.

A very important conclusion is so obtained.

For this type of hyperincursive discrete equations systems, many solutions exist in the direction of the future meanwhile only one solution exists in the reverse direction of the time. These systems behave with a memory of the choices made during their evolution towards the future.

### 4.2. Model 2

In 1973, I proposed (Dubois, 1973) to generate non-linear differential equations from an invariant, a Lyapounov function related to the Hamiltonian in Mechanics. Starting from such a well-known invariant (conservative systems) for the Lotka-Volterra model, I generate the other following canonical model written with discrete equations:

(23a) $X(t+\Delta t) = X(t) + \Delta t.\ln(Y(t))$
(23b) $Y(t+\Delta t) = Y(t) - \Delta t.\ln(X(t))$

where ln is the Neperian logarithm. These equations are only stable at the limit of $\Delta t=0$.

One hyperincursive version of these equations is (Dubois, 1996b):

(23c) $X(t+\Delta t) = X(t) + \Delta t.\ln(Y(t))$
(23d) $Y(t+\Delta t) = Y(t) - \Delta t.\ln(\mathbf{X(t+1)})$

The Figure 4 gives the numerical solutions for different initial conditions X(0)=1 to 0.15 and Y(0)=1, for $\Delta t$=0.45. The behaviour is qualitatively similar to the behaviour of the hyperincursive discrete Lotka-Volterra equations given in Figures 2a and 2c.

**Figure 4**

### 4.3. Model 3

Another hyperincursive Lotka-Volterra discrete equation model (Dubois, 1996b) is given by

(24a) $X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.\mathbf{X(t+\Delta t)}.Y(t)]$
(24b) $Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.\mathbf{X(t+\Delta t)}.Y(t)]$

for which eq. (24a) can be transformed in the recursive equation

(24c) $X(t+\Delta t) = X(t) + \Delta t.a.X(t)/(1 + \Delta t.b.Y(t))$

The simulations of these equations in the phase space (X, Y) are given in Figs 5ab, with a=b=c=d=1 with a time step Δt=1.. In Figure 5b, the different initial conditions are far from the steady state. The solutions are given by a **chaotic sea** with a lot of **islands** which look similar to the patterns of Figures 2a and 2c. In Figure 5a, different initial conditions are considered near the steady state. This Fig. 5a is a simulation corresponding to the bottom-left of Fig. 5b.

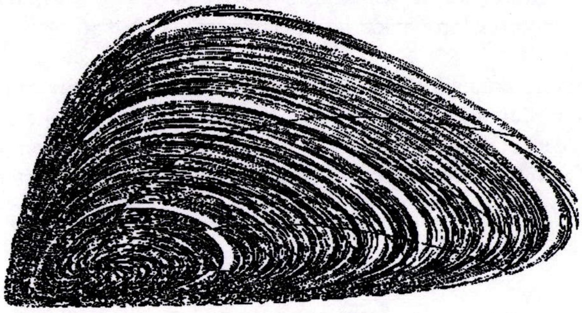

**Figure 5a**

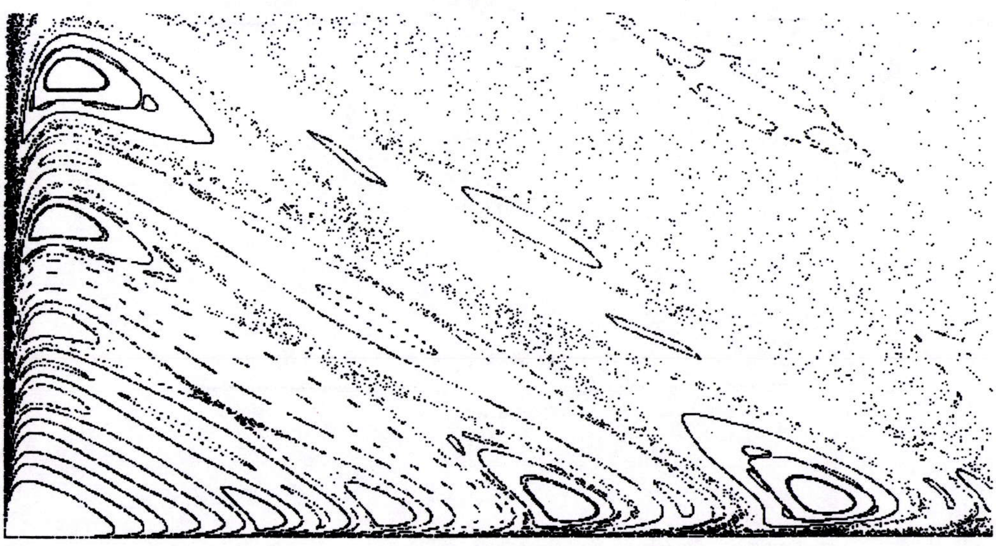

**Figure 5b**

## 5. Diffusive Chaos in Incursive Space-time Lotka-Volterra Model

With a space diffusion (Dubois, 1979), eqs. (15ab) are given by

(25a) $X(s,t+1) = X(s,t) + a.X(s,t) - b.X(s,t).Y(s,t) + D(s)[X(s-1,t)-2.X(s,t)+X(s+1,t)]$
(25b) $Y(s,t+1) = Y(s,t) -c.Y(s,t) + d.X(s,t).Y(s,t) + D(s)[Y(s-1,t)-2.Y(s,t)+Y(s+1,t)]$

The simulation of these space-time Lotka-Volterra equations with spatial diffusion gives rise to wave propagation: when two waves meet, they annihilate (D. Dubois, 1975).

With a space diffusion (Dubois, 1996a), the incursive eqs. (19ab) are given by

(26a) $X(s,t+1) = X(s,t) + a.X(s,t) - b.X(s,t).Y(s,t) + D(s)[X(s-1,t)-2.X(s,t)+X(s+1,t)]$
(26b) $Y(s,t+1) = Y(s,t) -c.Y(s,t) + d.\mathbf{X(s,t+1)}.Y(s,t) + D(s)[Y(s-1,t)-2.Y(s,t)+Y(s+1,t)]$

Numerical simulations of these equations are given in Figs. 6ab, 7ab, 8ab, 9ab where the horizontal axis is the space variable s = 2 to 199 and the vertical axis the values of X and Y. The numerical values of the parameters are the following: the number of automata s=2 to 199, with periodical conditions, a=0.648; b=0.00432; c=0.405; d=0.00405; the diffusion constant D=0.35 and the initial conditions are space homogeneous X(s,0)=150, except a perturbation for X(100,0)=150+15.
Figs. 6ab give the numerical simulations of X(s,t) and Y(s,t), respectively, for s = 1 to 200 at time iterate t=150. A first bifurcation appears which spreads in two opposite direction. Indeed, in Figs. 7ab, continuations of Figs 6ab at time t=300, the bifurcation disappears behind the bifurcation propagating fronts giving rise to a continuous emerging space pattern.

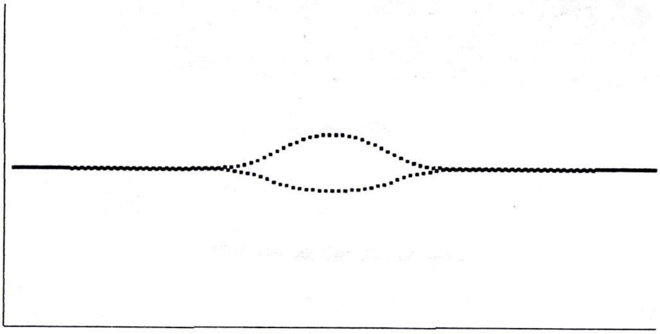

**Figure 6a**

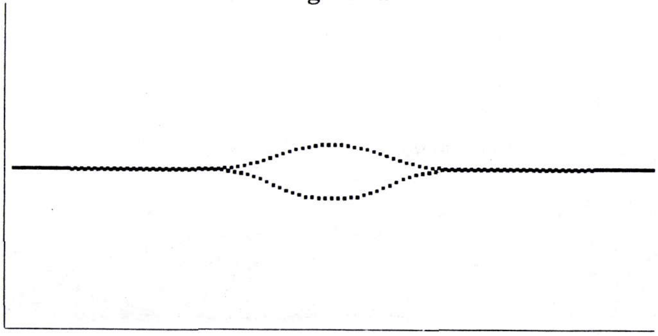

**Figure 6b**

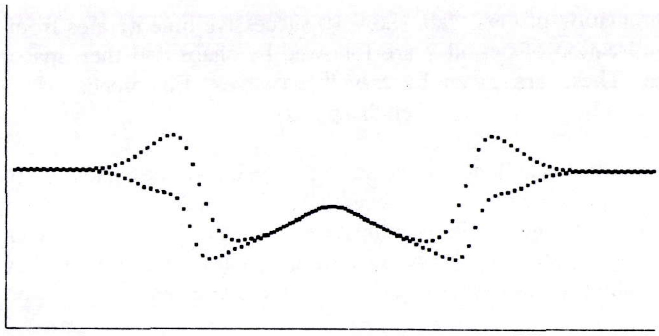

**Figure 7a**

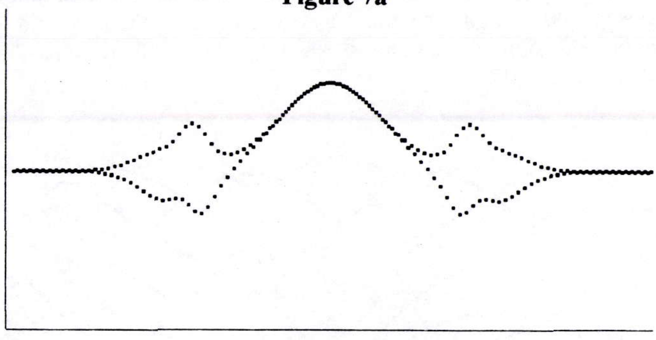

**Figure 7b**

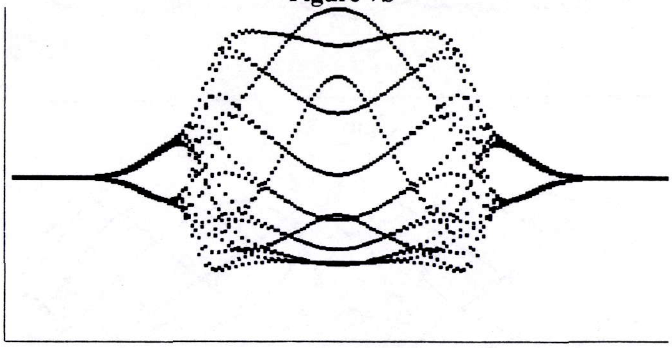

**Figure 8a**

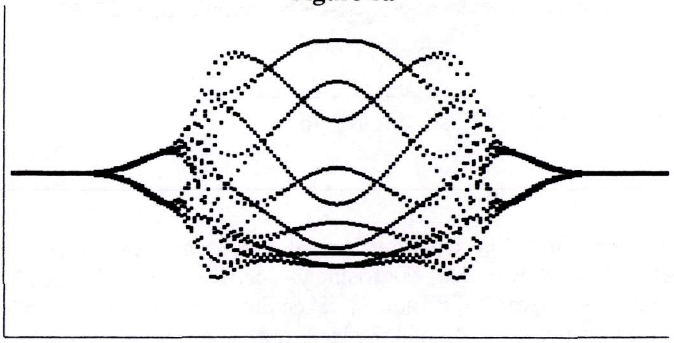

**Figure 8b**

Figures 8ab, continuation of Figs. 7ab, show 10 successive time iterates from t = 291 to 300. Two opposite bifurcation of period 2 are followed by chaos and then space-time structures, called patchiness. These are given by travelling waves. Emergence of space continuous patchiness is initiated by chaos which then disappears.

This remarkable result means that the cause, represented by the chaos, of the emergence of travelling waves, which is the effect, disappears after the installation of these travelling waves. This is well shown in Figs. 9ab, continuation of Figs. 8ab, where chaos disappeared. Patchiness, an emergent property, is given by stationary travelling waves. Any perturbation in the continous travelling waves initiates again transient bifurcations.

Let us point out that without space diffusion, the incursive Lotka-Volterra model shows orbital stability for the chosen values of parameters. This confirms, as in the space-time Pearl-Verhulst model, that the spatial diffusion initiates chaos, which I called "diffusive chaos".

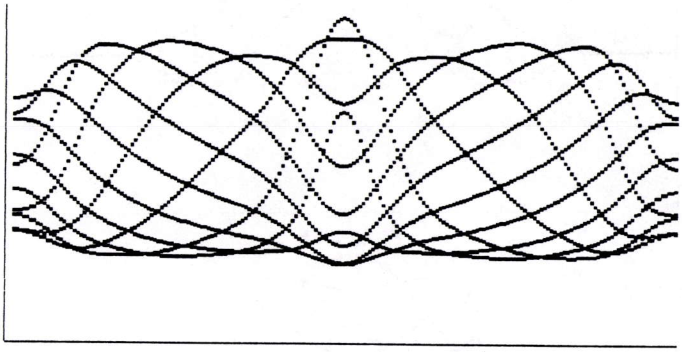

**Figure 9a**

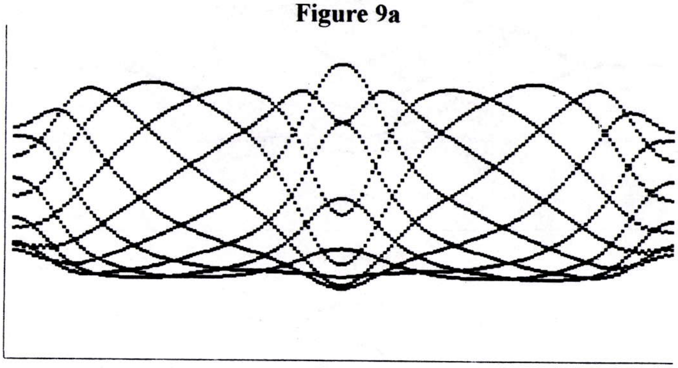

**Figure 9b**

## 6. Conclusion

In Pearl-Verhulst's finite difference equation, fractal chaos emerges for large values of the command parameter. It is shown that, surprisingly, chaos emerges for small values of the command parameter when spatial diffusion is taken into account. For small diffusion the computing space is uniform and stable, and for large diffusion, discrete travelling waves occur due to bifurcations, and then chaos, what I propose to call "diffusive chaos". The mathematical demonstration of the first bifurcation of diffusive chaos is made from a new incursive method.

It is also shown that advection can also give rise to bifurcations and chaos, what I call "advective chaos". From space-time Lotka-Volterra incursive discrete equations, it was shown that continuous space patterns emerge with discrete time iterates after a chaotic behaviour initiated by a perturbation in the initial conditions given by an homogeneous space distribution. Without space diffusion, the incursive Lotka-Volterra model shows orbital stability for the chosen values of parameters. A space-time chaotic propagating front is followed by spatial continuous travelling waves. Any perturbation in the continous travelling waves initiates again transient bifurcations. So, in conclusion, this paper shows that diffusion initiates a chaotic space-time structure and the incursion synchronises the automata along the space. This "diffusive chaos" is a new type of diffusive instability different from the Turing's morphogenesis by "diffusive instability".

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