Is Special Relativity Logically Prior to Quantum Mechanics?

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Abstract Special Relativity uses Einstein's two postulates to derive the Lorentz transformation, setting the stage for the Minkowski Spacetime that informs the dynamics of massive particles. How massive particles extract information from spacetime is not specified. We propose a method of specification in a simple two dimensional model that enforces Lorentz covariance by a local rule of preserving spacetime area. The model agrees with the canonical spacetime prescription on large scales but on small scales has the advantage that a particle's mass is evident in the fine-scale geometry of its world-line. This has the interesting feature that quantum propagation arises as a consequence of special relativity.
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1 Introduction

Historically, the development of Special Relativity initiated a radical change in the concepts of space and time. However, the magnitude of the speed of light is so great compared to terrestrial speeds, and the Newtonian low-speed limit so intuitive and well-developed that it is tempting to think of relativistic mechanics as an extension of Newtonian mechanics rather than vice-versa. Indeed, relativistic mechanics inherited many features of the Newtonian picture including the concept of a smooth world-line embedded in a background space designed to mimic the physical environment of a particle. Although non-relativistic quantum mechanics has shown that the picture of a completely smooth world-line is not tenable on fine scales, the ontological ambiguity of the wavefunction seems to prevent a specification as to how the world-line picture should be modified to show a clear relation between quantum mechanics and relativity, supposing one exists.

The replacement of particles by waves in quantum mechanics suggests that even the dimensionality of the world-'line' is ultimately in question. In the non-relativistic path integral, it is well-known that the uncertainty principle forces Feynman paths to have a Fractal dimension of two, so in a sense area rather than length is indicated for paths[1, 2, 3, 4]. However, the Lorentz transformation that motivates Minkowski spacetime is also extracted from a statement about 'areas'. For example:

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2 \tag{1}$$

The square of a small timelike displacement in spacetime is invariant through all inertial frames. This observation is conventionally enforced by defining a length metric with an odd signature so that the Newtonian picture of a smooth world-line in a Euclidean space is neatly transplanted into a smooth world-line in a pseudo-Euclidean space.

Conventionally, one takes equation(1) as motivation for the definition of an infinitessimal element of length ds in Minkowski space and assumes that a massive particle moves in this spacetime with mass being a background feature informed by the ambient Minkowski spacetime. However if "Spacetime tells matter how to move"[1] one might ask: How does it do this if *not* through geometry?

In this talk we adjust the two relativity postulates to allow us the freedom to implement a background spacetime through fine-scale geometry, following [6] closely. In the two dimensional model we discuss there are *considerable* advantages in this approach. The local rule for enforcing Lorentz covariance ultimately removes the formal clothing separating special relativity and quantum propagation. By the time we have explored the usual consequences of special relativity in light of the local rule, we find that quantum propagation is a natural feature that is discovered by paying close attention to path-dependent proper time. In this model neither formal quantization nor the invocation of a background Minkowski spacetime are needed to discover quantum propagation or Lorentz invariance. Both appear through the underlying geometry using only direct counting arguments from classical statistical mechanics.

In section 2 we review the modified relativity postulates and introduce a hypothetical particle called an 'EAPP' for Euclidean Area Preserving Particle, discussed in . We discuss the EAPP and how it approximates a conventional particle with a smooth world line.

In the following section we take EAPPs into the realm of stochastic processes. Here we see that the existence of the path-dependent proper time, if maintained in the continuum limit results in the Dirac and Schroedinger equations.

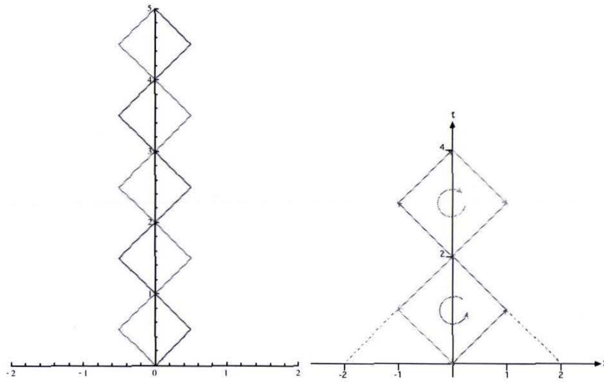
In the last section we summarize the advantages and disadvantages of this model and suggest directions for further work.

2 Euclidean Area Preserving Particles

In special relativity, the speed of light is a characteristic speed of deep significance. It is the speed of photons in free space, but it is also 'known' to massive particles through the famous relation to energy $E = mc^2$. In most expositions of special relativity this fact comes out when one considers conservation of momentum and energy in light of the Lorentz transformation[7, 8, 9]. Roughly speaking, Einstein's two postulates imply the necessity of relating inertial frames through the Lorentz

[1] "Spacetime tells matter how to move. Matter tells spacetime how to curve." J. A. Wheeler. [5]

transformation. This suggests that material objects must move in a spacetime with odd signature with the result that Newtonian momentum and energy conservation must be suitably modified.



(a) A chain of spacetime areas. (b) The basic repeating unit.

Fig. 1: (a) A particle at rest as a chain of spacetime areas. Here $c = 1$ and the particle, from the point of view of measuring its instantaneous speed, can be identified with the right-hand boundary. The sequence of crossing points and a smooth interpolant between them is the EAPP equivalent of a world line. (b) Two links in a chain of oriented areas. The orientation arises from a peculiarity of the Minkowski metric that for example links the $(x, t) = (1, 1)$ to the $(-1, 3)$ event.

The concept of the world-line of a massive particle is not itself imbued with any information regarding a particle's mass. Mass is simply a background attribute assigned to a world-line so that it can correctly model the behaviour of a real particle in connection with other particles and forces.

This is where we depart from standard approaches. The dynamics of a real-world particle reveals its mass and it is convenient to have mass as part of the geometry of the worldline. The idea is that there should be a simple fine-scale feature of the worldline that distinguishes a particle's mass. The feature has to be fine-scale so that on large scales we can revert to the picture of a smooth worldline informed by an ambient spacetime. We thus commit ourselves to saying exactly *how* spacetime tells particles how to move.

We can do this in a simple way by exploiting Nature's seeming predilection for area and its universal recognition of the speed of light. We shall require particle world lines to have instantaneous speeds $\pm c$ almost everywhere. Average speeds $v < c$ are then be generated by employing fine-scale motion that preserves an intrinsic area. As variants of the usual relativity postulates we propose:

1. The laws of physics are identical in all inertial frames on large scales.

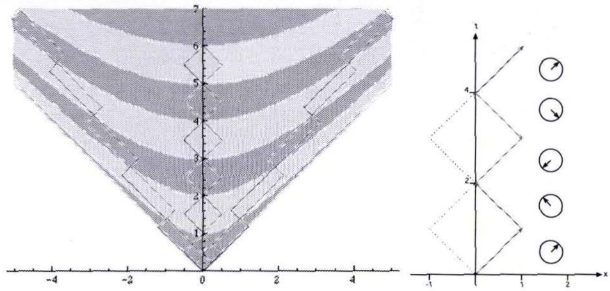
2. All material particles move with speed c almost everywhere.

We have weakened the first postulate to allow some flexibility with regard to scale. The strengthened second postulate forces a fine scale motion on massive particles. In a two dimensional spacetime the postulates force particle trajectories to have a zig-zag appearance as in Fig.(1(a)).

For reasons that will become clear, we shall think of such paths as occurring in pairs that form a chain of oriented spacetime areas. The oriented feature of areas in spacetime is a local mechanism for enforcing the Minkowski metric that would appear in a conventional approach, and must appear here on large scales. In the Minkowski metric, spacetime events that may be connected by a light-like path are equivalent in the sense that the metric distance between them is zero. In Fig.(1(b)) we see that the spacetime point $(x,t) = (1,1)$ is on the same null geodesic as the point $(x,t) = (-1,3)$. It is as if the space coordinates are interchanged from one area to the other. This feature is accounted for in an EAPP by orienting the areas of the two successive links. If the lower area is oriented positively according to the right-hand rule, the upper area is oriented negatively by the same rule if we consider the blue path from $(0,0)$ to $(4,4)$ to be directed as in the figure with the red path inheriting the appropriate direction to orient the two areas..

By analogy with worldlines we call the sequence of areas *world-chains*, the figure-of-eight pictured in Fig.(1(b)) providing the basic repeated unit. The EAPP pictured in Fig.(1) can be thought of as an approximation to a massive particle at rest. The ubiquitous presence of c is facilitated by the fact that each link in the chain has boundaries with slope $\pm c$. The 'at rest' feature is a manifestation of the fact that if we construct a linear interpolation of the crossing points of the chain, the result is the conventional world line of a particle at rest. The qualification of 'large scales' in our first postulate means scales much larger than the distance between crossing points of the chain. To give an idea of scale in the figure, if the EAPP is to mimic an electron, the time interval between crossing points is of the order of 10^{-22} seconds and the 'width' of the chain is of the order of 10^{-12} centimeters, both scales well below the effective limits of the classical behaviour of the electron. For simplicity we employ the same unit of measurement for both space and time, absorbing c into the t variable so light-like paths have slope ± 1 on spacetime diagrams and massive particles have average velocities $-1 < v < 1$.

We want the EAPP to respect Lorentz invariance and it is not too difficult to see how we can do this using Eucidian areas. Let us assume that if we observe the EAPP in Fig(1) from an inertial frame moving with respect to the lab frame with velocity $-v$, it will look just like a particle moving with velocity $+v$ on the same spacetime diagram. The fact that an EAPP has velocity $+v$ means that the interpolant connecting the chain of crossing points will be a straight line with slope $\triangle x / \triangle t = v$. To be in accord with the second postulate the boundaries of the links must still have slope $\pm c$. Furthermore, the first crossing point that was located at, say $(x,t) = (0,T)$ in the lab frame must be mapped onto $(x,t) = (X',T')$ where



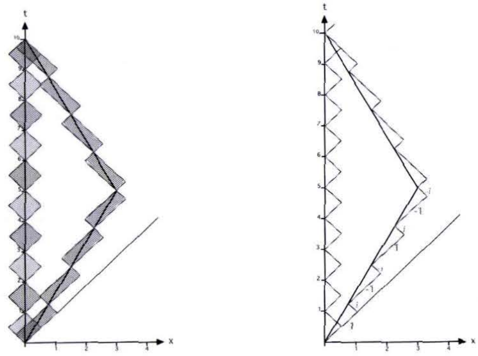
(a) Three EAPPs and oriented areas. (b) Clocks and corners.

Fig. 2: (a) A few EAPPs aligned at the origin. The ensemble of these world-chains representing free particles partitions spacetime into areas of opposite orientation. (b) The enumerative paths of an EAPP are such that the oriented paths provide a 'digital watch' that ticks with each corner in the path. This provides us with a convenient way of measuring proper time using complex numbers. In the figure is the right enumerative path of an EAPP. On the right of each link is the digital watch that ticks at each corner, keeping parallel to the oriented path. The association with complex numbers is that if the watch hand is a unimodular complex number, each tick represents multiplication by i.

$T'^2 - X'^2 = T^2$. A little geometry suggests that for the EAPP to obey the two postulates it has to preserve the *Euclidean area* enclosed by each link in the chain, since $T'^2 - X'^2$ is just twice this area. To allow arbitrarily small velocity changes, the orientation of the area must also be preserved. A sketch of a few EAPPs with the resulting oriented areas appears in Fig.(2(a)).

The ensemble of crossing points provides a sketch of a grid of points in Minkowski space. For example the locus of points of first crossings of all our particles of arbitrary v consists of all points whose time-like distance from the origin in the Minkowski metric is T. Our world-chains understand the Lorentz transformation through the imposition of the local and frame independent requirement that the Euclidean area of our chains be preserved. The ensemble of EAPPs also partitions spacetime into regions of positive and negative orientation, a feature that is not apparent in the smooth-worldline paradigm. The extra feature of orientation of areas means that the entire ensemble of free particle paths agree on the orientation of spacetime areas within the future light cone of the origin. This is illustrated in Fig.(2(a)) through the alternate shading of areas of positive and negative orientation.

The crossing points and corners of the EAPPs can be thought of as the ticks of an intrinsic clock carried by the particle. In Fig.(2(b)) we see that the basic repeating unit of two areas has four possible paths from bottom to top corresponding to two possibilities for the bottom two links and two possibilities for the top two. The two paths that maintain colour and direction in the figure have two corners. The two



(a) The Twin Paradox. (b) Digital Time.

Fig. 3: (a)The 'Twin Paradox' with a reflected EAPP. A chain of oriented areas 'at rest' at the origin represents the inertial twin in its rest frame. The rocket twin moving at speed 3/5 the speed of light is represented by a chain that is reflected back at the spacetime point $(x,t) = (3,5)$ in the lab frame. The moving EAPP's wrist watch time is 8 compared to the inertial time of 10. This can be seen by counting the areas in both paths. In this particular case the two twins have a different age, but they agree on the orientation of the areas when they meet. Other paths with differing times will have differing orientation when they meet. (b) Counting corners in the twin paradox example. Associating the unit imaginary with every corner in a path keeps track of the number of ticks of the clock as the argument of the exponential $e^{i\theta}$. As we progress along the right enumerative path of the rocket twin our digital clock ticks at the corners as $\{e^0, e^{i\pi/2}, e^{3i\pi/2}, \ldots\}$. The rule of i for every corner of the path will appear later when we consider the Dirac equation.

outer boundaries that change colour at the crossing point each contain three corners. The principal of maximal ageing suggests that these two, of the four possible, should count the proper time of the particle. We call these paths 'enumerative' for their role in counting proper time. The right hand boundary of a chain has links with directions that rotate counterclockwise with period four. This useful feature can be used as a clock.

The proper time of an EAPP differs from the usual proper time of special relativity in that it is digital. To distinguish it from conventional proper time we shall borrow the expression wristwatch time[9] to remind us that it is carried with the particle. As we shall see, the interval between ticks is determined by the mass of the particle, the clock itself being encoded in the spacetime geometry of the enumerative path. To see this we look at the twin paradox for EAPPs since it illustrates an important feature that does not appear explicitly in the usual formulation.

In Fig.(3(a)) two chains are compared. The inertial twin is at rest in the lab frame, the rocket twin moves at speed $v = 3/5$ out to the point $x = 3$ and back to the origin. As expected the wristwatch time of the rocket twin is 8 compared to 10 in the rest frame. This is evident by just counting time as units of area in the two chains. A feature that will be important later can be seen in the figure. Where the world lines cross at $t = 10$, the chains have an overlapping area. The areas of overlap have the same orientation. However orientation is clearly *path dependent* and paths that cross may intersect with areas of opposite orientation.

The feature that EAPPs have an internal structure that counts their wristwatch time is very useful and worth exploring. *It lies at the heart of the ubiquitous presence of complex numbers in quantum propagation, and the odd signature of spacetime in this model.* Along a world chain, the wristwatch ticks at each corner of the right-hand boundary, and a complex number may be used to represent a vector that stays parallel to the oriented right-hand boundary of the EAPP Fig(2(b)). The association of complex numbers with the four directions of oriented areas allows us to associate i with every tick of the clock (Fig.(3(b))). *This allows us to count the wrist-watch time of the particle using multiplication by the unit imaginary.* A tick corresponds to multiplication by i. The argument of $e^{i\theta}$ counts the ticks in units of $\frac{\pi}{2}$. Real or imaginary determines right or left. This is illustrated in (Fig. 3(b)) for the first part of the rocket twin path where each link is assigned one of the fourth roots of unity. Let us use the unit imaginary and the corners in the paths to count time for the twins. For the inertial twin the right enumerative path from $(x, t) = (0^+, 0^+)$ to $(0^+, 10^+)$ has 20 corners resulting in i^{20} for a proper time of 10 ($i^{20} = e^{10\pi i}$), an orientation of $+1$ ($i^{20} = 1$) and and a final enumerative direction along the right light cone (i^{20} is real). Similarly the rocket twin has a path with 16 corners for a proper time of 8 with a final orientation as for the inertial twin. The ultimate reason that complex numbers are implicated in the counting process here is the fact that we are dealing with oriented *areas* rather than the smooth curves of conventional world-lines. The oriented areas have orientation ± 1 so *the counting process involved in the statistical mechanics involves a periodic use of subtraction as well as addition.* The translation of this counting from area to length through a square root then invokes the unit imaginary. The rule itself ("associate i with every corner in the path") is a clever trick discovered by Feynman and will reappear later in association with his Chessboard Model.

Thus far EAPPs fulfill the kinematic requirements of special relativity on scales greater than the chainlink size. The area preserved is the product of the projections of the enclosed area onto the light-cone boundaries. For free particles, the crossing-point ticks are determined by two fixed frequencies, one on each of the cones. An EAPP always sees these two frequencies as equal on his wristwatch. An observer in a lab frame moving with respect to the EAPP will see two different frequencies Fig.(4).

To define an energy for an EAPP we borrow from the photoelectric effect. Let

$$E = h\nu \tag{2}$$

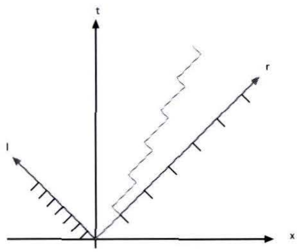


Fig. 4: The right-enumerative path of an EAPP moving at constant speed. The frequency of direction changes on the left and and right light cones depends on the macroscopic velocity of the particle. If the particle is at rest the two frequencies are the same. If the particle is moving the two frequencies differ.

where h is some fixed positive constant and ν is a frequency. The frequency we have in mind is the sum of frequencies of direction changes projected onto the light cones. That is

$$\nu = (\nu_r + \nu_l) \tag{3}$$

where the subscripts l and r refer to the left and right light cones. ν_r and ν_l are just inversely proportional to the lengths of the zigs and zags. If ν_0 is the frequency in the rest frame then the moving frame frequencies are:

$$\nu_r = \sqrt{\frac{1-v}{1+v}}\,\nu_0/2, \quad \nu_l = \sqrt{\frac{1+v}{1-v}}\,\nu_0/2 \tag{4}$$

so

$$\nu = \frac{\nu_0}{\sqrt{1-v^2}} = \gamma\,\nu_0. \tag{5}$$

Our proposed energy is then

$$E = h\frac{\nu_0}{\sqrt{1-v^2}} \approx h\nu_0 + \frac{1}{2}h\nu_0\,v^2 \tag{6}$$

the latter being the case in the event that $v << 1$. If we identify the EAPP rest mass with $m = h\nu_0$, equation (5) gives the correct velocity dependence of the relativistic mass. Similarly if we write $p = m\,\gamma\,v$ we get the required $E^2 = m^2 + p^2$.

These arguments show that the oriented area construction is sufficient to have EAPPs behave on large scales as if they were massive particles moving in Minkowski space. In the next section we consider more closely the effect of orientation on fine scales.

3 Sum Over Paths

When discussing the twin paradox of Fig(3(a)) we noted that orientation is *path-dependent* and is a function of the particle's wristwatch time. From Fig(2(b)) it seems that a convenient measure of orientation is a complex number that gives us a digital readout of the path's wristwatch time. From Fig(3(b)) it is clear that along any given path, starting at the origin, the orientation at the end of the path will be i^R where R is both the number of corners in the path and the number of ticks of the particle's wristwatch.

Let us now introduce a stochastic element. Consider a lattice with spacing $\epsilon << T$ where T is the first crossing point of our free particle EAPP. We generate a stochastic EAPP in the following way. We always step along diagonals on the lattice in the positive t direction. At each step we usually maintain our current direction but very occasionally switch direction introducing a corner in the path, with probability $\epsilon\, m << 1$. Such paths look just like our enumerative paths except the individual links ultimately have lengths governed by the exponential distributions. Now consider the following sum:

$$K(b, a, \epsilon) = \sum_R N(R)(i\epsilon m)^R \tag{7}$$

where b is a positive timelike distance from a. The sum here is over all the stochastic 'Chessboard-like' paths between a and b. The paths are partitioned with respect to the number of corners R in each path. In terms of EAPPs, R is the wristwatch time along the path, i^R is the orientation at the end of the path and $(\epsilon\, m)^R$ is the probability that a particular path has R corners. Clearly, all R-paths have the same wristwatch time and consequently the same digital watch state. There are only four such states, but the sum over R will mix them as linear combinations of the four complex numbers. The result will be another complex number that will interpolate between the original set of four watch-ticks. Equation(7) simply calculates an expected value of the orientation over all possible lattice paths, the variation of orientation being a result of path dependent proper time!

The sum in equation (7) is in fact well known. It is the same sum as the Chessboard model due to Feynman[2] . In the limit as $\epsilon \to 0$ it approaches the propagator for the Dirac Equation[10]. The formulation in (7) is Feynman's version of of his sum-over paths for a relativistic particle in two dimensions.

Formula (7) is usually the starting point for a demonstration of the sum over paths formulation for the Dirac Equation [10, 11, 12], the relation to Kac's model of the telegraph equations through analytic continuation[13], the interpretation of world chains as a single path[14, 15, 16] or an exploration and development of discrete physics[17]. It is a formula with a small but noticeable place in the history of quantum mechanics. In all these contexts Feynman's rule of i for every corner of

[2]Reference [1] problem 2.6

the path appears simply as a feature that eventually ties the model into the Dirac equation. The difference here is that we have arrived at (7) as a stochastic variant of a model that requires massive particles to preserve Lorentz invariance through local geometry. After the continuum limit the sum has a conventional interpretation as a formula for a free particle in Minkowski space after quantization. Prior to the continuum limit the paths themselves and the method of counting are nothing more than a stochastic version of EAPPs. Our local rule for preserving Lorentz invariance for a massive particle seems to have done more than requested.

We started with the intent to use oriented area to make sure worldlines had microscopic structure that would enable them, through a local rule, to behave as if they moved in Minkowski space. By providing our particle with a local rule about preserving spacetime area in order to satisfy Lorentz invariance, we ultimately get the Dirac equation for free.

Two for the price of one is economical, but it also suggests the possibility that attempted marriages of relativity and quantum mechanics may miss common causes. If we perform the sum in (7) and take the limit as $\epsilon \to 0$ we get, in the non-relativistic approximation $v << 1$, in conventional units

$$K(b,a) = \exp[-imc^2(t_b - t_a)/\hbar]\Big(\big(\frac{2\pi i\hbar(t_b - t_a)}{m}\big)^{-1/2} \exp\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\Big) \tag{8}$$

We can now read this formula in terms of EAPPS! The product of the exponentials is just the expected orientation based on wristwach time over the ensemble of paths using the approximation $(1 - v^2)^{1/2} \approx 1 - v^2/2$. In the conventional picture the formula is just Feynman's non-relativistic kernel multiplied by a very rapidly varying rest mass term that acts like a carrier wave. When we remove the rest mass term, the remainder obeys the Schroedinger equation. From the perspective of EAPPS, *Schroedinger's equation aquires its form as a diffusion equation with an imaginary diffusion constant as an inheritance from Lorentz invariance and the resulting path dependent proper time!* We do not find c explicitly in the Schroedinger equation simply because it drops out in the first order term in the small v expansion of $m\gamma c^2$.

From the EAPP point of view, the non-relativistic path integral also makes perfect sense. The usual Feynman kernel, in brackets in equation(8) is essentially the usual Gaussian kernel of the Wiener process that you would get from the Kac model of diffusion[18, 19], except the corner weight of 1 in the Kac model is replaced by i to count the proper time of the path. The path-dependent phase of Feynman paths in the non-relativistic path integral implements the path-dependent proper time of EAPPS in the non-relativistic approximation!

4 Discussion

One of the motivations for this approach was the observation that smooth world lines and quantum mechanics appear incompatible. A conventional approach where we try to quantize a system while invoking an ambient Minkowski space may be essentially taking two separate continuum limits where there should only be one. If we insist on a world-line picture in non-relativistic quantum mechanics we find that between the deBroglie and Compton scales, Feynman paths have dimension 2, but unlike Wiener paths, are assigned a complex number instead of a real positive weight. We also noted that Special Relativity starts with a metric statement that prefers area to length. When we wrote equation (1) we noted the usual restriction that we were referring to a timelike interval. Had we left out this qualification the equation would have been written:

$$(\Delta s)^2 = \pm((\Delta t)^2 - (\Delta x)^2) \tag{9}$$

allowing for both space and timelike displacements. At a qualitative level equation (9), as a statement about areas, suggests that areas are potentially *signed.* If we are discussing large objects on coarse scales we can, as is usually done, partition events into spacelike and timelike-separated and choose the length metric appropriately, invoking spacetime as the arbiter of the proper behaviour of massive particles. However, we know that in a path-oriented view of quantum mechanics, the uncertainty principle ultimately forces paths to be light-like on fine scales[3, 4], so with quantum mechanics in mind, we cannot afford the luxury of choosing the usual timelike/spacelike form of the length metric beforehand if we are going to model massive particles on fine scales.

Our choice of model, the EAPP, brings an intriguing unity to Nature's apparent preference for area over length. The signed area in (9) is invoked in the EAPP through areas oriented by their boundaries. The principle of maximal ageing picks out the enumerative paths associated with the chain of areas and shows that Feynman's corner rule just counts wristwatch time. This brings a representation of complex numbers into the description as we go from oriented area to length. The implication of complex numbers here comes from the fourth roots of unity associated with (9) when we take a square root. When we put in stochastic paths we see that areas of different orientation overlap. The path manifestation of this is that the varying wristwatch times over the various paths interpolate between the fourth roots of unity associated with each path. The result is the Dirac equation. When we look at the whole picture in the non-relativistic approximation, the paths on scales above the Compton length are Wiener-like except they inherit the complex phase that measures wristwatch time.

The moral of this story is that the *mathematical convenience* of the invocation of spacetime in two dimensions comes at a price. As originally intended, it allows one to transplant the smooth worldline picture of Newton into the relativistic domain. However once invoked one then has to *force* quantization on the picture through a formal quantization procedure. This model shows that in two dimensions this is not only unnecessary, it obscures the fact that relativity and quantum mechanics here share a common cause. By replacing smooth world lines with chains of oriented areas we enforce Lorentz covariance through local geometry in a way that both invites, and illuminates Feynman's chessboard model and the associated Dirac equation.

There can be two contrasting speculations about this.

1. The picture is likely to be an artifact of two dimensions made possible by the fact that there are only four possible directions in spacetime. The four possible directions can be linked to the fourth roots of unity resulting in the connection demonstrated by EAPPS. In a four dimensional spacetime the simplicity of only four directions is lost, making the situation more complicated and removing the common connection. This is reflected in the lack of any consensus on an extension of the Chessboard model to four dimensions.

2. The picture is likely to be more general than the 2D model. The arguments that motivate the model do not depend on dimension. The result of the model is that the path dependence of the proper time of special relativity manifests itself as the path dependent phase of quantum mechanics. Since proper time is a Lorentz scalar the result may be expected to carry over to four dimensions.

We cannot resolve these two possibilities here although the author favours the second conclusion and will publish an extension of the Chessboard model featuring oriented areas in due course.

There is a large literature on Zittebewegung in the Dirac equation with opinions on the phenomenon varying from considering it an artifact and a distraction, to being a phenomenon central to quantum mechanics [20]. This model sides strongly with the latter view in that the EAPP is a specific model of Zitterbewegung in two dimensions. As the proponants of Geometric Algebra continually point out, the efficacy of GA to provide a compact description of both spacetime on large scales, and the Dirac equation with its link to Zitterbewegung on small scales, is a feature that warrants thorough investigation[21, 22]. This work would suggest that the efficacy of GA relates directly to the encoding of the idea of oriented areas in a graded algebra. The manifestation of this on large scales is the immediate encoding of an odd spacetime signature appropriate to relativity. On small scales, the particular encoding of the odd signature through vectors allows for a compact and geometric treatment of the Dirac equation.

The Chessboard model, for which this paper could be considered a preamble, was developed by Feynman in a period when he was trying to understand the Dirac equation from as many points of view as possible. Regarding this process he commented to his friend T. A. Welton:[23]

> I dislike all this talk of others [of there] not being a picture possible, but we only need know how to go about calculating any phenomenon. True we only *need* calculate. But a picture is certainly a *convenience* & one is not doing anything wrong in making one up. It may prove to be entirely haywire while the equations are nearly right — yet for a while it

> helps. The power of mathematics is terrifying — and too many physicists finding they have correct equations without understanding them have been so terrified they give up trying to understand them. I want to go back & try to understand them. What do I mean by understanding? Nothing deep or accurate — just to be able to see some of the qualitative consequences of the equations by some method other than solving them in detail.

The EAPP model is much in the same spirit. It shows that Feynman's 'corner rule', that ultimately implicates the ubiquitous phase of wavefunctions, is itself a consequence of classical counting arguments from Special Relativity. This suggests that amending the smooth-worldline picture in Special Relativity may ultimately lead to a better understanding of Quantum Mechancs.

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