Intelligent Network Structures and Algorithms for Solving Multiobjective Control Problems

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Abstract Discrete multiobjective control problems with varying time of states transactions of dynamical system are formulated and studied. Nash equilibria conditions for the considered problems are given and algorithms based on dynamic programming for determining optimal solutions in the sense of Nash and Pareto are proposed and proved. We exploit a certain (so-called) intelligent network structure to underline a constructive approach.
Keywords : Time-Discrete Dynamical System, Discrete (Multi-)objective Control, Nash Equilibria, Pareto Optima

1 Introduction

The paper is concerned with a game-theoretical approach for the discrete optimal control problems with varying time of states' transitions of dynamical system. We formulate and study discrete multiobjective control problems which extend and generalize the following single-objective control problem.

Let a time-discrete system L with finite set of states $X \subseteq R^n$ be given. At every discrete moment of time $t = 0, 1, 2, \ldots$ the state of L is $x(t) \in X$. Two states x_0 and x_f are given in X, where $x_0 = x(0)$ is the starting state and x_f is the final state of L. The dynamical system should reach the final state x_f at the time moment $T(x_f)$ such that

$$T_1 \leq T(x_f) \leq T_2,$$

where T_1 and T_2 are given. The control of the system L at each time-moment $t = 0, 1, 2, \ldots$ for an arbitrary state $x(t)$ is realized by using the control vector $u(t)$ which is characterized by a feasible set $U_t\,(x(t))$, i.e. $u(t) \in U_t\,(x(t))$. In addition we assume that for an arbitrary t and $x(t)$ on $U_t\,(x(t))$ an integer function

$$\tau : \; U_t\,(x(t)) \rightarrow N$$

is introduced. To each control $u(t) \in U_t\,(x(t))$ an integer value $\tau\,(u(t))$ is related.

This value represents the time of system's passage from the state $x(t)$ to the state $x\big(t+\tau\,(u(t))\,\big)$ if the control $u(t) \in U_t\,(x(t))$ has been chosen at the time moment t for the given state $x(t)$.

Assume that the dynamics of the system is described by the following system of difference equations

$$\begin{cases} t_{j+1} & = & t_j + \tau\,(u(t_j))\,; \\ x(t_{j+1}) & = & g_{t_j}\,(x(t_j), u(t_j))\,; \\ & & u(t_j) \in U_{t_j}\,(x(t_j))\,; \\ & & j = 0, 1, 2, \dots, \end{cases} \tag{1}$$

where

$$t_0 = 0, \quad x(t_0) = 0 \tag{2}$$

is a starting representation of the dynamical system L. We suppose that the functions g_t and τ in (1) are known and t_{j+1} and $x(t_{j+1})$ are determined uniquely by $x(t_j)$ and $u(t_j)$ at every step $j = 0, 1, 2, \dots$. Let $u(t_j)$, $j = 0, 1, 2, \dots$, be a control, which generates the trajectory

$$x(0), x(t_1), x(t_2), \dots, x(t_k), \dots$$

Then either this trajectory passes through the final state x_f and $T(x_f) = t_k$ represents the time-moment when the final state x_f is reached or this trajectory does not pass trough x_f. For an arbitrary control $u(t)$ we define the quantity

$$F_{x_0 x_f}\,(u(t)) = \sum_{j=1}^{k-1} c_{t_j}\big(x(t_j), g_{t_j}\,(x(t_j), u(t_j))\,\big)$$

if the trajectory $x(0), x(t_1), x(t_2), \dots, x(t_k), \dots$ passes through the final state x_f at the time-moment $t_k = T(x_f)$ such that $T_1 \le T(x_f) \le T_2$; otherwise we put

$$F_{x_0 x_f}\,(u(t)) = \infty.$$

Here $c_{t_j}\big(x(t_j), g_{t_j}\,(x(t_j), u(t_j))\big) = c_{t_j}\,(x(t_j), x(t_{j+1}))$ represents the cost of system L to pass from the state $x(t_j)$ to the state $x(t_{j+1})$ at the stage $[j, j+1]$.

We consider the problem of finding the time-moments $t = 0, t_1, t_2, \dots, t_k$ and the vectors of control parameters $u(t_0), u(t_1), u(t_2), \dots, u(t_k)$ which satisfy conditions (1), (2) and minimize the functional $F_{x_0 x_f}\,(u(t))$.

In the case $\tau(u(t)) \equiv 1$ this problem becomes the discrete control problem from [1, 3] for which dynamic programming algorithms are proposed in [1, 3, 4]. Algorithms based on dynamic programming for the problem with varying time of states transitions of dynamical system have been developed in [9]. In this paper we extend these results from [5, 6, 7] and study the game theoretic version of the considered discrete control problem.

The main results are concerned with determining Nash equilibria and Pareto optima for the multiobjective control problem. Algorithms for finding optimal strategies of the players in such problems are proposed and proved. Furthermore we introduce a special network structure (intelligent network) to support a constructive approach.

2 Multiobjective Control Based on Concept of Noncooperative Games and Determining Nash Equilibria

In this section we formulate the discrete multiobjective control problem with varying time of states transitions using the concept of noncooperative games. Furthermore we propose an approach for determining Nash equilibria in the case when the alternate players' control condition holds.

2.1 The Problem Formulation

Assume that the dynamics of system L is controlled by p players. To each player $i \in \{1, 2, \ldots, p\}$ it is associated a vector of control parameters $u^i(t) \in R^{m_i}$ and at every discrete moment of time $t = 0, 1, 2, \ldots$ for an arbitrary state $x(t) \in X$ a feasible set $U_t(x(t))$ is given such that $u^i(t) \in U_t(x(t))$. In addition for an arbitrary state $x = x(t) \in X$ at every discrete moment of time t the time-transition function

$$\tau = \tau\left(u^1(t), u^2(t), \ldots, u^p(t)\right),$$

is defined on the set

$$U_t(x(t)) = \prod_i U_t^i(x(t)).$$

The dynamics of the system L is assumed to be described by the following system of difference equations:

$$\begin{cases} t_{j+1} &= t_j + \tau\left(u^1(t_j), u^2(t_j), \ldots, u^p(t_j)\right); \\ x(t_{j+1}) &= g_{t_j}\left(x(t_j), u^1(t_j), u^2(t_j), \ldots, u^p(t_j)\right); \\ & \quad u^i(t_j) \in U_{t_j}^i(x(t_j)), \ i = \overline{1,p}; \end{cases} \tag{3}$$

$$j = 0, 1, 2, \ldots,$$

where

$$t_0 = 0, \ x(t_0) = 0 \tag{4}$$

is a starting representation of the dynamical system L.

In the following we consider that the state $x(t_{j+1})$ of the system L at the time-moment t_{j+1} is obtained uniquely if in (3) the state $x(t_j)$ at the time-moment t_j is known and players $1, 2, \dots, p$ independently fix there vectors of control parameters $u^1(t_j), u^2(t_j), \dots, u^p(t_j)$, respectively, where

$$u^i(t_j) \in U^i_{t_j}\left(x(t_j)\right),\ i = 1, 2, \dots, p;\ j = 0, 1, 2, \dots.$$

Let a control

$$\begin{cases} u^1(t_0), u^2(t_0), \dots, u^p(t_0), \\ u^1(t_1), u^2(t_1), \dots, u^p(t_1), \\ u^1(t_2), u^2(t_2), \dots, u^p(t_2), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u^1(t_j), u^2(t_j), \dots, u^p(t_j), \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{5}$$

be given where

$$\begin{aligned} &t_{j+1} = t_j + \tau\left(u^1(t_j), u^2(t_j), \dots, u^p(t_j)\right),\quad j = 0, 1, 2, \dots; \\ &t_0 = 0. \end{aligned}$$

Then for a given control either a unique trajectory

$$x_0 = x(0), x(t_1), x(t_2), \dots, x(t_k) = x\left(T(x_f)\right) = x_f$$

from x_0 to x_f exists and $t_k = T(x_f)$ represents the time-moment when the state x_f is reached or such a trajectory does not pass through x_f. We denote by

$$F^i_{x_0 x_f}\left(u^1(t), u^2(t), \dots, u^p(t)\right) = \sum_{j=1}^{k-1} c^i_{t_j}\left(x(t_j), g_{t_j}\left(x(t_j), u^1(t_j), u^2(t_j), \dots, u^p(t_j)\right)\right)$$

the integral-time cost of system's passage from x_0 to x_f for the player i, $i \in \{1, 2, \dots, p\}$ if vectors $u^1(t_j), u^2(t_j), \dots, u^p(t_j)$ satisfy conditions (3), (4) and generate a trajectory

$$x_0 = x(0), x(t_1), x(t_2), \dots, x(t_k) = x\left(T(x_f)\right) = x_f$$

from x_0 to x_f such that $T_1 \le T(x_f) \le T_2$; otherwise we put

$$F^i_{x_0 x_f}\left(u^1(t), u^2(t), \dots, u^p(t)\right) = \infty.$$

In an analogous way as in Section 1 we consider that T_1 and T_2 are given.

Here $c^i_{t_j}\left(x(t_j), g_{t_j}\left(x(t_j), u^1(t_j), u^2(t_j), \dots, u^p(t_j)\right)\right) = c^i_{t_j}\left(x(t_j), x(t_{j+1})\right)$ represents the cost of system's passage from the state $x(t_j)$ to the state $x(t_{j+1})$ at step j.

We consider the problem of finding the vector of control parameters

$$u^{1^*}(t), u^{2^*}(t), \dots, u^{i-1^*}(t), u^{i^*}(t), u^{i+1^*}(t), \dots, u^{p^*}(t)$$

which satisfies the condition

$$F^i_{x_0 x_f}\left(u^{1*}(t), u^{2^*}(t), \dots, u^{i-1^*}(t), u^{i^*}(t), u^{i+1^*}(t), \dots, u^{p^*}(t)\right) \le$$

$$\le F^i_{x_0 x_f}\left(u^{1^*}(t), u^{2^*}(t), \dots, u^{i-1^*}(t), u^{i}(t), u^{i+1^*}(t), \dots, u^{p^*}(t)\right),$$

$$\forall\ u(t) \in R^{m_i},\ t = 0, 1, 2, \dots;\ i = 1, 2, \dots p.$$

So, we are seeking for a Nash equilibrium [10].

This problem in the case of unit time of states transactions of dynamical system has been studied in [5, 6, 7].

2.2 Determining Nash Equilibria for Multiobjective Control Problems in Positional Form

In order to formulate the existence theorem of Nash equilibria for the considered multiobjective control problem we will use the alternate players control condition from [6].

We assume that an arbitrary state $x(t_j) \in X$ of the dynamical system L at the time-moment t_j represents a position $(x, t_j) \in X \times \{0, 1, 2, \dots\}$ for one of the players $i \in \{1, 2, \dots, p\}$. This means that in the control process the next state $x(t_{j+1}) \in X$ is determined (chosen) by the player i if the dynamical system L at the time-moment t has the state $x(t_j)$ which corresponds to the position (x, t_j) of the player i. This situation corresponds to the case when the expression

$$g_{t_j}(x(t_j), u^1(t_j), u^2(t_j), \dots, u^{i-1}(t_j), u^i(t_j), u^{i+1}(t_j), \dots, u^p(t_j))$$

in (3) for given t_j depends only on the control vector $u^i(t_j)$, i.e.

$$g_{t_j}(x(t_j), u^1(t_j), u^2(t_j), \dots, u^{i-1}(t_j), u^i(t_j), u^{i+1}(t_j), \dots, u^p(t_j)) =$$

$$= \overline{g}_{t_j}(x(t_j), u^i(t_j)).$$

Moreover, we assume that for the given state $x(t_j)$ at the given moment of time t_j the transition-time function

$$\tau = \tau(u^1(t_j), u^2(t_j), \dots, u^{i-1}(t_j), u^i(t_j), u^{i+1}(t_j), \dots, u^p(t_j))$$

also depends only on the vector of control parameters $u^i(t)$, i.e.

$$\tau(u^1(t_j), u^2(t_j), \dots, u^{i-1}(t_j), u^i(t_j), u^{i+1}(t_j), \dots, u^p(t_j)) = \overline{\tau}(u^i(t_j)).$$

The alternate players' control condition mentioned above means that at every moment of time $t_j = 0, 1, 2, \ldots, T_2$ for the set of states X there exists a partition

$$X = X(t_j) = X_1(t_j) \cup X_2(t_j) \cup \cdots \cup X_p(t_j) \ (X_k(t_j) \cap X_l(t_j) = \emptyset, \ k \neq l)$$

such that the second equation in (3) can be represented as follows

$$x(t_{j+1}) = g^i_{t_j}(x(t_j), u(t_j)) \text{ if } x(t_j) \in X_i(t_j).$$

The following theorem holds.

Theorem 2.1. *Let us assume that for the multiobjective control problem with varying time of states' transitions there exists a trajectory*

$$x_0 = x(0), x(t_1), x(t_2), \ldots, x(t_k) = x\left(T(x_f)\right) = x_f$$

from starting state x_0 to the final state x_f generated by the vectors of control parameters (5), where $t_0 = 0,\ t_{j+1} = t_j + \tau\left(u^1(t_j), u^2(t_j), \ldots, u^p(t_j)\right),\ j = 0, 1, 2, \ldots,$ $u^i(t_j) \in U_{t_j}\left(x(t_j)\right),\ i = 1, 2, \ldots, p,$ and $T_1 \leq T(x_f) \leq T_2$. Moreover, we assume that the alternate players control condition is satisfied. Then for this problem there exists the optimal solution in the sense of Nash.

Proof. We prove the theorem by using the idea and the constructive scheme of the main theorem from [7] concerning the existence of Nash equilibria for the multiobjective control problem with unit time of states' transitions. So, we give a construction which allows us to reduce our problem to a problem of finding the optimal stationary strategies of the players in a dynamic c-game on the network $(\overline{G}, Y_1, Y_2, \ldots, Y_p, c^1, c^2, \ldots, c^p, z_0, \overline{z}_f)$, where $\overline{G} = (Y \cup \{\overline{z}_f\}, \overline{E})$ is an acyclic directed graph with sink vertex $\overline{z}_f$. The set of vertices Y consists of $T_2 + 1$ copies of the set of states X, i.e.,

$$Y = X_0 \cup X_1 \cup \cdots \cup X_{T_2},$$

where $X_t = (X, t)$ is a set of states which correspond to the time moment $t = 0, 1, 2, \ldots, T_2$ (see Fig. 1). We call this graph an intelligent network in order to express that the structure acts as a brain in the constructive approach.

In $\overline{G}$ two vertices (x, t_j) and (y, t_{j+1}) are connected with a directed edge $((x, t_j), (y, t_{j+1})) \in \overline{E}$ if for the given position $z = (x, t_j) = x(t_j) \in Y_i$ there exists a control $u^i(t_j) \in U^i_{t_j}(x(t_j))$ such that

$$y = x(t_{j+1}) = g^i_t(x(t_j), u^i(t_j)).$$

Additionally, in $\overline{G}$ for $t = T_1, T_1 + 1, \ldots, T_2$ we add the edges $((x, t), \overline{z}_f)$.

To each edge $((x, t_j), (y, t_{j+1})) \in \overline{E}$ we associate p costs

$$c^i((x, t_j), (y, t_{j+1})) = c^i_{t_j}(x(t_j), y(t_{j+1})), \ i = \overline{1, p},$$

and for each edge $((x, t), \overline{z}_f)$, $t = T_1, T_1 + 1, \dots, T_2$, we put

$$c^i((x, t), \overline{z}_f) = 0, i = \overline{1, p}.$$

It is easy to observe that between the set of feasible trajectories of the dynamical system L from the starting state $x_0 = x(0)$ to the final state x_f in the multiobjective control problem and the set of directed paths from $(x_0, 0)$ to $\overline{z}_f$ in $\overline{G}$ there exists a bijective mapping such that the corresponding costs of the payoff functions of the players in the multiobjective control problem and in the dynamic c-game on the network are the same. Taking into account that for the dynamic c-game on the acyclic network there exists Nash equilibria for arbitrary costs on the edges (see [2, 7]) we may conclude that for our multiobjective control problem there exists a Nash equilibrium. □

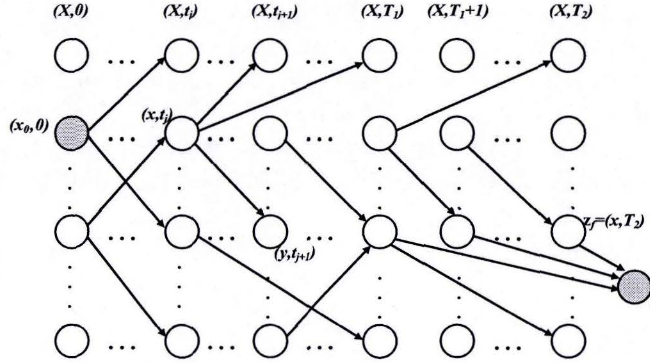

Fig. 1: Intelligent Network

Based on the constructive proof of Theorem 1 we can propose the following algorithm for determining Nash equilibria in the multiobjective control problem when the alternate players control condition holds.

Algorithm

1. Construct the auxiliary network $(\overline{G}, Y_1, Y_2, \ldots, Y_p, c^1, c^2, \ldots, c^p, z_0, \overline{z}_f)$;

2. Find optimal stationary strategies of the players in a dynamic c-game on the auxiliary network;

3. Find optimal solutions in the sense of Nash for the multiobjective control problem using a one-to-one correspondence between the set of strategies of the players in the dynamic c-game on $\overline{G}$ and the set of feasible control parameters of the players in the multiobjective control problem.

2.3 Determining Stationary Strategies of the Players in a Dynamic c-game on Networks with Transition-Time Functions on the Edges

In the following we formulate the dynamic c-game in the general form, i.e. we consider the c-game on a network which may contain cycles and transition-time functions on the edges depending on time.

Let $G = (X, E)$ be the graph of states' transition for the dynamical system L with a finite set of states X. So, an arbitrary edge $e = (x, y) \in E$ indicates the possibility of system L to pass from the state $x = x(t)$ to the state $y \in X_G(x) = \{y \in X | (x, y) \in E\}$ at every moment of time $t = 0, 1, 2, \ldots$. Two states x_0, x_f are given in X, where $x_0 = x(0)$ is the starting state of the dynamical system L and x_f is the final state of L. The final state x_f should be reached at the time moment $T(x_f)$ such that $T_1 \leq T(x_f) \leq T_2$, where T_1 and T_2 are known.

Assume that the system L is controlled by p players and the control on G is made by the following way. The vertex set X is divided into p subsets

$$X = X_1 \cup X_2 \cup \cdots \cup X_p \ (X_i \cap X_j = \emptyset, i \neq j),$$

where vertices $x \in X_i$ are regarded as positions of the player $i \in \{1, 2, \ldots, p\}$. The control starts at position $x_0 = x(t_0)$, where $t_0 = 0$. If $x(t_0) \in X_{i_1}$, then player i_1 transfers system L from the state x_0 to state $x_1 = x(t_1)$, where $t_1 = t_0 + \tau_{e_0}(t_0)$, $e_0 = (x_0, x_1) \in E$. If $x(t_1) \in X_{i_2}$, then player i_2 transfers system L from the state x_1 to state $x_2 = x(t_2)$, where $t_2 = t_1 + \tau_{e_1}(t_1)$, $e_1 = (x_1, x_2) \in E$ and so on. If at time moment t_k the final state x_f is reached, i.e. $x(t_k) = x_f$, then STOP. After that the players calculate their integral-time costs

$$H^i_{x_0 x_f} = \sum c^i_{(x(t_j), x(t_{j+1}))}(t_j), i = \overline{1, p},$$

if $T_1 \leq t_k \leq T_2$; otherwise put $H^i_{x_0 x_f} = \infty$.

Here $c_e^i(t_j)$ represents the cost function on the edge $e = (x, y) \in E$, which expresses the cost of system's passage from the state $x = x(t_j)$ to the state $y = x(t_j + \tau_e(t_j))$. In the control process players intend to minimize their integral-time costs by a trajectory $x(t_0), x(t_1), \ldots, x(t_k)$.

In this dynamic game we assume that players use only stationary strategies. We define the stationary strategies of the players $1, 2, \ldots, p$ as maps:

$$\begin{aligned} s_1 &: x \to y \in X(x) \text{ for } x \in X_1 \setminus \{x_f\}; \\ s_2 &: x \to y \in X(x) \text{ for } x \in X_2 \setminus \{x_f\}; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_p &: x \to y \in X(x) \text{ for } x \in X_p \setminus \{x_f\}. \end{aligned}$$

Let $s_1, s_2, \ldots, s_p$ be an arbitrary set of the strategies of the players and $G_s = (X, E)$ represents the subgraph of G generated by edges $e = (x, s_i(x))$ for $x \in X \setminus \{x_f\}$, $i = \overline{1,p}$. Then for fixed $s_1, s_2, \ldots, s_p$ either a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f exists in G_s or such a path does not exist in G_s. For fixed strategies $s_1, s_2, \ldots, s_p$ and given x_0 and x_f we define the quantities

$$H^1_{x_0x_f}(s_1, s_2, \ldots, s_p), H^2_{x_0x_f}(s_1, s_2, \ldots, s_p), \ldots, H^p_{x_0x_f}(s_1, s_2, \ldots, s_p)$$

by the following way.

Let us assume that the path $P_s(x_0, x_f)$ exists in G and we assign to its edges numbers $0, 1, 2, \ldots, k_s$ starting with the edge that begins in x_0. Then we can calculate the time $t_{e_k} = t_{e_k}(s_1, s_2, \ldots, s_p)$, where $t_{e_0} = 0$, $t_{e_j} = t_{e_{j-1}} + \tau_{e_j}(t_{e_{j-1}})$, $j = \overline{1, k_s}$. We put

$$H^i_{x_0x_f}(s_1, s_2, \ldots, s_p) = \sum_{j=0}^{k_s} c_{e_j}(t_{e_k}(s_1, s_2, \ldots, s_p)),$$

if $T_1 \le |E(P_s(x_0, x_f))| \le T_2$; otherwise we put $H^i_{x_0x_f}(s_1, s_2, \ldots, s_p) = \infty$.

We consider the problem of finding maps $s_1^*, s_2^*, \ldots, s_p^*$ for which the following conditions are satisfied

$$H^i_{x_0x_f}(s_1^*, s_2^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_p^*) \le$$

$$\le H^i_{x_0x_f}(s_1^*, s_2^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_p^*), \ \forall s_i \in S_i, i = \overline{1,p}.$$

The intelligent network for the dynamic c-game with transition time functions is denoted by $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), \tau(t), x_0, x_f, T_1, T_2)$, where $\tau(t) = (\tau_{e_1}(t), \tau_{e_2}(t), \ldots, \tau_{e_{|E|}}(t))$. In an analogous way as for the stationary dynamic c-game from [6] here T_1 and T_2 satisfy conditions: $0 \le T_1 \le |X|-1$, $T_1 \le T_2$. If $T_1 = 0$, $T_2 = \infty$, then we will use the notation $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), \tau(t), x_0, x_f)$.

The following theorem holds.

Theorem 2.2. *Let $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), \tau(t), x_0, x_f, T_1, T_2)$ be a dynamic network for which in G there exists a directed path $P_s(x_0, x_f)$ with integral time $T(x_f)$ such that $T_1 \le T(x_f) \le T_2$ $(0 \le T_1 \le |X| - 1,\ T_1 \le T_2)$. In addition vectors $c^i = (c^i_{e_1}, c^i_{e_2}, \dots, c^i_{e_{|E|}})$, $i \in \{1, 2, \dots, p\}$, and $\tau = (\tau_{e_1}, \tau_{e_2}, \dots, \tau_{e_{|E|}})$ have positive and constant components. Then in the dynamic c-game on the network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), \tau(t), x_0, x_f,$ $T_1, T_2)$ there exists an optimal solution in the sense of Nash.*

This theorem in the case of integer functions τ_e follows from [6]. Indeed, the dynamic c-game on the network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), \tau(t), x_0, x_f, T_1, T_2)$ can be reduced to an auxiliary dynamic c-game with unit transition time functions on the edges on an auxiliary network $(\overline{G}, \overline{X}_1, \overline{X}_2, \dots, \overline{X}_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{x}_0, \overline{x}_f, T_1, T_2)$, where the graph $\overline{G}$ is obtained from G if each directed edge $e = (x, y) \in E$ is changed by a sequence of $|\tau_e|$ edges $e^1 = (x, x^1)$, $e^2 = (x^1, x^2)$, $\dots$, $e^{|\tau_e|} = (x^{|\tau_e|}, y)$ with the costs $\overline{c}_{(x,x^1)} = c_e$, $\overline{c}_{(x^1,x^2)} = \overline{c}_{(x^2,x^3)} = \cdots = \overline{c}_{(x^{|\tau_e|},y)} = 0$. We define the partition $\overline{X}_1, \overline{X}_2, \dots, \overline{X}_p$ in $\overline{G}$ by the following way: The corresponding sets $X_1, X_2, \dots, X_p$ are associated with $\overline{X}_1, \overline{X}_2, \dots, \overline{X}_p$ and,respectively, and all new vertices x^i, $i = \overline{1, |\tau_e|}$ are associated with $\overline{X}_1$.

According to [6] there exists in the dynamic c-game on

$$(\overline{G}, \overline{X}_1, \overline{X}_2, \dots, \overline{X}_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), \overline{x}_0, \overline{x}_f, T_1, T_2)$$

a Nash equilibrium if $T_1 \le |\overline{E}(P_s(\overline{x}_0, \overline{x}_f))| \le T_2$. Taking into account that between the set of strategies in the auxiliary dynamic c-game and the set of strategies in the initial dynamic c-game there exists a bijective mapping which preserves the integral-time costs of strategies we obtain the proof of Theorem 2.

So, for finding optimal stationary strategies in the dynamic c-game with transition-time functions on the edges algorithms from [6, 7] on the auxiliary time-expanded network can be used (as for the problem from the previous section).

2.4 Determining Nonstationary Strategies of the Players in a Dynamic c-game on Networks

Now we formulate and study the problem of finding the optimal nonstationary strategies of the players in a dynamic c-game on the networks with varying time of the states transitions. We define nonstationary strategies of the players as maps:

$$\begin{aligned} u_1 &: (x, t) \to (y, t + \tau_{(x,y)}(t)) \text{ for } x \in X_1 \setminus \{x_f\} \\ u_2 &: (x, t) \to (y, t + \tau_{(x,y)}(t)) \text{ for } x \in X_2 \setminus \{x_f\} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_p &: (x, t) \to (y, t + \tau_{(x,y)}(t)) \text{ for } x \in X_p \setminus \{x_f\} \end{aligned}$$

where $y \in X(x)$ and $t = 0, 1, 2, \dots$.

Using the same concept from [6, 7] for an arbitrary set of nonstationary strategies $u_1, u_2, \dots, u_p$, which generate a trajectory $x(t_0), x(t_1), x(t_2), \dots, x(t_k), \dots$, we can define integral-time costs

$$F^i_{x_0 x_f}(u_1, u_2, \dots, u_p) = \sum_{j=0}^{t_{k-1}} c^i_{(x(t_j), x(t_{j+1}))}(t_j), i = \overline{1, p}$$

if $T_1 \le t_k \le T_2$; otherwise we put $F^i_{x_0 x_f}(u_1, u_2, \dots, u_p) = \infty$. Here $t_0 = 0$, $t_{j+1} = t_j + \tau_{(x(t_j), x(t_{j+1}))}(t_j)$, $t_k = T(x_f)$ and $x(t_{j+1}) = u_i(x(t_j))$ if $x(t_j) \in X_i$, $i = \overline{1, p}$.

The same technique of the time-expanded network from [8] can be developed for reducing the nonstationary problem to the stationary case with constant costs and constant transition times on the edges in an acyclic auxiliary network as for the transition problem from Section 2.2.

2.5 Remark on Determining Pareto Optima for Multiobjective Control Problems with Varying Time of States' Transitions

The concept of cooperative games for multiobjective control problems with varying time of the states' transition of the dynamical system can be used in an analogous way as in [6, 7]. Here we should take into account that the dynamics of the system is described by the following system of difference equations

$$\begin{aligned} t_{j+1} &= t_j + \tau\left(u(t)\right), \\ x(t_{j+1}) &= g_{t_j}\left(x(t_j), u(t_j)\right), \\ &\quad u(t_j) \in U_t\left(u(t_j)\right), \\ &\quad j = 0, 1, 2, \dots \end{aligned}$$

where

$$x_0 = 0, \quad x(t_0) = x_0$$

is a starting representation of dynamical system L.

In the control process p players participate, which coordinate their actions by using common vector of control parameters $u(t) \in R^m$.

Let $u(t_j)$, $j = 0, 1, 2, \dots$, be a cooperative control of the dynamical system, which generates a trajectory

$$x(t_0), x(t_1), x(t_2), \dots,$$

where $t_0 = 0$, $t_{j+1} = t_j + \tau\left(u(t_j)\right)$, $j = 0, 1, 2, \dots$

We define

$$F^i_{x_0 x_f}(u(t)) = \sum_{j=0}^{k-1} c^i_{t_j}\left(x(t_j), g_{t_j}(x(t_j), u(t_j))\right), i = 0, 1, 2, \ldots, p$$

the integral-time cost of the system's passage from $x_0 = x(0)$ to $x_f = x(t_k) = x(T(x_f))$ if

$$T_1 \leq T(x_f) \leq T_2;$$

otherwise we put

$$F^i_{x_0 x_f}(u(t)) = \infty.$$

A Pareto solution for this cooperative game can be obtained on the basis of the modified time-expanded network method by using algorithms from [7].

The auxiliary network for the problem in this case is obtained in the analogous way as in Section 2.2. Here we should not take into account the partition $X = X_1 \cup X_2 \cup \cdots \cup X_p$, i.e. graph $\overline{G}$ is obtained in the same way and two vertices (x, t_j), (y, t_{j+1}) are connected if there exists a control $u(t)$ such that $y = g_t(x(t), u(t))$. In addition in $\overline{G}$ we add also edge $((x,t), \overline{z})$ for $t = T_1, T_1 + 1, \ldots, T_2$ if for a given state $x = x(t)$ there exists the control $u(t)$ such that $x_f = g_t(x(t), u(t))$.

3 Conclusion

The results described in this paper allows us to conclude that the game-theoretical concept for single-objective control problems with varying time of states' transitions of the dynamical system can be applied in the same way as for the problem with constant unit time of states' transitions. Moreover, the algorithm based on the dynamic programming technique for determining the optimal strategies of the players for multiobjective control problems (stationary and nonstationary cases) can be grounded in an analogous way as for the problem with unit time of states' transitions.

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