An Extension of Fuzzy Set Theory Encompassing Inconsistency and Paracompleteness

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Abstract
Fuzzy Set Theory and Paraconsistent Annotated Logics are subjects much researched nowadays, with an immense number of applications in a variety of themes. In this paper we discuss how to extend the Fuzzy set theory in order to deal with inconsistent and/or paracomplete data. For the task we use a special kind of paraconsistent annotated logic and the resulting theory, the paraconsistent annotated set theory.
Keywords: Fuzzy set theory, paraconsistent annotated logic, inconsistent data, paracomplete data.

1 Introduction

Fuzzy set theory has provided in a most important theory to deal with imprecision data without using statistical or probabilistic methods. However, more and more inconsistent and paracomplete data are becoming important to deal with more general and formal tools. In this sense, in this paper we propose an extension of fuzzy sets to encompass inconsistent and paracomplete data in a 'natural' way. In order to accomplish these ideas, we lean on a new class of non-classical logic, namely the paraconsistent annotated evidential logic Eτ. This paper extends ideas studied in (Abe 1992).

2 Paraconsistent Annotated Set Theory Q$\tau\in$

In this paragraph we present a particular form of annotated set theory (among several possibilities). This is made trough some set structures called 'normal structures'. The most convenient to develop this is in some formal set theory, such as ZF (Zermelo-Fraenkel-Skolem) theory.

Let us consider the annotated logic $Q\tau\in$ which is a family of first-order logics, called annotated set first-order predicate calculi. For a detailed account for annotated logics, see (Abe 1992). They are defined as follows: throughout this work, $\tau = <|\tau|, \leq,$

~> will be some arbitrary, but fixed, finite lattice of truth values with operator. The least element of τ is denoted by $\perp$, while its greatest element by $\top$. We also assume that there is a fixed unary operator $\sim: |\tau| \to |\tau|$ which constitutes the "meaning" of our negation. $\vee$ and $\wedge$ denote, respectively, the least upper bound and the greatest lower bound operators (of τ) .

The language L of $Q\tau\in$ is a first-order language whose primitive symbols are the following:

1. Individual variables: a denumerably infinite set of variable symbols.

2. Logical connectives: $\neg$, (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and $\to$ (conditional).

3. The predicate symbol $\in$.

4. Quantifiers: $\forall$ (for all) and $\exists$ (there exists).

5. Annotated constants: each member of τ is called an annotational constant.

6. Auxiliary symbols: parentheses and commas.

We observe that the atomic formulas have the form $\in_\lambda(t_1, t_2)$, where t_1, t_2 are terms and λ is an annotational constant. We write $t_1 \in_\lambda t_2$ instead of $\in_\lambda(t_1, t_2)$, and it is read: t_1 belongs to t_2 with degree λ. Intuitively, $\in$ is the membership symbol and λ is 'degree' of membership.

Definition 1. Let X be a non-empty set. A normal structure on X is any function f: $X \times X \to \tau$.

A normal structure is basically an interpretation of $\in$.

Let's introduce the Annotated Set Theory. Let us add to ZF two individual constants U and τ and the following axiom schemata:

(A_1) $(\forall X)(X \in U \to X \in U)$

(A_2) $\tau \in U$ and τ is a lattice.

The scheme (A_1) implies that U is a transitive set.

Definition 2. A function e: $U \times U \to \tau$ is called a normal structure on U. We write $x\ \mathrm{e}_\lambda\ y$ instead $\mathrm{e}(x, y) = \lambda$.

Definition 3. If X, $Y \in U$, then

$X^{(\lambda,\, \mathrm{e})} = \{Y: Y \in U \wedge Y\, \mathrm{e}_\lambda X\}$

$X^{[\lambda,\, \mathrm{e}]} = \{Y: Y \in U \wedge (\exists\mu)(\mu \in \tau \wedge \mu \leq \lambda \wedge (Y\, \mathrm{e}_\lambda X)\}$

$\mathrm{F}_X = \{f: f{:}X \to \tau \wedge (\exists \mathrm{e})(\mathrm{e}$ is a normal structure $\wedge\ (\forall\lambda)(\forall Y)((\lambda \in \tau \wedge Y \in U) \to (f(Y) = \lambda \leftrightarrow (Y\, \mathrm{e}_\lambda X)))\}$

If $\lambda \in \tau$, $X =_\lambda^\theta Y =_{\mathrm{Def.}} (\forall Z)((Z \in U) \to (Z\, \mathrm{e}_\lambda X \leftrightarrow Z\, \mathrm{e}_\lambda Y))$.

Definition 4. A non-empty set X is called strongly transitive if X is transitive and $(\forall Y)(Y \in X \to P(Y) \in X)$ (where $P(Y)$ indicates the power set of Y). A set X is called universe if X is strongly transitive and for each function $f{:}Y \to X$ such that $Y \in X$, we have $\cup\mathrm{Im}(f) \in X$, where $\mathrm{Im}(f) = \{z: (\exists t)\ (t \in Y \wedge f(t) = z)\}$. A universe U is called complete, if $\{0, 1, 2, \ldots\} \in X$.

Theorem 1. If X is a universe, then X is a model of ZF axioms with the possible exception of Infinity Axiom.

Theorem 2. X is a complete universe iff there is an infinite set A such that $A \in X$.

Theorem 3. Let X be a universe, $\lambda \in \tau$ and $X \in U$. Then

1. $X^{(\lambda, e)} \in U$
2. $X^{[\lambda, e]} \in U$
3. $F_X \in U$
4. $\{Y\text{: } F(Y) \wedge Y \mathrm{e}_\lambda X\} \in U$, where F is any formula of ZF.

Theorem 4. $(\forall X)(\forall Y)((X, Y \in U \rightarrow (\forall e)(\forall \lambda)((e$ is a normal structure and $\lambda \in \tau) \rightarrow X =_\lambda^\theta Y) \leftrightarrow X = Y)$.

Definition 5. Let U be a set. A fuzzy set of U is a function u: $U \rightarrow [0, 1]$. By $F(U)$ we denote the set of all membership functions on U. For all $u, v \in F(U)$ and $x \in U$, we put

$(u \vee v)(x) = \sup\{u(x), v(x)\}$; $(u \wedge v)(x) = \inf\{u(x), v(x)\}$; $u'(x) = 1 - u(x)$.

Definition 6. Let $u, v \in F(U)$. We put $u = v \leftrightarrow (\forall x \in U)(u(x) = v(x))$. The distinguished elements 1_U and 0_U are respectively the membership functions

$1_U(x) = 1$; $0_U(x) = 0$ for all $x \in U$.

Theorem 5. The structure $[F(U), \wedge, \vee]$ constitutes a complete lattice with the infinite distributive property.

Theorem 6. The structure $[F(U), \wedge, \vee, ']$ constitutes an algebra, which in general it is not Boolean.

Theorem 7. Every fuzzy set $u \in F(U)$ can be identified with a normal structure v based on the set $U \cup [0, 1]$ such that $\tau = \{\perp, \top\}$ and

$$v(x, y) = \begin{cases} \top, & \text{if } x \in U \textit{ and } y \in [0, 1] \textit{ and } y = u(x) \\ \perp, & \text{if it is not the case} \end{cases}$$

Thus, the Theorem above show us that the normal structure theory encompasses the fuzzy set theory. If U is an universe, then the above definition of fuzzy set in terms of normal structures can be simplified.

3 Another Extension

In the sequel, we present briefly another version of annotated set theory, based on paraconsistent annotated logic $E\tau$ (Abe 1992). Let X be a non-empty set. Any function $f{:}X \rightarrow [0, 1]^2$ is called a (paraconsistent) membership function. An order relation is defined on $[0, 1]^2$: $(\mu_1, \lambda_1) \leq (\mu_2, \lambda_2) \Leftrightarrow \mu_1 \leq \mu_2$ and $\lambda_1 \leq \lambda_2$, constituting a lattice that will be symbolized by τ.

$f(x) = [\mu(x), \lambda(x)]$, $x \in X$.

The expression $f(x) = [\mu(x), \lambda(x)]$ can be intuitively read: "It is assumed that $(x \in f)$'s favorable evidence is $\mu(x)$ and contrary evidence is $\lambda(x)$." Thus,

- $f(x) = [1, 0]$ can be read $(x \in f)$ is a true proposition.
- $f(x) = [0, 1]$ can be read as $(x \in f)$ is a false proposition.
- $f(x) = [1, 1]$ can be read as $(x \in f)$ is an inconsistent proposition.
- $f(x) = [0, 0]$ can be read as $(x \in f)$ is a paracomplete (unknown) proposition.
- $f(x) = [0.5, 0.5]$ can be read as $(x \in f)$ is an indefinite proposition.

We observe that with paraconsistent membership function we can represent inconsistent and/or paracomplete membership. Moreover, fuzziness and all degrees tending to inconsistency and paracomplete, as well as to true and false concepts can be introduced.

Usual operations as union (maximization), intersection (minimization), and complementation can be introduced. Following ideas of the logic Eτ, the complement of the set f is $f':X \rightarrow [0, 1]^2$ defined as

$f'(x) = [\lambda(x), \mu(x)], x \in X.$

Given the sets $f:X \rightarrow [0, 1]^2$ and $g:X \rightarrow [0, 1]^2$ we define

Union: $(f' \cup g)(x) = [\mu_1(x), \lambda_1(x)] \cup [\mu_2(x), \lambda_2(x)] = [max\{\mu_1(x), \mu_2(x)\}, max\{\lambda_1(x), \lambda_2(x)\}$.

Intersection: $(f \cap g)(x) = [\mu_1(x), \lambda_1(x)] \cap [\mu_2(x), \lambda_2(x)] = [min\{\mu_1(x), \mu_2(x)\}, min\{\lambda_1(x), \lambda_2(x)\}$.

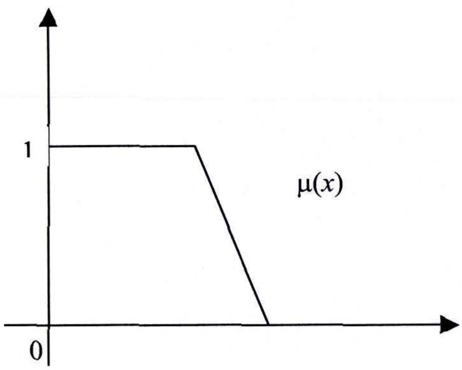

Figure 1: Membership function of favorable evidence

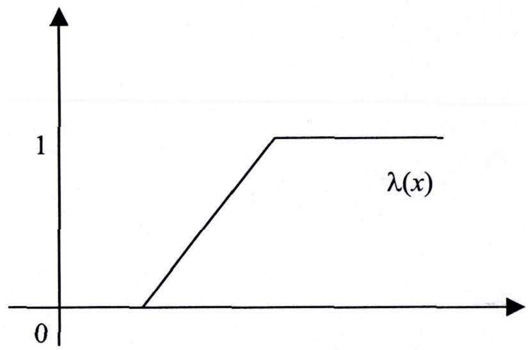

Figure 2: Membership function of contrary evidence

Depending on the application, we can also define the complement as $f'(x) = [1 - \mu(x), 1 - \lambda(x)], x \in X$; other definitions are possible.

Also we can define suitable inference rules for each case, as well as a correspondent concept for 'defuzzification' of Fuzzy Set for annotated set theory.

Figure 3 displays an example of annotated set. Its projection on plane of favorable evidence gives the membership function of favorable evidence, while its projection on plane of contrary evidence gives the membership function of contrary evidence.

It is easy that this treatment encompasses the ordinary Fuzzy Set Theory, constituting an interesting generalization of it.

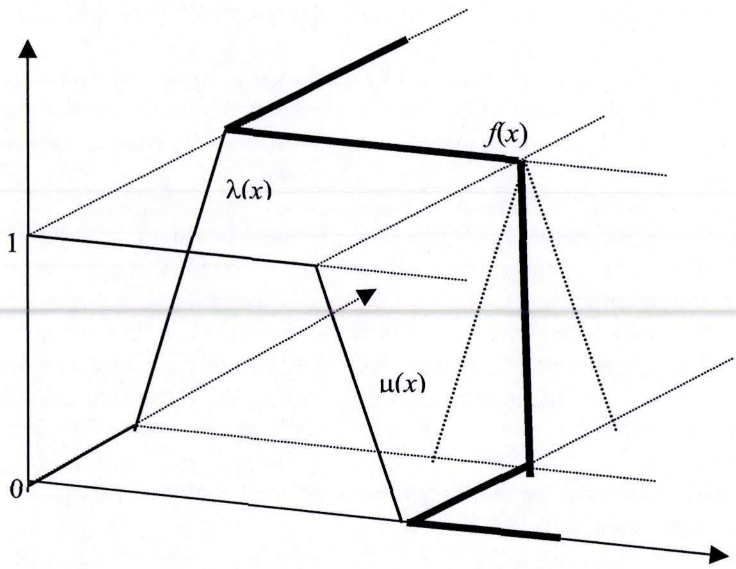

Figure 3: Membership function of annotated set theory.

4 Conclusions

The ideas presented in this paper can be adapted to other set theories. For instance, the theory of flou sets and L-sets (Negoita & Ralescu 1975) can be simplified as particular cases of the general theory of normal structures when we extend a little, the concept of normal structures. Also the rough sets can be viewed as particular case of normal structures (with adaptations). Annotated set theory is capable to represent fuzzy, inconsistent and paracomplete concepts which are more and more common in applications nowadays, constituting in an interesting alternative theory with own merit.

Acknowledgements
The authors are grateful for the opportunity in presenting these ideas in the Invited Session "Non-classical Logics and Decision Making Techniques", CASYS'07.

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