An Algebraic Description of Development of Hierarchy

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Abstract
We propose an algebraic description of emergence of new levels in trophic level networks. Trophic level networks are described by directed graphs. Their properties are surveyed in terms of an adjunction on a subcategory of the category of directed graphs. In particular, it is shown that trophic level networks are invariant under the composition of the right adjoint functor and the left adjoint functor. This invariance of trophic level networks can be broken by introducing the notion of time into the left adjoint functor. This leads to changes in trophic level networks. We show that the left adjoint functor consists of an intra-level process and an inter-level process. An inconsistency between them arises by the introduction of time. Negotiation between the intra-level process and the inter-level process can resolve the inconsistency at a level, however, a new inconsistency can arises at an emerged new level. Thus our algebraic description can follow indefinite development of trophic hierarchy.
Keywords : inconsistency, negotiation, hierarchy, ecosystem, trophic level.

1 Introduction

Ecosystems consist of biotic communities, abiotic factors and interrelationships between them. Interrelationships in an ecosystem are often characterized by energy flows between taxa [19, 20]. In particular, hierarchical nature of an ecosystem can be revealed by focusing on a trophic level network [19]. As an ecosystem develops, new trophic levels emerges from the existing trophic level network. Statistical physicists often define emergence of new trophic levels by a stochastic process [2]. However, the purpose of this paper is providing an algebraic description of such emergence of hierarchy.

Trophic level networks can be described by directed graphs as other many biological or social networks can be [6, 7, 22]. The directed graph representation primarily emphasizes the timeless structure of a network, on which certain dynamics of energy flows occurs. It is convenient to introduce a framework in which common properties of directed graphs structure can be investigated. Category theory [8] provides such a framework. In section 2 we work with the category of directed graphs in order to survey algebraic properties of trophic level networks. However, changes in trophic level

International Journal of Computing Anticipatory Systems, Volume 20, 2008
Edited by D. M. Dubois, CHAOS, Liège, Belgium, ISSN 1373-5411 ISBN 2-930396-07-5

networks cannot be treated by focusing on only its timeless structure. The implicit assumption of the categorical treatment of directed graphs is globally controlled synchronization of interrelationships between energy flows. This is unrealistic since all physically realizable interactions take finite time [9]. Interrelationships between energy flows undergo not a global control but local regulations [10]. The notion of time is needed in order to address local regulations of energy flows.

The introduction of time into directed graph framework leads to a distinction between an intra-level process and an inter-level process. Since any consistency between the two processes are not guaranteed a priori, an inconsistency can arise. Negotiation between the intra-level process and the inter-level process toward a consistency attempts to remove inconsistency. However, negotiation itself can generate a new inconsistency by its local character [3, 4, 9]. Thus trophic level networks can undergo changes indefinitely. Since any concrete change in a system occurs under some constraints, what constraints are available in trophic level networks should be addressed [11, 18]. We will show that under an appropriate realistic constraint we can follow a development of trophic hierarchy by our algebraic setting.

This paper is organized as follows. In section 2 we review a categorical treatment of directed graphs in terms of an adjunction [5]. Trophic level networks are defined as directed graphs and their responses to the adjunction are concerned. In section 3 the notion of time is introduced in order to address changes in trophic level networks. In section 4 we discuss how emergence of new trophic levels can be described by our algebraic formalism. Finally we give conclusions in section 5.

2 Duality between Decomposition and Gluing

Organizations of biological or social systems are often described by graphs. Components (e.g. proteins, genes, metabolites, individuals, populations and so on) in a system are usually represented by nodes and interactions between components are represented by arrows (for example, [6, 7, 22]). Meanwhile there are in general multiple biological components for a single interaction, biological or social networks are indeed hypergraphs in which an arrow can connect more than two nodes. Therefore it is a matter how to derive appropriate graphs from real hypergraphs [1]. However, we here limit ourselves to discussing usual directed graphs since our primary concern in this paper is trophic level networks that can be described by directed graphs.

A trophic level network can be described by a directed graph. Each node represents a trophic level. A trophic level is defined as distance from producers (i.e. plants) in a ecosystem [19]. The least level consists of plants. Harbivores belong to the second level. They are the primary consumers. Carnivores that eat harbivores belong to the third level (the secondary consumers). Note that this definition is a functional definition. An individual organism can belong to multiple trophic levels. For example, omnivores belong to more than one trophic level. We put arrows between contiguous levels. The direction of an arrow is from lower to upper level which

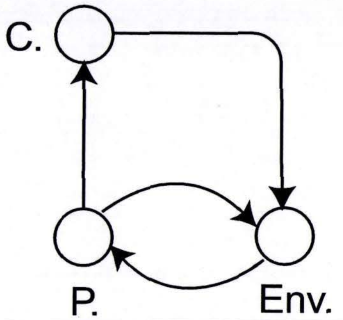

Fig. 1: A trophic level network consisting of two levels, producers (P.) and the primary consumers (C.). Each level has an energy flow to environment (Env.).

indicates energy flow. We add another node to the trophic level network that represents environment. Environment includes not only external factors for the ecosystem such as sun light, air, water, soil and other ecosystems but also the detritus food chain in the ecosystem. Hence decomposers belong to environment. Every level has an arrow to environment. There exists an arrow from environment to producers that indicates assimilation of energy from environment. Figure 1 shows a trophic level network consisting of producers, the primary consumers and environment.

Directed graphs are formally defined as follows. A quadruplet $G = (A, O, \partial_0, \partial_1)$ is called directed graph. A is a set of arrows. O is a set of nodes. $\partial_i (i = 0, 1)$ are maps from A to O. ∂_0 sends each arrow to its source. ∂_1 sends each arrow to its target. The category of directed graphs $\mathcal{G}rph$ is defined as a category with its objects are directed graphs. The morphisms in $Grph$ are homomorphisms of directed graphs. A homomorphism D between directed graphs $G = (A, O, \partial_0, \partial_1)$ and $G' = (A', O', \partial'_0, \partial'_1)$ is a pair of maps D_A and D_O. D_A is a map from A to A' and D_O is a map from O to O'. They must satisfy the equations $D_O \partial_i = \partial'_i D_A (i = 0, 1)$. That is, homomorphisms of directed graphs are mappings that preserve both sources and targets.

Each node in a directed graph has functions that connect one arrow to another arrow. For example, producers assimilate energy from environment. A part of them are transferred to the primary consumers by their feeding and the remaining parts go back to environment. These two flows are connected to the flow from environment to producers at producers. In order to analyze these functions it is convenient to consider an operation that decompose a node into its functions of connecting arrows [12, 16, 17]. This operation can be defined as a functor from the category of directed graphs $\mathcal{G}rph$ to itself.

The operation of decomposition $\mathcal{R}$ is defined as an operation that transforms given directed graph $G = (A, O, \partial_0, \partial_1)$ into a new directed graph

$$\mathcal{R}G = (\mathcal{R}A, \mathcal{R}O, \partial_0^{\mathcal{R}}, \partial_1^{\mathcal{R}}),$$

where

$$\begin{aligned} \mathcal{R}A &= \{(f,g) \in A \times A | \partial_1 f = \partial_0 g\} \\ \mathcal{R}O &= A \\ \partial_0^{\mathcal{R}}(f,g) &= f \;\; \partial_1^{\mathcal{R}}(f,g) = g \text{ for } (f,g) \in \mathcal{R}A. \end{aligned}$$

$\mathcal{R}$ is a functor from $\mathcal{G}rph$ to itself.

Each node of a directed graph can be reconstructed by gluing its functions [12]. This operation of gluing can be also defined by a functor. However, a gluing functor cannot be defined on the category of directed graphs $\mathcal{G}rph$. We must concern the operation of gluing on a subcategory of $\mathcal{G}rph$ on which it becomes a functor. A subcategory $\mathcal{H}$ of $\mathcal{G}rph$ is defined as follows. Each object is a directed graph that satisfies the condition that for all $x \in O$ there exist $f, g \in A$ such that $\partial_1 f = x = \partial_0 g$. That is, there exists an incoming arrow and an outgoing arrow for any node. Morphisms of $\mathcal{H}$ are homomorphisms of directed graphs. The gluing operation $\mathcal{L}$ that transforms a directed graph $G = (A, O, \partial_0, \partial_1)$ to a new directed graph $\mathcal{L}G = (\mathcal{L}A, \mathcal{L}O, \partial_0^{\mathcal{L}}, \partial_1^{\mathcal{L}})$ defined as follows is a functor from $\mathcal{H}$ to itself.

$$\begin{aligned} \mathcal{L}A &= O \\ \mathcal{L}O &= T/\sim \\ T &= \{(x,y) \in O \times O | \exists f \in A \; \partial_0 f = x, \partial_1 f = y\} \end{aligned}$$

$\sim$ is an equivalence relation generated by a relation R on T defined by

$$(x,y)R(z,w) \Leftrightarrow x = z \text{ or } y = w.$$

$\sim$ is the transitive closure of R. That is, $(x,y) \sim (z,w)$ holds if and only if there exist $t_1, t_2, \cdots, t_n \in T$ such that $(x,y) = t_1 R t_2 R \cdots R t_n = (z,w)$. The relation R implies that two arrows are glued if they have a common source or target when G is a binary graph (a directed graph containing at most one arrow between each ordered pair of nodes). Source and target maps are defined as follows.

$$\partial_0^{\mathcal{L}} x = [(\partial_0 f, \partial_1 f)]_\sim \;\; \partial_1^{\mathcal{L}} x = [(\partial_0 g, \partial_1 g)]_\sim$$

where $\partial_1 f = x, \partial_0 g = x, \; f, g \in A$ and $[\alpha]_\sim$ is an equivalence class that includes α. It is proved that $\mathcal{H}$ is the largest subcategory of $\mathcal{G}rph$ on which $\mathcal{L}$ becomes a functor. Note that all directed graphs that represent trophic level networks are in $\mathcal{H}$. The functor $\mathcal{L}$ can be extended to a functor on $\mathcal{G}rph$ by an appropriate modification, however, we do not concern this aspect in this paper since it is enough to work on the category $\mathcal{H}$ to discuss emergence of new levels in trophic level networks.

The functors $\mathcal{R}$ and $\mathcal{L}$ constitute a special kind of duality, called an adjunction.

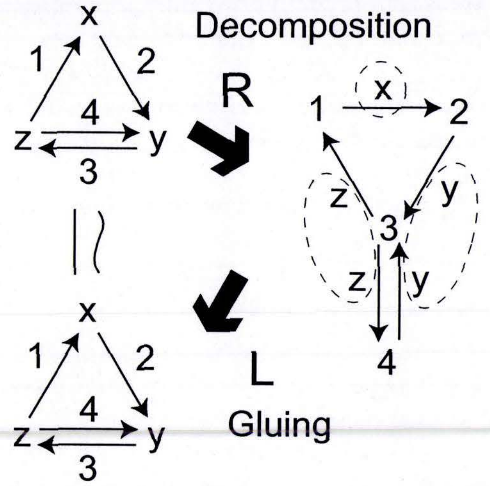

Fig. 2: The trophic level network in Figure 1 is decomposed into functions of connecting arrows (from left to right) by the functor $\mathcal{R}$. The network is reconstructed by gluing the decomposed functions (from right to left), the functor $\mathcal{L}$.

Theorem 1. $\mathcal{L}$ is a left adjoint to $\mathcal{R}$. That is, we have a natural isomorphism

$$\mathcal{H}(\mathcal{L}G, G') \cong \mathcal{H}(G, \mathcal{R}G')$$

for any pair of directed graphs G, G', where $\mathcal{H}(G_1, G_2)$ for directed graphs G_1, G_2 is the set of all morphisms from G_1 to G_2 in $\mathcal{H}$.

The proof is given in [5]. See also [14].

By analyzing the adjunction, one can find that the counit $\eta : \mathcal{LR} \to \mathcal{I}$ of the adjunction is a natural isomorphism, where $\mathcal{I}$ is the identity functor on $\mathcal{H}$, which sends each directed graph to itself [5]. That is, for any directed graph G in $\mathcal{H}$ we have a directed graph isomorphism $\mathcal{LR}G \cong G$. This means that a directed graph can be fully reconstructed from information about its functions of nodes that connect arrows. Note that the unit $\epsilon : \mathcal{I} \to \mathcal{RL}$ is not a natural isomorphism. However, we can find a necessary and sufficient condition for $G \cong \mathcal{RL}G$ for a directed graph G. Note also that when $\mathcal{L}$ is extended to $\mathcal{G}rph$, the counit is not also a natural isomorphism. See [5] for full explanations of these issues. After all the fact that we need in this paper is that $\mathcal{LR}G \cong G$ holds for any directed graph G in $\mathcal{H}$.

Figure 2 shows how the trophic level network in Figure 1 is recovered by $\mathcal{LR}$. Here producers, the primary consumers and environment are represented by the nodes labeled z, x and y, respectively. Note that both operations of decomposition $\mathcal{R}$ and $\mathcal{L}$ operate on the whole network simultaneously. The globally controlled synchronization of all the parts of the network is implicitly assumed. This is because they are defined as a mathematical operation, functor. There is no change in a trophic level network as long as the implicit globally controlled synchronization is not removed. However, real trophic level networks do not work in this manner. Synchronization of parts must be achieved by local regulations since it takes a finite duration for parts to interact with each other [9]. In order to formalize local regulations for synchronization between parts, we here focus on the gluing operation. In particular, we introduce the notion of time into the gluing operation. Then the gluing operation becomes a gluing process. In the next section we examine how time can be introduced into the gluing operation.

3 Introduction of Time into Gluing Operation

In the following we assume that we are working with binary graphs when we discuss the gluing operation $\mathcal{L}$. Binary graphs are special directed graphs in which there is at most one arrow between an ordered pair of nodes. Note that $\mathcal{R}G$ is always a binary graph for any directed graph G. The gluing operation $\mathcal{L}$ consists of two operations. The first one is gluing arrows by taking the transitive closure of a relation defined on the set of arrows. This operation constructs a set of nodes in a directed graph. A new node is a set of old arrows. The second operation is making new arrows between new nodes. Mathematically this corresponds to defining a source and target maps $\partial_0^{\mathcal{L}}, \partial_1^{\mathcal{L}}$.

Let us suppose these two operations take finite time to complete their works. When we take into account time explicitly we do not say 'operation' but 'process'. Timeless gluing operation $\mathcal{L}$ becomes a gluing process. The gluing process is also denoted by $\mathcal{L}$ as the gluing operation is. What is the difference between the gluing operation and the gluing process? The gluing operation always reconstructs a given directed graph G in $\mathcal{H}$ from $\mathcal{R}G$, which contains only information about how each node connects arrows in $\mathcal{G}$ due to implicit global synchronization of the whole. On the other hand, the gluing process does not necessarily reconstruct the original directed graph. Suppose that the process of gluing arrows takes d_1 to complete its task and the process of making new arrows between new nodes takes d_2 to complete its task in an arbitrary time unit. If $d_1 \leq d_2$ holds then the process of gluing arrows always finishes before the process of making new arrows between new nodes does. Hence the original directed graph can be reconstructed by the gluing process. On the other hand, if $d_1 > d_2$ then the process of gluing arrows cannot finish before the process of making new arrows between new nodes finishes. Some of old arrows to be glued remain to be unglued. This implies that a node in the original directed graph G are broken up into multiple nodes in $\mathcal{LR}G$, where $\mathcal{L}$ is the gluing process. Note that the durations d_1, d_2 just introduced are virtual durations. They are not durations measured in real time. They are defined in order to represent a kind of logical inconsistency in the gluing process.

How can d_1 and d_2 be estimated? Without loss of generality, we can assume that d_2 is a constant since only ratio d_1/d_2 is the matter. Let G be a directed graph in $\mathcal{H}$. The process of gluing arrows in G are defined by the transitive closure of the binary relation R on T. R and T are the same as those defined in section 2. We assume that d_1 correlates with a cost of the process of gluing arrows. If the same computational process is repeated with serial manner and durations between processes can be ignored then time needed to finish all the processes is simply (the number of repeat) × (time needed to finish a single process). In such a case, the cost of a computational process can be evaluated by the number of computational steps. The transitive closure of a binary relation is an example of this case. Thus we define d_1 by a increasing function of m_G, the maximum of the least number of transition in the transitive closure of R. That is,

$$m_G = \max_{(x,y)\sim(z,w)\in T} \{\min\{n-1 | (x,y) = t_1 R t_2 R \cdots R t_n = (z,w)\}\}.$$

For a finite directed graph, it is clear that m_G is always finite. If there exists a directed graph G' in $\mathcal{H}$ such that $G = \mathcal{R}G'$ for a directed graph G then the following claim holds.

Theorem 2. $m_{\mathcal{R}G} \leq 2$ for any directed graph G in $\mathcal{H}$.

Proof. Put $G = (A, O, \partial_0, \partial_1)$. By the definitions of $\mathcal{R}$ and $\mathcal{L}$,

$$\begin{aligned} \mathcal{R}G &= (T, A, \partial_0^{\mathcal{R}}, \partial_1^{\mathcal{R}}), \\ \mathcal{L}\mathcal{R}G &= (A, T/\sim, \partial_0^{\mathcal{L}\mathcal{R}}, \partial_1^{\mathcal{L}\mathcal{R}}), \end{aligned}$$

where $T = \{(f,g) \in A \times A | \partial_1 f = \partial_0 g\}$, $\sim$ is the transitive closure of the binary relation R on T defined by $(f,g)R(k,h) \Leftrightarrow f = k$ or $g = h$. Define two auxiliary relations R_l and R_r on T by

$$(f,g)R_l(k,h) \Leftrightarrow f = k, \quad (f,g)R_r(k,h) \Leftrightarrow g = h.$$

It is clear that $R_l \circ R_l = R_l$ and $R_r \circ R_r = R_r$. If $(s_1,t_1)R_l(s_2,t_2)R_r(s_3,t_3)R_l(s_4,t_4)$ then

$$\partial_0 t_1 = \partial_1 s_1 = \partial_1 s_2 = \partial_0 t_2 = \partial_0 t_3 = \partial_1 s_3 = \partial_1 s_4 = \partial_0 t_4,$$

since we have $s_1 = s_2, t_2 = t_3, s_3 = s_4$ and $(s_i, t_i) \in T$ for $i = 1,2,3,4$. Hence $(s_1, t_4) \in T$ holds. It follows that $R_l \circ R_r \circ R_l \subset R_l \circ R_r$ since $(s_1,t_1)R_l(s_1,t_4)R_r(s_4,t_4)$. We can show that $R_r \circ R_l \circ R_r \subset R_r \circ R_l$ by the same way. Thus we obtain $\sim = R_r \cup R_l \cup (R_l \circ R_r) \cup (R_r \circ R_l) = R \cup (R \circ R)$ since $R = R_l \cup R_r$. The claim follows immediately. ∎

When does the equality $m_{\mathcal{R}G} = 2$ hold? Suppose there exists a node that have two incoming arrows and two outgoing arrows in a directed graph G (Figure 3 (a),

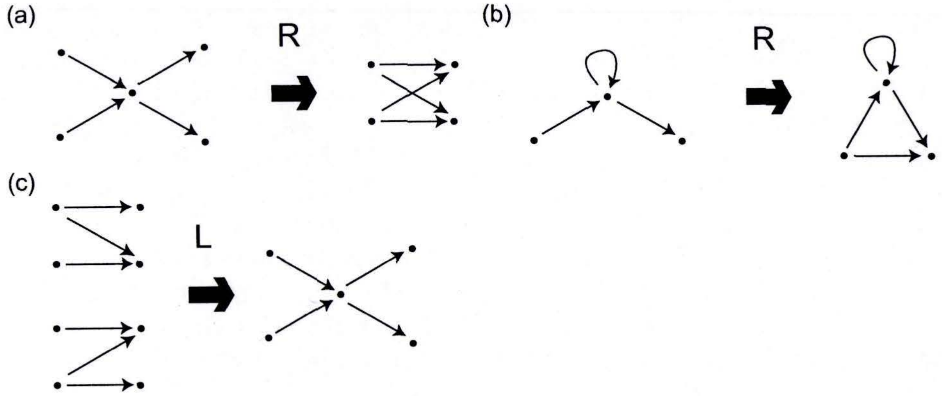

Fig. 3: (a) If there are two incoming arrows and two outgoing arrows for a node then two transitions of the relation R are necessary in order to glue all arrows that can be glued in $\mathcal{R}G$. (b) Either one of the two incoming arrows and either one of the two outgoing arrows can be the same in (a). (c) $\mathcal{L}$ sends both graphs at the left to the same graph.

left hand side). In $\mathcal{R}G$, the node is decomposed into four arrows (Figure 3 (a), right hand side). Two of them are drawn in parallel and the other two are crossed in Figure 3. In order to glue the two parallel arrows we need two transitions of the relation R. Thus if a directed graph G in $\mathcal{H}$ contains such a node it follows that $m_{\mathcal{R}G} = 2$. This is ture when either one of the two incoming arrows and either one of the two outgoing arrows are the same (i.e. when a loop is attached to the node with the other incoming and outgoing arrows, Figure 3 (b)). On the contrary, if $m_{\mathcal{R}G} = 2$ holds then $\mathcal{R}G$ must contain one of the two subgraphs shown in Figure 3 (c) (left hand side). All three arrows must be distinct although nodes can be degenerated in each case. If the three arrows in each case are glued by the gluing operation $\mathcal{L}$ then the corresponding new node has two incoming arrows and two outgoing arrows (Figure 3 (c), right hand side). Since $\mathcal{L}\mathcal{R}G \cong G$ for a directed graph G in $\mathcal{H}$, G must contain a node with two incoming arrows and two outgoing arrows. Since there is no such node in any trophic level network, we have $m_{\mathcal{R}G} < 2$ for any directed graph G that represents a trophic level network defined in section 2.

In trophic level networks the process of gluing arrows is an intra-level process and the process of making arrows between nodes is an inter-level process. The inequality $d_1 > d_2$ suggests the existence of an inconsistency between the intra-level process and the inter-level process. In the next section we concern how such an inconsistency arises, can be resolved and leads to development of trophic hierarchy.

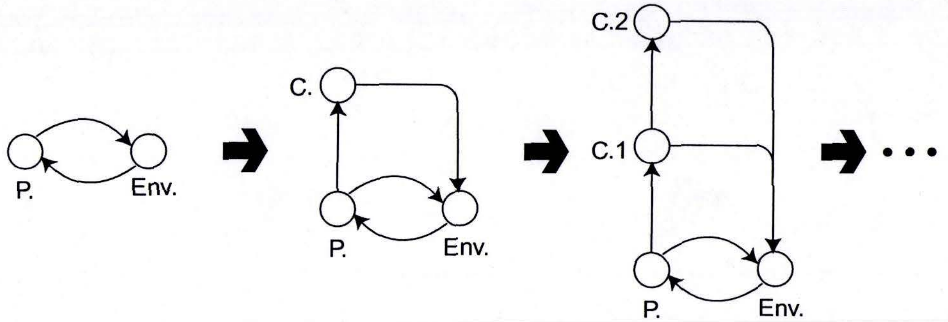

Fig. 4: Development of trophic hierarchy.

4 Emergence of New Levels

Development of ecosystems can encompass appearances of new trophic levels (Figure 4). On the primordial earth there exist trophic level networks consisting of only producers and environment. The producers are prokaryotes. The appearance of eukaryotes indicates the invention of predation, which implies an appearance of a new trophic level, the primary consumers. The secondary consumers come into being along with the organic evolution. In principle, this process of emergence of new levels continues indefinitely. If a geological isolation of a ecosystem is dissolved then a new trophic level can emerge caused by a exogenous factor such as immigration. However, all appearances of new levels cannot be caused by only exogenous factors. For example, the first appearance of the primary consumers in a ecosystem in Precambrian age must have endogenous factors since no exogenous ecological factor is imaginable at this case. Endogenous factors for emergence of new levels are more fundamental than exogenous ones since they work without any exogenous factor.

Without exogenous causes, any new trophic level must be latent in the existing trophic level networks. How can such latency be represented in directed graphs? Since arrows in a trophic level network represents energy flows between nodes (trophic levels and environment), the representation of latency should also be considered in terms of energy flows. It seems that this can be done by adding a loop to a trophic level that represents an intra-level energy flow. On the other hand, by the functional definition of the trophic level, a new trophic level emerges from only the highest level. for example, an appearance of new species of plant does not change the existing trophic level network. Therefore we here suggest that the existence of a latent new level can be represented by adding a loop to the highest level. If a directed graph G in $\mathcal{H}$ that represents a trophic level network is modified in such a way then the value of $m_{\mathcal{R}G}$ changes from 0 or 1 to 2. This leads to increase in d_1 the time needed to complete the process of gluing arrows in $\mathcal{R}G$.

The increase in d_1 results in two different situations. The first situation is un-

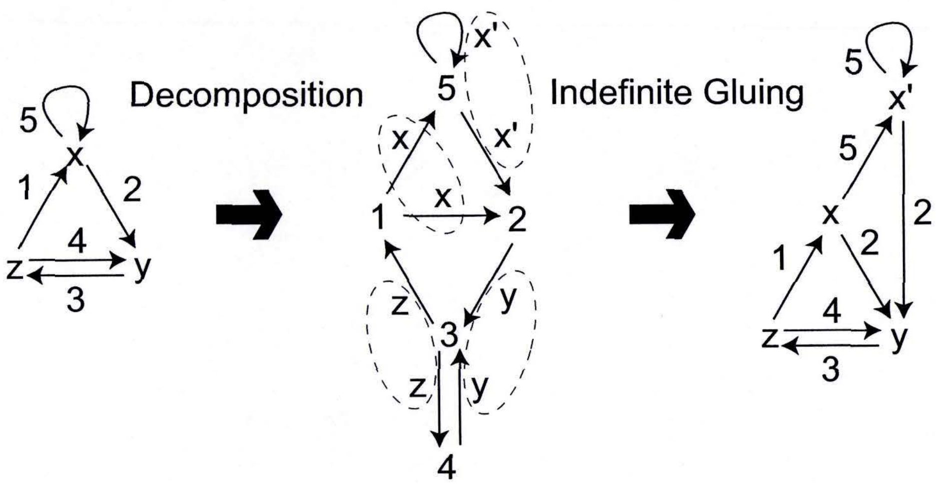

Fig. 5: The loop at the highest level in the trophic level network at the left hand side indicates a new level is latent in the existing trophic level network. The latent new level emerges explicitly by indefinite gluing.

interesting case, the inequality $d_1 \leq d_2$ remains to be held. In this case the gluing process cannot be distinguished from the gluing operation, hence there is no change in the trophic level hierarchy. In the second situation, the inequality $d_1 \leq d_2$ is broken. An inconsistency between the intra-level process of gluing arrows in $\mathcal{R}G$ to make nodes in $\mathcal{LR}G$ and the inter-level process of making arrows in $\mathcal{LR}G$ arises. Since the former is slower than the latter, arrows in $\mathcal{R}G$ to be glued remain not to be glued. A node in G must be broken up into multiple nodes in $\mathcal{LR}G$ if its function of connecting arrows cannot be fully glued. This is a negotiation between the intra-level process and the inter-level process in order to retain logical consistency. We call such a gluing process the indefinite gluing. There are several possibilities in the shape of the resultant directed graph $\mathcal{LR}G$ without any constraint. However, we here imposes a constraint that restrict the possibilities. Since we are interested in development of trophic hierarchy, we assume that the resultant directed graph has a property that all trophic networks have. It is a constraint that any trophic level has a energy flow to environment. In fact we can show that just one possibility is acceptable by this constraint and the resultant directed graph represents a trophic level network in which the number of trophic levels increases by one with a loop at the highest level. We see this by an example in the following.

Let G be a trophic level network that consists of two levels. We assume that without a loop at the highest level $G \cong \mathcal{LR}G$ holds where $\mathcal{L}$ is the gluing process. In this case we have $m_{\mathcal{R}G} = 1$. Hence we also assume that when a loop exists at the highest level (Figure 5, left hand side), the gluing of arrows with a single transition of the relation R on the set T of arrows in $\mathcal{R}G$ can finish before the process of making arrows between nodes in $\mathcal{LR}G$. Thus two pairs of arrows $\{3 \to 1,\ 3 \to 4\}$ and $\{2 \to 3,\ 4 \to 3\}$ in $\mathcal{R}G$ can be glued respectively (Figure 5, center). Suppose two arrows $5 \to 2$ and $1 \to 2$ are glued. $1 \to 5$ cannot be glued to them since it needs two transition of R to glue $1 \to 5$ to $5 \to 2$. However, a node in $\mathcal{LR}G$ (which is an equivalence class of arrows in $\mathcal{R}G$) containing $1 \to 5$ have no arrows to environment (node 2 represents energy flows to environment). This is impossible because of the assumed constraint. Hence $5 \to 2$ and $1 \to 2$ belong to different equivalence classes. $1 \to 5$ cannot be glued to $5 \to 2$ since the gluing needs two transition of R via $5 \to 5$. If $1 \to 5$ is not glued to $1 \to 2$ then the equivalence class containing $1 \to 5$ does not have any arrow to environment. Hence $1 \to 5$ must be glued to $1 \to 2$. By the similar argument, $5 \to 5$ is glued to $5 \to 2$. Thus we obtain the set of nodes in $\mathcal{LR}G$ as $\{x, x', y, z\}$ where $x = \{1 \to 2,\ 1 \to 5\}, x' = \{5 \to 2,\ 5 \to 5\}$, $y = \{2 \to 3,\ 4 \to 3\}$ and $z = \{3 \to 1,\ 3 \to 4\}$. Arrows in $\mathcal{LR}G$ are defined as follows. Let a, b are nodes in $\mathcal{LR}G$. We put an arrow from a to b if there exists $i \to j$ in a and $k \to l$ in b such that $j = k$. This is consistent with the definition in the gluing operation $\mathcal{L}$. The resultant directed graph is a trophic level network with a loop at the highest level (Figure 5, right hand side). The number of levels in $\mathcal{LR}G$ increases by one from that in G. Finally note that the process of development of trophic hierarchy described above can continue indefinitely from the simplest network that consists of only producers and environment.

5 Conclusions

In this paper we provided an algebraic description of emergence of trophic hierarchy. Introduction of time into functors gives rise to an inconsistency between an intra-level process and an inter-level process. Negotiation between the intra-level process and the inter-level process under a realistic constraint leads to an appearance of a new level. A new level is latent at the highest level even after the inconsistency between the intra-level process and the inter-level process is resolved at a lower level. Hence development of trophic hierarchy can continue indefinitely in principle.

In real ecosystems, the number of trophic levels is limited by constraints such as the history of community organization, resource availability, the type of predator-prey interactions, disturbance and ecosystem size [13]. Extinctions can decrease the number of trophic levels [15]. Non-trophic effects are important to understand trophic relationships [21]. In our algebraic framework, these issues are included in the question that how the inconsistency between the intra-level process and the inter-level process arises. In this paper we did not treat this problem and only provided a description that what happens if the inconsistency arises.

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