

Study on the Wave Nature of Mass

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Abstract

In the wave-particle duality, a free particle can be considered as a wave packet. Is there a wave property that corresponds to the rest mass of the particle? We suggest that this problem can be approached by treating the rest mass on the same footing as energy and momentum. Here we demonstrate that, by assuming that the matter wave of a particle is an excitation of a real physical field in the vacuum, one could derive the mass-energy relation from the solution of a simplified wave equation describing a free particle. This solution suggests that the rest mass of a particle is associated with a “transverse wave number”, which characterizes the radial variation of the wave function in the transverse plane. This model has several appealing features. For example, it predicts that a massless particle must travel at the constant speed of light.

Keywords: Mass, wave-particle duality, quantum mechanics, wave mechanics, relativity

1. Introduction

We know that matter has a dual character, with both wave and corpuscular properties [1]. In the Newtonian mechanics, a particle is regarded as a corpuscular object, the motion of which is determined by its mass and the external force. Clearly, the mass is considered here as a basic property of the particle in this corpuscular view. Later, it was suggested from the special theory of relativity (STR) that mass is not a constant but varies with the particle speed [2]. Then, only the rest mass can be regarded as a basic property of the particle.

During the development of quantum mechanics, we further learned that a particle can also be considered as a wave [1]. This concept of wave-particle duality raises an interesting question: Is there a wave property that is connected to the rest mass of the particle?

We think that there ought to be such a connection. As suggested from the relativistic theory, mass is closely related to energy and momentum [2]. It is well known that the momentum (a corpuscular property) of a particle is connected to its wave vector (a wave property) by the de Broglie relation [3], and the energy of a particle is connected to its wave frequency by the Planck's relation. If mass is to be treated on the same footing as momentum and energy, should not the rest mass also be connected to the magnitude of some sort of “intrinsic wave vector”?

This work presents a simple idea that suggests a possible way to find such a connection. We will use a very simple model (i.e., a system containing a single free particle) to demonstrate our conceptual approach.

2. Details of the Proposed Model

2.1. Proposing a Simplified Wave Equation that Describes the Asymptotic Wave Properties of a Free Particle in the Vacuum

Following the spirit of many pioneers in modern physics, including Einstein and deBroglie, we proposed that the matter wave of a particle with non-zero rest mass behaves very much like a photon. More specifically, we assume that:

- (1) *Like the photon, a particle is not a point-like object; instead, it behaves like a wave packet.*
- (2) *Like the photon, the matter wave of a particle is an excitation of a real physical field.*

These assumptions imply that the solution of the wave equation does not only give the probability of finding the presence of a particle, it actually represents an oscillation of the force field. These assumptions thus propose a slight modification from the traditional Copenhagen interpretation of the wave function [4]. This proposal, however, is not unreasonable, since the current interpretation on the physical meaning of the wave function was dependent more on philosophy rather than on physical evidence. While the statistical interpretation of the Copenhagen school had been strongly supported by major contributors of the quantum theory, including Bohr and Heisenberg, it was not universally agreed. In fact, many well-known physicists at that time, including Einstein, Schrödinger, and de Broglie, had opposed such an interpretation [4]. Here, we will simply hypothesize that, like the photon, the matter wave of a particle with non-zero rest mass is a real physical wave.

Our hypothesis may be justified by a number of observations. For example, it is well known that electrons can form interference pattern upon diffraction from a crystal. And, the wave nature of the electron is clearly demonstrated in the operation of an electron microscope. These examples strongly suggest that the electron can behave like a physical wave. Then, what is the nature of the physical field (which is sometimes referred to as the “vacuum”) that gives rise to this matter wave?

At present, there are only four known interaction fields, i.e., electro-magnetic (EM), weak interaction, strong interaction and gravitational. According to the standard model of elementary particle today, the first three fields may not be regarded as truly independent and can be unified in extremely short distance (or high energy) [5]. Among these four different fields, only the EM field appears to have sufficient long range and strength to support the wave of a free particle. The gravitational field is extremely weak, while both the strong and weak forces are applicable only in very short range (10^{-13} cm and 10^{-16} cm, respectively). Since the wave packet representing a free particle (such as an electron) must be larger than the inter-atomic distance in a crystal (10^{-8} cm), if the

matter wave is a real physical wave, it is most likely to be associated with an oscillation of the EM field.

This argument may be unorthodox, but it is not unreasonable. First, at least one type of particle (photon) is already known to be the excitation of the EM field. Second, since electrons and positrons can be converted to photons (by annihilation) and vice versa (by pair creation), their mutual convertibility may suggest that electrons and positrons are also excitations of a vacuum, the nature of which is related to the EM field. A similar argument can also be made on muons and their anti-particles since they can decay to become other leptons and photons. Hence, it is not unreasonable to propose that different solutions of the wave equation of a vacuum (which behaves like the EM field in the long range) could represent particles of different rest masses.

At this time, we do not have sufficient knowledge about the detailed properties of the vacuum to allow us to write down an exact wave equation for it. In this work, we can only aim at finding a reasonable approximation that may describe the basic wave properties of a free particle. Thus, we will take advantage of the knowledge that, in the range longer than 10^{-13} cm, the vacuum behaves mainly like an EM field. To find the wave equation of a vacuum, we may start by considering the EM field first. From the Maxwell's equations, one can derive [6]

$$\square \phi = -\rho / \varepsilon_0 \quad (1A)$$

and
$$\square \mathbf{A} = -\mathbf{j} / \varepsilon_0 c^2 \quad , \quad (1B)$$

where $\square \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ is the wave operator or the "D'Alembertian", ϕ and \mathbf{A} are the scalar and vector potentials of the EM field, ρ and \mathbf{j} are the charge density and current, respectively, ε_0 is the dielectric constant of the vacuum, and c is the speed of light. Using the 4-vector notation $A_\mu = (\phi, \mathbf{A})$ and $j_\mu = (\rho, \mathbf{j})$ and adopting the natural unit in which $c = 1$, Eq. (1) can be simplified to become

$$\square A_\mu + j_\mu / \varepsilon_0 = 0. \quad (2)$$

Now, if one wants to include the contribution from the strong and weak forces to the vacuum, additional terms may be added, and the wave equation may look like

$$\square A_\mu + j_\mu / \varepsilon_0 + F^{St} + F^{Wk} = 0. \quad (3)$$

At this time, it is not possible to determine the exact form of the contribution from the strong (F^{St}) and weak force (F^{Wk}). We only know that they are very short ranged. But, if we consider the simplest case in which the system contains only a single free particle (travelling in velocity \mathbf{v}), these two terms become less important because there is no particle-particle interaction. Furthermore, since these forces are negligibly small except at very short distance ($\leq 10^{-13}$ cm), they can be ignored if we are interested in solving the wave equation only in the asymptotic region (at distance $> 10^{-13}$ cm).

In this one-particle system, ρ and \mathbf{j} represent the self-charge and the self-current associated with the particle. For a particle with no electrical charge, such as a photon, ρ and \mathbf{j} of course are equal to zero. For a charged particle, such as an electron, ρ and \mathbf{j} are very complicated. Although there is still a lack of satisfactory theory in calculating the distribution of the self-charge in an electron, it is known that the charge of the particle is highly localized in a small core [7]. Several earlier theoretical works had estimated that the charge of an electron is confined within a region having a very short radius (r_o), which is in the order of 10^{-13} cm [7]. The wave of an electron, on the other hand, is supposed to occupy a much larger area ($> 10^{-8}$ cm); otherwise, it would not be possible to generate a diffraction pattern from a crystal. Then, we can simplify Eqs.(3) by considering only the asymptotic region (i.e., $r > r_o$), where ρ and \mathbf{j} can be assumed to be practically zero. Under this condition, we have

$$\square A_\mu = 0 . \quad (4)$$

The simplest solution of Eq. (4) is to let all components of this 4-vector to vary with space and time in the same manner. That is, if one can find a scalar function ψ , which satisfies

$$\square \psi = 0 , \quad (5)$$

then the 4-vector $A_\mu = (\phi_o \psi, \mathbf{A}_o \psi)$ will automatically satisfy Eq. (4). Here, the coefficients \mathbf{A}_o and ϕ_o are subject only to the Lorentz gauge condition [8]

$$\nabla \cdot \mathbf{A} + \partial \phi / \partial t = 0 . \quad (6)$$

We will regard Eq.(5) as the asymptotic wave equation of a free particle in the vacuum.

2.2. Solutions of the Proposed Wave Equation

The simplest solution of Eq.(5) is a plane wave

$$\psi_{\mathbf{k}} \sim e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} , \quad (7)$$

where \mathbf{k} and ω are the wave vector and frequency, respectively. This solution represents the well-known wave function of a photon. This plane wave solution, however, does not properly describe the properties of a particle with nonzero rest mass, which behaves like a mass point in the classical limit. Since such a particle must have a limited "size", the probability of detecting the particle (i.e. $|\psi|^2$) in the transverse plane should not be uniform. That is, the probability of finding the particle is expected to be highest at its trajectory. This expectation suggests that the wave function of a free particle should depend not only on the coordinate parallel to its trajectory (i.e., $\hat{\mathbf{k}} \cdot \mathbf{x}$), but also on the coordinates in the transverse plane ($\hat{\mathbf{k}} \times \mathbf{x}$).

Furthermore, since the trajectory of a free particle is a straight line, only one direction (i.e. the direction of motion, $\hat{\mathbf{k}}$) is specified. The wave function must have a

cylindrical symmetry. Thus, one can assume that the general wave function representing a free particle has the form

$$\psi_{\hat{k}}(\mathbf{x}, t) = \psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) \psi_T(\hat{\mathbf{k}} \times \mathbf{x}), \quad (8)$$

where ψ_L is the longitudinal component of the wave function which describes the travelling wave along the particle's trajectory, and ψ_T is the transverse component of the wave function which determines the probability density of the particle at the transverse plane. Substituting Eq. (8) into Eq. (5), one has

$$\nabla^2 \left[\psi_T(\hat{\mathbf{k}} \times \mathbf{x}) \psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) \right] - \frac{\psi_T(\hat{\mathbf{k}} \times \mathbf{x})}{c^2} \frac{\partial^2}{\partial t^2} \psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) = 0. \quad (9)$$

After expanding the ∇^2 term (keeping in mind that $\nabla \psi_T \cdot \nabla \psi_L$ vanishes) and dividing the whole equation by $\psi_T \cdot \psi_L$, one can rearrange Eq. (9) to obtain

$$\frac{1}{\psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t)} \left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) = \frac{-1}{\psi_T(\hat{\mathbf{k}} \times \mathbf{x})} \nabla^2 \psi_T(\hat{\mathbf{k}} \times \mathbf{x}). \quad (10)$$

The left-hand side of this equation is a function only of $\hat{\mathbf{k}} \cdot \mathbf{x}$ and t , while the right-hand side of this equation is a function only of $\hat{\mathbf{k}} \times \mathbf{x}$. Equation (10) holds only if both sides equal a constant, which we denote as ℓ^2 . Then, Eq. (10) becomes two simultaneous equations

$$\left\{ \begin{array}{l} \left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) = \ell^2 \psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) \end{array} \right. \quad (11)$$

$$\left\{ \begin{array}{l} \nabla^2 \psi_T(\hat{\mathbf{k}} \times \mathbf{x}) = -\ell^2 \psi_T(\hat{\mathbf{k}} \times \mathbf{x}), \end{array} \right. \quad (12)$$

which can be solved separately for ψ_L and ψ_T . The solution of Eq. (12) is

$$\psi_T(\hat{\mathbf{k}} \times \mathbf{x}) \propto J_n(\ell r) e^{\pm i n \theta}, \quad (13)$$

where J_n is the Bessel function of the first kind, and n is an integer or half integer; r and θ represent the amplitude and the azimuthal angle of the vector $\hat{\mathbf{k}} \times \mathbf{x} \times \hat{\mathbf{k}}$, respectively. The solution of Eq. (11) is a plane wave

$$\psi_L(\hat{\mathbf{k}} \cdot \mathbf{x}, t) \propto e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (14)$$

where $\mathbf{k} = k \hat{\mathbf{k}}$ is a vector parallel to $\hat{\mathbf{k}}$ and

$$\omega = (k^2 + \ell^2)^{1/2} c. \quad (15)$$

By combining Eqs. (8), (13), and (14), the wave function thus becomes

$$\psi_{\vec{k}}(\mathbf{x}, t) = a J_n(\ell r) e^{\pm i n \theta} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (16)$$

(where a is a normalizing constant). As expected, the wave function of a free particle behaves like a travelling plane wave moving along the direction of its trajectory. But because of an added phase factor $n\theta$, the particle wave actually propagates in a helical fashion. The wave function as a whole thus behaves like a vortex. Also, $\psi_{\vec{k}}$ varies in a diminishing oscillating manner in the directions perpendicular to the particle's trajectory. We have calculated the variation of the wave function in the transverse plane; the results (for $n = 0, 1, 2$) are shown in Fig. 1.

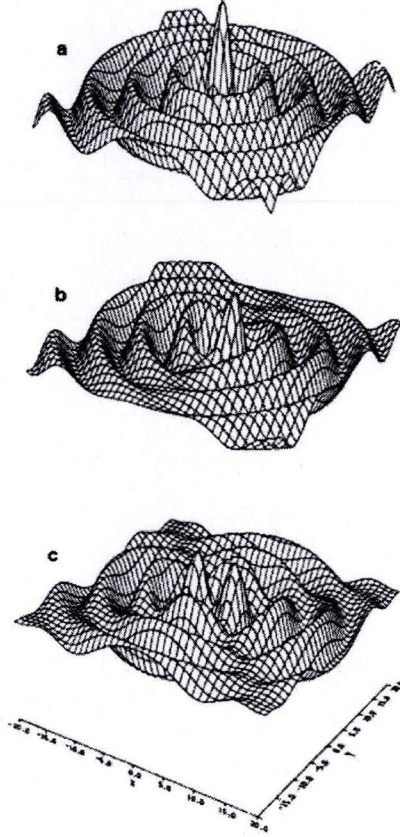


Figure 1: Variation of the real component of the transverse wave, $\psi_T = J_n(\ell r) e^{i n \theta}$, is plotted as a function of the coordinates on the transverse plane (represented here as the horizontal X,Y plane). Here, $X = \ell r \cos \theta$ and $Y = \ell r \sin \theta$. (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$.

2.3. Interpreting the Physical Meaning of the Wave Parameters

The wave function of (16) contains four parameters, ω , \mathbf{k} , ℓ and n . What are their physical meanings? From the correspondence principle [9], the energy (E) and momentum (\mathbf{p}) of a particle in the classical limit can be obtained from the expectation values of the $E \rightarrow i\hbar \partial / \partial t$ and $\mathbf{p} \rightarrow -i\hbar \nabla$ operators, i.e.,

$$E = \int \psi^* i\hbar \frac{\partial}{\partial t} \psi d^3x, \quad (17A)$$

and

$$\mathbf{p} = \int \psi^* \frac{\hbar}{i} \nabla \psi d^3x. \quad (17B)$$

(Here \hbar is Planck's constant divided by 2π). Substituting Eq. (16) into Eq. (17A), one can easily show that

$$E = \hbar \omega, \quad (18)$$

which, of course, is just the Planck's relation. Similarly, by substituting Eq. (16) into Eq. (17B), one can obtain the de Broglie relation

$$\mathbf{p} = \hbar \mathbf{k}. \quad (19)$$

But what is the physical meaning of ℓ in the classical limit? From Eq. (15), we know ℓ is closely related to ω and \mathbf{k} . By combining Eqs. (15), (18), and (19), we can obtain

$$E^2 = c^2(p^2 + \hbar^2 \ell^2). \quad (20)$$

It is well known in wave mechanics that the particle velocity (v) is determined by the group velocity of the wave packet [10], that is,

$$v = \frac{\partial \omega}{\partial k} = \frac{\partial E}{\partial p}. \quad (21)$$

Combining Eqs. (20) and (21), one can solve for E or \mathbf{p} and obtain

$$E = \frac{\hbar \ell c}{(1 - v^2/c^2)^{1/2}} \quad (22)$$

and

$$\mathbf{p} = \left[\frac{\hbar \ell / c}{(1 - v^2/c^2)^{1/2}} \right] \mathbf{v}. \quad (23)$$

In the classical limit, the momentum (\mathbf{p}) is equal to the mass (M) times the velocity (\mathbf{v}). Hence, we can identify the quantity within the bracket on the right-hand side of Eq. (23) as mass, that is,

$$M = \frac{\hbar\ell/c}{(1-v^2/c^2)^{1/2}} . \quad (24)$$

At $v = 0$, M equals the rest mass, m . Eq. (24) then implies that

$$m = \frac{\hbar\ell}{c} , \quad (25)$$

which suggests that the parameter ℓ is associated with the rest mass of the particle. This result appears to make good sense, since when we substitute Eq. (25) into Eq. (20), we have

$$E^2 = p^2c^2 + m^2c^4 , \quad (26)$$

which agrees exactly with the energy-momentum relationship obtained from the classical treatment of the STR [2]. Furthermore, by substituting Eq. (25) into Eqs. (22), (23), and (24), we can obtain the other relativistic relations, i.e.

$$E = \gamma mc^2 , \quad (27)$$

$$\mathbf{p} = \gamma m\mathbf{v} , \quad (28)$$

and
$$M = \gamma m , \quad (29)$$

where $\gamma = (1-v^2/c^2)^{-1/2}$. Combining Eqs. (27) and (29), we have

$$E = Mc^2 \quad (30)$$

which is the well-known relation of mass and energy.

Finally, what is the physical meaning of the parameter n ? It appears that n is likely to be associated with the helicity of the free particle. First, n is a quantum number conjugate to the angular coordinate θ . Dimensional analysis thus suggests that n is associated with some sort of angular momentum. The helicity operator can be regarded as equivalent to an angular momentum operator about an axis of the particle's trajectory (in direction $\hat{\mathbf{k}}$) [11]. In our solution of the wave equation, the eigenvalue of this operator is $n\hbar$. Secondly, from Eq. (16), one can see that, because of the added phase factor $n\theta$, the wave function representing a free particle actually propagates in a helical fashion. It is known that helicity is related to the particle spin. One may assume that the wave function with an integer n represents a boson while the wave function with half integer n represents a fermion.

3. Discussion

The validity of this model, of course, will depend on future experimental tests. We believe that it deserves to be examined carefully because it appears to have a number of appealing features:

3.1. It Suggests a Possible Connection between the Special Theory of Relativity and Wave Mechanics.

One important result of this work is our demonstration that some of the well known relativistic relations can be derived using an approach of wave mechanics. STR is known to be a classical theory, which does not consider quantum effects. And thus, STR and quantum mechanics (or wave mechanics) are traditionally regarded as two totally independent physical theories. In this work, we show that, if we regard a particle as an excitation of the vacuum field, the relativistic relations between energy, momentum and mass can be derived naturally from the dispersion relation of the wave function representing a free particle. This result thus suggests that wave mechanics and STR could have a deeper philosophical connection in their theoretical roots.

3.2. It Naturally Leads to the Klein-Gordon Equation.

After demonstrating that ℓ is connected with m , it becomes possible to relate our wave function with those derived in the traditional wave mechanics. For example, using Eq. (25), we can see that Eq. (11) now becomes

$$\square \psi_L - \left(\frac{mc}{\hbar} \right)^2 \psi_L = 0, \quad (31)$$

which is identical to the “Klein-Gordon equation” [9]. This implies that the wave function derived from the Klein-Gordon equation is equivalent to the longitudinal component of the travelling wave representing a free particle in our model. This is consistent with our starting assumption that the wave function of a particle does not only represent the probability of finding the particle, it actually represents an oscillation of the force field. Furthermore, the fact that our model can naturally lead to the Klein-Gordon equation may indicate that our proposal of using Eq.(5) as the wave equation to describe the transporting properties of a free particle was a reasonable one.

3.3. It Predicts that any Massless Particle must always Travel in the Speed of Light.

As shown in Fig.1, the wave function given in Eq. (16) not only oscillates in the longitudinal direction, it also oscillates in the radial direction in the transverse plane. The “wavelength” of this radial oscillation is equal to $2\pi/\ell$. Thus, ℓ can be regarded as the “transverse wave number” of the free particle. From Eq. (25), we can see that the wavelength of this transverse oscillation is

$$\lambda = \frac{2\pi}{\ell} = \frac{h}{mc}, \quad (32)$$

which is identical to the “Compton wavelength” (λ_c) of the particle [12].

What is the consequence of connecting the rest mass with ℓ in the wave function of a free particle? Let us examine what happens to a particle with zero rest mass, such as the case of a photon. From Eq. (25), $m = 0$ implies $\ell = 0$. Hence, $J_n(\ell r)$ is a constant, and the wave function $\psi_{\vec{k}}$ given in (16) now becomes a plane wave. Thus, our model predicts that the EM wave of a photon in the vacuum is essentially a plane wave, which agrees well with the known results of the electromagnetic theory. Furthermore, when $\ell = 0$, Eq. (15) becomes

$$\omega = ck, \quad (33)$$

which implies that the group velocity of a massless particle must equal to c , i.e.,

$$v = \frac{\partial \omega}{\partial k} = c. \quad (34)$$

This provides a simple explanation to the fact that a photon must always travel in the speed of light.

3.4. It Predicts that the Rest Mass is Lorentz Invariant.

Finally, since the rest mass is known to be a relativistic scalar, in order to associate m with ℓ , one must show that the ℓ defined in our model also behaves like a scalar under a Lorentz transformation. Indeed, as shown in Appendix A, we found that the value of ℓ is independent of the inertial frame. In other words, ℓ does satisfy the requirement of being a relativistic scalar.

Since the wave equation we used to describe the asymptotic behavior of a particle wave is in a form similar to that of the EM radiation in a vacuum, the solution of our wave equation is analogous to that encountered in the studies of cylindrical waveguides in the classical electromagnetic theory [13]. It has been noted that, due to the form of the dispersion relation, the cut-off wavenumber (which is equivalent to our transverse wavenumber) in a waveguide could be viewed as an “effective mass” of the electromagnetic radiation. However, the physical concept of this work is fundamentally different from that of the waveguide problem. Our wave equation (i.e., Eq. (5)) describes the asymptotic behavior of the matter wave of a free particle in the vacuum, not the radiation wave within a metal compartment. In the later case, the cut-off wavenumber is determined by the imposed external boundary conditions (i.e., dimensions of the waveguide), which is not Lorentz invariant. Thus, the “effective mass” in the EM radiation within a cylindrical waveguide cannot be regarded as a true mass. In our model, the value of ℓ is a relativistic scalar.

4. Conclusion

Following the spirit of de Broglie and Einstein, we think the concepts of matter and radiation should be unified; both of them are excitation waves of the vacuum. We propose a model based on the following postulates: (1) Like the photon, a particle is not a point-like object; instead, it behaves like a wave packet. (2) Like the photon, the

matter wave of a particle is an excitation of a real physical field. Thus, Different types of particles are different excitation modes of a unified field in the vacuum. This model has several appealing features. For example, it predicts that a massless particle must travel at the constant speed of light and the rest mass is Lorentz invariant. Furthermore, this model suggests that the relativistic mass-energy relation can be derived from the wave properties of a particle, implying that the special theory of relativity and wave mechanics may have a common root. Indeed, it is shown that the Klein-Gordon equation can be derived naturally based on this model. Thus, this model appears to be able to explain a number of important physical concepts in a simple way.

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Appendix A

To show that the transverse wave number, ℓ , as defined in Eqs. (11) and (12) is a "relativistic scaler," we will demonstrate that ℓ is frame-independent. Consider two inertial frames S_1 and S_2 , where S_2 moves along the direction \hat{u} at a speed u relative to S_1 . Since the one-particle free-field equation (i.e., Eq. (5)) is Lorentz invariant, its solutions in both frames must be in the same form. That is, in the S_1 frame, we have

$$\psi_{\hat{k}_1}(\mathbf{x}_1, t_1) = a J_n(\ell_1 r_1) e^{\pm i n \theta_1} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 - \omega_1 t_1)}, \quad (\text{A1})$$

where
$$\omega_1 = (k_1^2 + \ell_1^2)^{1/2} c, \quad (\text{A2})$$

and in the S_2 frame, we have

$$\psi_{\hat{k}_2}(\mathbf{x}_2, t_2) = a J_n(\ell_2 r_2) e^{\pm i n \theta_2} e^{i(\mathbf{k}_2 \cdot \mathbf{x}_2 - \omega_2 t_2)}, \quad (\text{A3})$$

where
$$\omega_2 = (k_2^2 + \ell_2^2)^{1/2} c. \quad (\text{A4})$$

(Here we assume $n_1 = n_2 = n$ because the n -fold symmetry should not change with frame). The phase factor of the travelling wave is a dot product of two 4-vectors (ct, \mathbf{x}) and $(\omega/c, \mathbf{k})$ and thus is Lorentz invariant. Similarly, the norm of the 4-vector $(\omega/c, \mathbf{k})$ should also remain unchanged under a Lorentz transformation, that is,

$$\omega_1^2/c^2 - k_1^2 = \omega_2^2/c^2 - k_2^2. \quad (\text{A5})$$

Combining eqs.(A2), (A4) and (A5), we have

$$\ell_1 = \ell_2. \quad (\text{A6})$$

Hence, ℓ is independent of the inertial frame.

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