Incursive Algorithms for Newtonian and Relativistic Gravitations, and Simulation of the Mercury Orbit

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Abstract
This paper firstly recalls the mathematical model of the Newtonian gravitation law applied to the solar system dealing with the Sun and the Mercury planet. The orbits of all the planets are given by closed ellipses in the Newton paradigm. This paper continues with the introduction of the relativistic correction to this Newton gravitation law, for exhibiting the precession of orbits and, more particularly, the advance of the perihelion of the Mercury planet. The following section gives the Euler and the First Incursive algorithms for the classical Newton gravitation law and the relativistic correction of this Newtonian law. The last section of this paper gives the result of the numerical simulations of the Mercury orbit around the Sun. It is shown that the simulation with the Euler algorithm is not stable and does not give a closed ellipse to the Mercury orbit with the Newtonian law. The simulation with the First Incursive algorithm gives a perfect simulation of the Newtonian orbit in 88 days. Then, the simulation with this First Incursive algorithm of the relativistic Newtonian gravitation shows the correct Mercury precession angle that is equal to $\delta\theta = 180$ degrees, after 15,000 centuries, in agreement with the experimental data and Einstein relativity.
Keywords: incursive algorithms, Newtonian gravitation, relativistic gravitation, anomalous precession, Mercury orbit

1. Introduction

The Newton law of gravitation, based on an instantaneous propagation of gravity, for the solar system works rather well, except that it does not show the advance of the perihelia of the planets. The case of the Mercury planet is taken as example because its precession is the most important aberration in the solar system. This aberration is a relativistic effect as explained by Einstein with the General Relativity.
This paper is organized as follows. After a recall of the mathematical model of the Newton law of gravitation, a relativistic Newton law will be corrected to obtain an equivalent effective potential as in General Relativity.
Then, this paper will deduce the precession formula by linearization of this relativistic Newton law. This gives the precession amount for the Mercury planet.
Finally, this paper will give some numerical simulations of the trajectory of the Mercury planet with and without precession.
Let us firstly, recall the mathematical model of the Newton law of gravitation

2. The Mathematical Model of the Newton Law of Gravitation

The Newton law (see, for example, Landau and Lifchitz, 1966) of a material point S of mass m (the Mercury planet in this study) is given by

$$m\,\mathbf{a} = \mathbf{F} \tag{1}$$

with the acceleration $\mathbf{a} = d^2\mathbf{s}/dt^2$, where $\mathbf{s} = r\,\mathbf{e}_r$ is its position vector and $\mathbf{F}$ the force. In the case of a gravitational central force in r^{-2}, the force is given by

$$\mathbf{F} = -\,\mu\, m\, \mathbf{e}_r / r^2 \tag{2}$$

with $\mu = GM$, where G is the gravitational constant, M the mass of the attractive body at rest (M >> m) at the origin of axes (the Sun in this study). The distance between the two bodies is r. So, from eqs. 1 and 2, the Newton law of gravitation is given by

$$m\,\mathbf{a} = -\,\mu\, m\, \mathbf{e}_r / r^2 \tag{3}$$

or, in dividing the two members by m,

$$\mathbf{a} = -\,\mu/r^2\, \mathbf{e}_r \tag{4}$$

The vector product, ×, of the two members by **s**, in taking into account that the force is parallel to **s**, gives

$$\mathbf{s} \times \mathbf{a} = d[\mathbf{s} \times \mathbf{v}]/dt = 0 \tag{5}$$

with the velocity $\mathbf{v} = d\mathbf{s}/dt$, or

$$\mathbf{s} \times \mathbf{v} = \mathbf{h} \tag{6}$$

where the constant vector **h** represents the kinetic momentum by unity of mass of S (the Mercury planet). This shows that the vector position **s** and velocity **v** are constantly contained in a fixed plane perpendicular to **h**. The initial conditions of the movement, $\mathbf{s}_0$ and $\mathbf{v}_0$, determine this vector **h**

$$\mathbf{h} = \mathbf{s}_0 \times \mathbf{v}_0 \tag{7}$$

Due to this symmetry, the movement can be described in polar coordinates $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as

$$\mathbf{s} = r\,\mathbf{e}_r \tag{8}$$

$$\mathbf{v} = dr/dt\, \mathbf{e}_r + r\, d\mathbf{e}_r/dt = dr/dt\, \mathbf{e}_r + r\, d\theta/dt\, \mathbf{e}_\theta \tag{9}$$

$$\mathbf{a} = d^2r/dt^2\, \mathbf{e}_r + dr/dt\, d\theta/dt\, \mathbf{e}_\theta + dr/dt\, d\theta/dt\, \mathbf{e}_\theta + r\, d^2\theta/dt^2\, \mathbf{e}_\theta - r\,[d\theta/dt]^2\, \mathbf{e}_r \tag{10}$$

Introducing eq. 10 to eq. 4 gives the two following scalar equations of the movement

$$d^2r/dt^2 - r\,[d\theta/dt]^2 = -\,\mu/r^2 \tag{11}$$

$$2\, dr/dt\, d\theta/dt + r\, d^2\theta/dt^2 = (1/r)\, d[r^2\, d\theta/dt]/dt = 0 \tag{12}$$

This second eq. 12 gives the following first scalar integral

$$h = r^2 d\theta/dt \tag{13}$$

which is the constant scalar kinetic momentum, i.e. the projection on an axis $\mathbf{e}_z$, perpendicular to the movement plane, of the first vector integral (6),

$$\mathbf{h} = h\ \mathbf{e}_z \tag{14}$$

With eq. 13, eq. 11 can be written as

$$d^2r/dt^2 - h^2/r^3 = -\mu/r^2 \tag{15}$$

The integration of this eq. 15 gives the following law of the conservation of energy

$$E = (1/2)(dr/dt)^2 + h^2/2r^2 - \mu/r \tag{16}$$

or

$$E = T + V_{eff}(r) \tag{17}$$

where E is the constant total energy, defined by the initial conditions, T the kinetic energy and V_{eff} the effective potential energy

$$V_{eff}(r) = h^2/2r^2 - \mu/r = h^2/2r^2 + V(r) \tag{18}$$

where

$$V(r) = -\mu/r \tag{19}$$

is the gravitational potential energy, which derives from the force by unit mass

$$\mathbf{F}/m = -\nabla V(r) \tag{20}$$

For obtaining the analytical solution of the gravitational equation, it is useful to work with (u, θ) instead of (r, t), with the new variable u given by

$$u = 1/r \tag{21}$$

With this new variable u, the constant scalar kinetic momentum eq. 13 becomes

$$d\theta/dt = h\ u^2 \tag{22}$$

and the radial velocity and acceleration are given by

$$dr/dt = -h\ du/d\theta \tag{23}$$

$$d^2r/dt^2 = -h^2\ u^2\ d^2u/d\theta^2 \tag{24}$$

so eq. 15 becomes

$$-h^2\ u^2\ d^2u/d\theta^2 - h^2\ u^3 = -\mu\ u^2 \tag{25}$$

or, in dividing by $h^2 \neq 0$,

$$d^2u/d\theta^2 + u = \mu/h^2 \tag{26}$$

and the conservation of energy eq. 16 becomes

$$E = (1/2)(\ h\ du/d\theta)^2 + h^2u^2/2 - \mu u \tag{27}$$

The general solution of this eq. 26 is

$$u(\theta) = C_1 \cos\theta + C_2 \sin\theta + 1/p \qquad (28)$$

with

$$p = h^2/\mu \qquad (29)$$

and where C_1 and C_2 are constants of integration to be defined by initial conditions.
In choosing, as initial conditions, the perihelion at $\theta = 0$ for which $du/d\theta = 0$, eq. 27 becomes

$$E = h^2u^2/2 - \mu u \qquad (30)$$

and the maximum root of u is the value of the perihelion u_0 given by

$$u_0 = (1 + e)/p \qquad (31)$$

where e is the eccentricity

$$e = \sqrt{[1 + 2Eh^2/\mu^2]} \qquad (32)$$

With these initial conditions, the analytical solution eq. 28 becomes

$$u(\theta) = (e \cos\theta + 1)/p \qquad (33)$$

or

$$r(\theta) = p/(1 + e \cos\theta) \qquad (34)$$

For $E = -\mu^2/2h^2$, the movement is a circular orbit of radius $p = h^2/\mu$.
For $-\mu^2/2h^2 < E < 0$, the movement is given by an orbit which is an ellipse.
These orbits are closed ellipses. Indeed, it exists a third specific integral

$$\mathbf{v} \times (\mathbf{s} \times \mathbf{v}) - \mu\, \mathbf{s}/r = \mathbf{P} \qquad (35)$$

where the vector **P** is constant. Eq. 35 can be written as

$$r\, v^2\, \mathbf{e}_r - r\, v\, \mathbf{v} - \mu\, \mathbf{e}_r = \mathbf{P} \qquad (36)$$

where $v^2 = \mathbf{v}.\mathbf{v}$ (scalar product). In polar coordinates, this integral becomes

$$(h^2/r - \mu)\mathbf{e}_r - (dr/dt)h\mathbf{e}_\theta = \mathbf{P} \qquad (37)$$

Let us remark that this third supplementary integral is due to the degeneration of the movement. This only occurs for two types of central potential for which all the trajectories are closed. These are the potentials for which the potential energy is proportional to $1/r$ or to r^2, this second potential corresponding to the spatial oscillator.
This conservative vector **P** is along the major axis of the ellipse and directed to the perihelion. This is easy to show that **P** is constant by calculating the time derivative of eq. 23 as

$$d\mathbf{P}/dt = (-h^2/r^2)(dr/dt)\mathbf{e}_r + (h^2/r - \mu)(h/r^2)\mathbf{e}_\theta - (d^2r/dt^2)h\mathbf{e}_\theta + (dr/dt)(h^2/r^2)\mathbf{e}_r$$

$$= -\,(d^2r/dt^2 - h^2/r^3 + \mu/r^2)\, h\, \mathbf{e}_\theta = \mathbf{0} \qquad (38)$$

by using eq. 15, and the value of **P** is

$P = \mu\, e = \mu\, \sqrt{[1 + 2Eh^2/\mu^2]}$ (39)

where e is the eccentricity of the orbit (eq. 32).
As the perihelion is the minimum radius of the orbit for which the radial velocity is null, in choosing as initial condition a null radial velocity, dr/dt = 0, eq. 23 shows that **P** is parallel to $\mathbf{e}_r$ and remains constantly parallel to $\mathbf{e}_r$.
This means that there is no precession of the orbit, contrary to the experimental data in the solar system.
For the Mercury Planet, the experimental data are given in Table 1.

Table 1 : Experimental Data for the Mercury Planet

Observed precession per century	43.1 ± 0.5 arc seconds
Perihelion distance r_{min}	45.9 Mkm
Aphelion distance r_{max}	69.7 Mkm
Tangential velocity at perihelion v_{per}	5.11 Mkm/day
Tangential velocity at aphelion v_{aph}	3.365 Mkm/day
Mean velocity	4.128 Mkm/day
Period	87.97 days
Number of revolutions per century	414.93
Velocity of light	25920 Mkm/day
GM/c^2	1.48 10^{-6} Mkm
$\mu = GM$	994.332672 Mkm^3/day
Kinetic momentum $h = r_{min} v_{per} = r_{max}\, v_{aph}$	234.549 Mkm^2/day
Gravitational radius $r_0 = \mu/c^2$	1.48 10^{-6} Mkm
Radius of the circular orbit $p = h^2/\mu$	55.327 Mkm
Precession per revolution $\delta\theta = 2\, \pi\, 3\, r_0 /p$	0.5044 10^{-6} radian
Precession per century $\delta\theta(414.93)(360)(3600)/2\pi$	43.15 arc seconds
Mean delay duration $\tau = p/c$	3.1 minutes
Mean tangential delay distance $r_h = h/c$	9.050 10^{-3} Mkm

In this Table 1, the distances are given in Mkm = 10^6 km.
Let us recall that the perihelion is the shortest distance from Mercury to the Sun and the aphelion, the largest distance. At the perihelion and aphelion, the velocity of Mercury is a tangential velocity (the radial velocity is zero).
This paper continues with the introduction of the relativistic correction to this Newton gravitation law, for exhibiting the precession of orbits and, more particularly, the advance of the perihelion of the Mercury planet.
The next section deals with the relativistic correction to the Newtonian Gravitation.

3. The Relativistic Correction to the Newtonian Gravitation Law

In 1916, Einstein presented his theory of the General Relativity (GR) and explained the precession of the Mercury perihelion.
A few months later, Schwarzschild (1916) gave the solution of the Einstein equation for the geometry outside of a spherical star as follows:

$$ds^2 = -(1 - 2\mu/c^2r)(c\,dt)^2 + (1 - 2\mu/c^2r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (40)$$

where ds is the line element, c the speed of light, and (r, θ, ϕ) the spherical coordinates. It was shown that the solution is independent of time, t, and also independent of the angular coordinate ϕ. The two natural conserved quantities are thus the energy per unit rest mass and angular momentum per unit rest mass, respectively. The conservation of angular momentum leads to planar orbits (chosen for $\theta = \pi/2$)
The following "GR effective" potential was derived:

$$V_{GReff}(r) = (c^2/2)\,[\,(1 - 2\mu/c^2r)(1 + \ell^2/c^2r^2) - 1\,] \qquad (41)$$

where the angular momentum is given by

$$\ell^2 = (r^2 d\phi/dt)^2 \qquad (42)$$

Let us remark that this ϕ angle, in spherical coordinates, corresponds to the θ angle, in polar coordinates, in the preceding section. So eq. 42 is similar to eq. 13.
As this is not the purpose of this paper to demonstrate all the results of the General Relativity for the precession of the Mercury perihelion, let us give the well-known result of the precession amount which is given by

$$\delta\phi = 6\,\pi\,(\mu/hc)^2 \qquad (43)$$

The advance of the Mercury perihelion predicted by this formula 43 is 43 arc seconds per century and corresponds to the exact experimental data. Historically, this result was the first proof of the validity of the General Relativity who made Einstein so confident in his theory.
In view of comparing the effective potential of Newton and Einstein, let us first rewrite the eq. 41 as

$$V_{GReff}(r) = (c^2/2)\,[\,(1 - 2\mu/c^2r)(1 + h^2/c^2r^2) - 1\,] \qquad (44)$$

with eq. 42 as

$$h^2 = (r^2 d\theta/dt)^2 \qquad (45)$$

by renaming the variables ℓ by h and ϕ by θ. So the precession eq. 43 is rewritten as

$$\delta\theta = 6\,\pi\,(\mu/hc)^2 \qquad (46)$$

The GR effective potential can be developed as

$$V_{GReff}(r) = h^2/2r^2 - \mu/r - \mu h^2/c^2r^3 \qquad (47)$$

to be compared with the Newtonian effective potential eq. 18.

$$V_{eff}(r) = h^2/2r^2 - \mu/r \qquad (18)$$

So, the correction to make to the Newtonian potential is to add a deviation of the potential given by

$$\delta V = -\mu h^2/c^2r^3 \qquad (48)$$

With this correction, the Newton law of gravitation given by eq. 15, becomes

$$d^2r/dt^2 - h^2/r^3 = -\mu/r^2 - 3h^2r_0/r^4 \qquad (49)$$

$$h = r^2\, d\theta/dt \qquad (50)$$

with

$$r_0 = \mu/c^2 \qquad (51)$$

The corrected potential is

$$V(r) = -\mu/r - h^2r_0/r^3 \qquad (52)$$

and the corrected effective potential

$$V_{eff}(r) = h^2/2r^2 - \mu/r - h^2r_0/r^3 \qquad (53)$$

is identical to the effective potential (47) of the General Relativity.
With the variable $u = 1/r$, eq. 49 becomes, similarly to eq. 26,

$$d^2u/d\theta^2 + u = \mu/h^2 + 3r_0u^2 \qquad (54)$$

The linearization of this eq. 54 gives the formula of the precession of the perihelion of the Mercury planet.

The nonlinear relativist Newton eq. 54 can be written as

$$d^2u/d\theta^2 + u = 1/p + 3r_0u^2 \qquad (55)$$

with $1/p = \mu/h^2$ and $r_0 = \mu/c^2$.
For $d^2u/d\theta^2 = 0$, eq. 55 becomes

$$3r_0u^2 - u + 1/p = 0 \qquad (56)$$

and the following roots are two stationary solutions

$$u_1 = (1 - \sqrt{[1 - 12r_0/p]})/6r_0 \qquad (57)$$

$$u_2 = (1 + \sqrt{[1 - 12r_0/p]})/6r_0 \qquad (58)$$

As $r_0/p << 1$ is very small, the first term in the Taylor series can be used

$$u_1 = (1 - [1 - 6r_0/p])/6r_0 = 1/p \qquad (59)$$

$$u_2 = (1 + [1 - 6r_0/p])/6r_0 = 1/3r_0 \qquad (60)$$

The first root corresponds to a stable orbit and the second one to an unstable orbit.
Let us linearize the non-linear term u^2 around the stable orbit as

$$u^2 = (u - u_1 + u_1)^2 = (u - u_1)^2 + (u_1)^2 + 2(u - u_1)u_1 \approx 2uu_1 - u_1^2 \qquad (61)$$

so the linearized equation becomes

$$d^2u/d\theta^2 + u = 1/p + 3r_0(2uu_1 - u_1^2) \quad (62)$$

or

$$d^2u/d\theta^2 + (1 - 6r_0u_1)u = 1/p - 3r_0u_1^2 \quad (63)$$

or

$$d^2u/d\theta^2 + (1 - 6r_0/p)u = (1 - 3r_0/p)/p \quad (64)$$

This linear equation 64 represents an harmonic oscillator of frequency equal to

$$\omega^2 = (1 - 6\ r_0/p) \quad (65)$$

the period of which being given by

$$T = 2\pi/\omega = 2\pi/\sqrt{[1 - 6r_0/p]} \quad (66)$$

As $3r_0/p \ll 1$, the period 66 can be developed as

$$T = 2\pi[1 + 3r_0/p] \quad (67)$$

The period of the orbit without the relativist correction is $T = 2\pi$, so the correction gives an advance $\delta\theta$ of the perihelion

$$T = 2\pi + 6\pi r_0/p = 2\pi + \delta\theta \quad (68)$$

so the advance of the perihelion of Mercury is given by

$$\delta\theta = 6\ \pi\ r_0/p \quad (68)$$

which, with $r_0 = \mu/c^2$ and $1/p = \mu/h^2$, is identical to the precession eq. 46 predicted by the General Relativity.
The numerical data for the Mercury planet, given in Table 1, are as follows :

$$r_0 = 1.48\ 10^{-6}\ \text{Mkm} \quad (70)$$

$$p = 55.327\ \text{Mkm} \quad (71)$$

so

$$r_0/p = 2.675\ 10^{-8} \quad (72)$$

By century, there are 414.93 revolutions of the Mercury planet. Introducing these data in the eq. 69 of the precession, expressed in arc seconds by century, is given by

$$\delta\theta_c = (6\pi r_0/p)(414.93)(360)(3600)/2\pi = (3r_0/p)(414.93)(360)(3600)$$

$$= (3r_0/p)\ 5.3774928\ 10^8 = 43.15\ \text{arc seconds/century} \quad (73)$$

which is in agreement with the observed precession, 43.1 ± 0.5 arc seconds, of the perihelion of the Mercury planet.
The following section gives numerical algorithms for Newton law of gravitation and the relativistic Newton law of gravitation for the Mercury planet.

4. Numerical Algorithms for Newtonian and Relativistic Gravitations

It was shown (eqs. 1 to 7) that the vector position **s** and velocity **v** are constantly contained in a fixed plane perpendicular to the kinetic momentum **h**.
Let us define the position vector **s** with the two following components x_1 and x_2 in the orthonormal axes $\mathbf{e}_1$ and $\mathbf{e}_2$ defining the plane of the planet orbit

$$\mathbf{s} = r\ \mathbf{e}_r = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \qquad (74)$$

where the origin $x_1 = x_2 = 0$ defines the position of the Sun.
So the velocity is given by

$$\mathbf{v} = ds/dt = dx_1/dt\ \mathbf{e}_1 + dx_2/dt\ \mathbf{e}_2 = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \qquad (75)$$

and the acceleration is defined by

$$\mathbf{a} = dv/dt = dv_1/dt\ \mathbf{e}_1 + dv_2/dt\ \mathbf{e}_2 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \qquad (76)$$

The square of the radial distance, between the Sun and the planet, is

$$\mathbf{s.s} = r^2 = x_1^{\ 2} + x_2^{\ 2} \qquad (77)$$

The Newtonian equation for the gravitational field, given by eq. 4 is then re-written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 = -\ \mu/r^2\ \mathbf{e}_r = -\ \mu/(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)/r^3 \qquad (78)$$

or, with eq. 76, the Newton law of gravitation in the orthonormal axes is given by

$$dv_1/dt = -\ \mu\ x_1/\ r^3 \qquad (79\text{-a})$$

$$dx_1/dt = v_1 \qquad (79\text{-b})$$

$$dv_2/dt = -\ \mu\ x_2/\ r^3 \qquad (80\text{-a})$$

$$dx_2/dt = v_2 \qquad (80\text{-b})$$

with $r^2 = (x_1^{\ 2} + x_2^{\ 2})$ and μ = GM, where G is the gravitational constant, M the mass of the attractive body at rest (M >> m) at the origin of axes (the Sun in this study).
With the relativistic correction of the potential, given by eq. 49, the eqs. 79-a and 80-b are transformed to the following relativistic Newton law of gravitation

$$dv_1/dt = -\ \mu\ x_1/\ r^3 - \mu\ 3\ (h/c)^2\ x_1/\ r^5 = (-\ \mu\ x_1/\ r^3)(1 + 3(h/c)^2\ /\ r^2) \qquad (81\text{-a})$$

$$dx_1/dt = v_1 \qquad (81\text{-b})$$

$$dv_2/dt = -\ \mu\ x_2/\ r^3 - \mu\ 3\ (h/c)^2\ x_2/\ r^5 = (-\ \mu\ x_2/\ r^3)(\ 1 + 3(h/c)^2\ /\ r^2) \qquad (82\text{-b})$$

$$dx_2/dt = v_2 \qquad (82\text{-b})$$

where c is the velocity of light and h the kinetic momentum.
In a recent paper, I have deduced this relativistic correction to the Newton law of gravitation from an anticipative effect (Dubois, 2005).
Let us now introduce the Euler and the First Incursive Algorithms for the Newtonian and relativistic gravitations equation systems. For a review of Euler, and First and Second Incursive Algorithms, see the paper of Dubois (2000).

4.1. Euler Algorithm for the Newton Law of Gravitation

The Euler algorithm for the Newtonian equation system 79-ab and 80-ab, is given as follows:

$v_1(t + \Delta t) = v_1(t) + \Delta t\,(-\mu\, x_1(t)/\, r(t)^3)$ (83-a)

$x_1(t + \Delta t) = x1(t) + \Delta t\, v_1(t)$ (83-b)

$v_2(t + \Delta t) = v_2(t + dt) + \Delta t\,(-\mu\, x_2(t)/\, r(t)^3)$ (84-b)

$x_2(t + \Delta t) = x_2(t) + \Delta t\, v_2(t)$ (84-b)

4.2. Euler Algorithm for the Relativistic Newton Law of Gravitation

The Euler algorithm of the Newtonian equation system 81-ab and 82-ab, with the relativistic correction, is given as follows:

$v_1(t + \Delta t) = v_1(t) + \Delta t\,(-\mu\, x_1(t)/\, r(t)^3)(1 + 3(h/c)^2 / r(t)^2)$ (85-a)

$x_1(t + \Delta t) = x_1(t) + \Delta t\, v_1(t)$ (85-b)

$v_2(t + \Delta t) = v_2(t + dt) + \Delta t\,(-\mu\, x_2(t)/\, r(t)^3)(\, 1 + 3(h/c)^2 / r(t)^2)$ (86-b)

$x_2(t + \Delta t) = x_2(t) + \Delta t\, v_2(t)$ (86-b)

4.3. First Incursive Algorithm for the Newton Law of Gravitation

The first incursive algorithm of the Newtonian equation system 79-ab and 80-ab, is given as follows:

$v_1(t + \Delta t) = v_1(t) + \Delta t\,(-\mu\, x_1(t)/\, r(t)^3)$ (87-a)

$x_1(t + \Delta t) = x_1(t) + \Delta t\, \mathbf{v_1(t + \Delta t)}$ (87-b)

$v_2(t + \Delta t) = v_2(t + dt) + \Delta t\,(-\mu\, x_2(t)/\, r(t)^3)$ (88-b)

$x_2(t + \Delta t) = x_2(t) + \Delta t\, \mathbf{v_2(t + \Delta t)}$ (88-b)

4.4. First Incursive Algorithm for the Relativistic Newton Law of Gravitation

The first incursive algorithm of the Newtonian equation system 81-ab and 82-ab, with the relativistic correction is given as follows:

$v_1(t + \Delta t) = v_1(t) + \Delta t\,(-\mu\, x_1(t)/\, r(t)^3)(1 + 3(h/c)^2 / r(t)^2)$ (89-a)

$x_1(t + \Delta t) = x1(t) + \Delta t\, \mathbf{v_1(t + \Delta t)}$ (89-b)

$v_2(t + \Delta t) = v_2(t + dt) + \Delta t\,(-\mu\, x_2(t)/\, r(t)^3)(\, 1 + 3(h/c)^2 / r(t)^2)$ (90-b)

$x_2(t + \Delta t) = x_2(t) + \Delta t\, \mathbf{v_2(t + \Delta t)}$ (90-b)

The following last section gives the simulations of the Newton law of gravitation and the relativistic Newton law of gravitation for the Mercury planet.

5. Simulations of the Newtonian and Relativistic Mercury Orbits

The numerical simulations of the revolutions of the Mercury planet were performed with the exact observed data (see Table 1).

Figure 1 is the framework where the simulations of the orbits of the Mercury planet around the Sun will be drawn in the following figures.

Figure 2 gives the simulation, with the First Incursive algorithm of the Newton law, of successive revolutions of the Mercury planet around the Sun, which is a closed ellipse, demonstrating so, that this incursive algorithm is stable.

Figure 3 gives the simulation with the Euler algorithm of the Newton law, with the parameters of figure 2, of the 1^{rst} and the 10^{th} revolutions of the Mercury orbit around the Sun, which do not give a closed ellipse, because the Euler algorithm is not stable.

Figure 4 gives the simulation, with the First Incursive algorithm of the relativistic Newton law, of a first revolution (88 days), and a second revolution after 15,000 centuries of the Mercury orbit. The figure shows an advance of the perihelion of the Mercury planet equal to an angle of $\delta\theta = 180$ degrees.

Figure 5 gives the simulation, with the First Incursive algorithm of the relativistic Newton law, of a Mercury orbit with 58 revolutions around a Sun with a bigger mass.

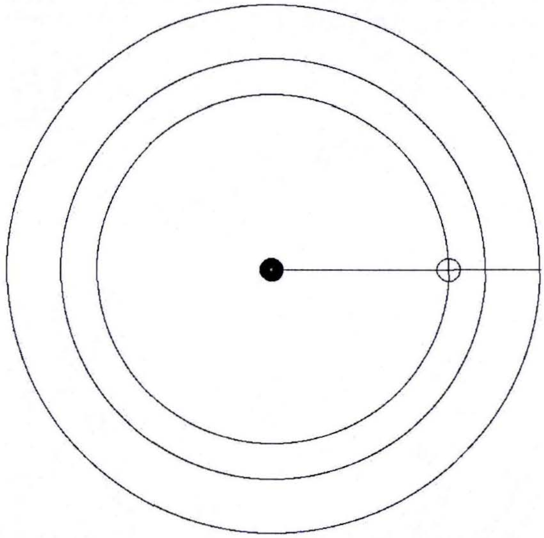

Figure 1: Framework where the simulations of the orbits of the Mercury planet around the Sun are given in the following figures. The black circle represents the position of the Sun at the origin (0, 0) of the orthogonal axes ($\mathbf{e}_1$, $\mathbf{e}_2$), where the horizontal line gives the $\mathbf{e}_1$ axis. The open circle represents the initial position of the Mercury planet at the perihelion. The three successive circles have a radius respectively equal to the perihelion $r_{per} = 45.9$ Mkm, the parameter $p = h^2/\mu = 55.327$ Mkm, and the aphelion $r_{aph} = 69.7$ Mkm.

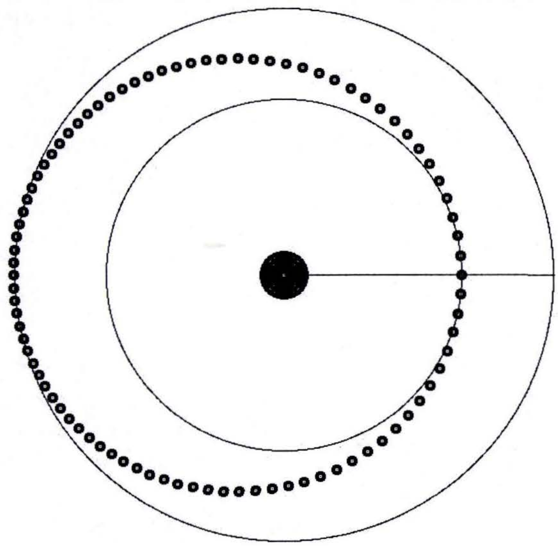

Figure 2: Numerical simulation, with the First Incursive algorithm, of successive revolutions of the Mercury planet around the Sun with the Newton law. The 88 successive positions of the Mercury Planet represent the daily positions during one revolution of 87.97 days, in a closed ellipse, so this incursive algorithm is stable.

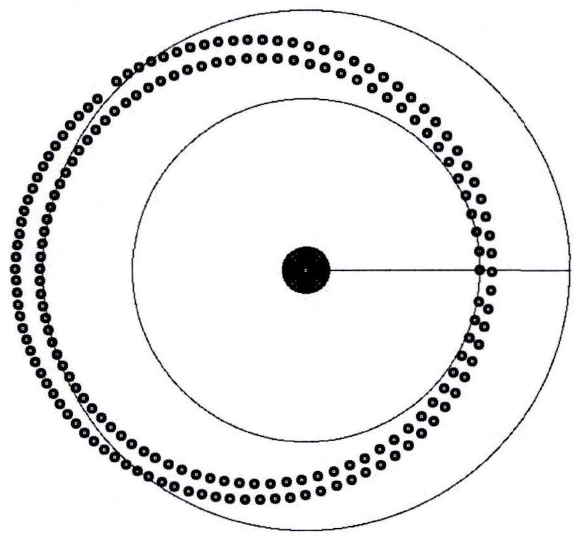

Figure 3: Numerical simulation, with the Euler algorithm of the Newton law of gravitation. This figure shows the 1^{rst} and the 10^{th} revolutions of the Mercury planet around the Sun The orbit is not a closed ellipse, so the Euler algorithm is not stable.

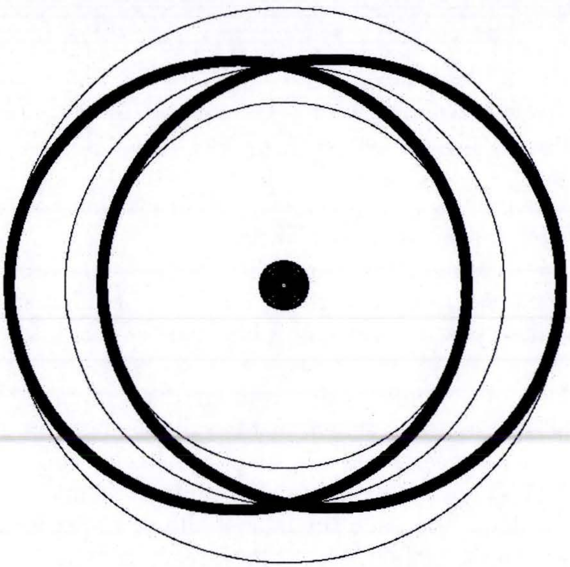

Figure 4: Initial revolution (88 days) of the Mercury planet around the Sun and the revolution after 15,000 centuries. This simulation was performed with the First Incursive algorithm of the relativistic Newton law, which shows an advance of the perihelion of the Mercury planet equal to the correct angle of $\delta\theta = 180$ degrees.

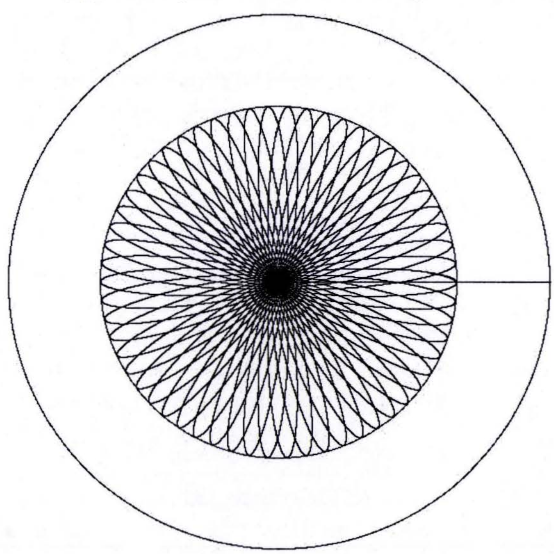

Figure 5: Simulation with the First Incursive algorithm of the relativistic Newton law of gravitation with particular values of the parameters. The mass of the Sun is multiplied by 2,500 and the tangential initial velocity of the Mercury planet is multiplied by 5.025. There are 58 revolutions.

6. Conclusion

This paper firstly recalls the mathematical model of the Newtonian gravitation law applied to the solar system dealing with the Sun and the Mercury planet. The orbits of all the planets are given by closed ellipses in the Newton paradigm.
This paper continues with the introduction of the relativistic correction to this Newton gravitation law, for exhibiting the precession of orbits and, more particularly, the advance of the perihelion of the Mercury planet.
The following section gives numerical algorithms for the simulation of the orbit of the Mercury planet around the Sun. The Euler and the First Incursive algorithms are detailed for the classical Newton gravitation law and for the relativistic correction of this Newtonian law.
The last section of this paper gives the result of the numerical simulations of the Mercury orbit around the Sun. It is shown that the simulation with the Euler algorithm is not stable and does not give a closed ellipse to the Mercury orbit with the Newtonian law. The simulation with the First Incursive algorithm gives a perfect simulation of the Newtonian orbit in 88 days. Then, the simulation with this First Incursive algorithm of the relativistic Newtonian gravitation shows the correct precession of the Mercury orbit. It is demonstrated numerically, with the First Incursive algorithm, that the Mercury precession angle is equal to $\delta\theta = 180$ degrees, after 15,000 centuries, in agreement with the experimental precession angle $\delta\theta_c = 180°/15{,}000 = 43{,}2$ degrees/century.

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