Retarded-advanced Differential Equation in Optimal Economic Growth Models

Adam Krawiec
Institute of Economics and Management, Jagiellonian University,
Gronostajowa 3, 30-387 Kraków, Poland;
Mark Kac Complex Systems Research Centre, Jagiellonian University,
Reymonta 4, 30-059 Kraków, Poland

Marek Szydłowski
Department of Theoretical Physics, Catholic University of Lublin,
Al. Racławickie 14, 20-950 Lublin, Poland;
Mark Kac Complex Systems Research Centre, Jagiellonian University,
Reymonta 4, 30-059 Kraków, Poland

Abstract We analyse the dynamics of simple class of neoclassical growth models with time-to-build. Time-to-build comes from the difference between the investment decisions and delivery of finished capital goods, as it was proposed by Tinbergen and Kalecki. This kind of delay in production of capital goods influence the optimal path of consumption of infinitely living economy. The optimal saving and consumption of households is chosen by the social planner in the way of solving the optimization problem with delay. Due to Kolmanovskii and Myshkis (1999) the classical Pontryagin maximum principle of dynamical optimization can be extended on the class of systems with time delay. The Hamiltonian for such systems can be simple constructed and the optimality condition can be derived. As a result we obtain a forward-looking Euler type equation. We compare the dynamics of economic systems with delay with the dynamics of their counterparts without the delay to show that both admit saddle type solutions. The paper points out the importance of the retarded-advanced dynamical systems in the economic theoretical investigations.
Keywords : economic growth, optimization, retarded-advanced differential equations

1 Introduction

The economic growth is in the centre of interest of economic theory. The history of last three centuries of capitalistic countries shows the enormous increase of life quality. The determinants of growth are therefore important for understanding the dynamics of nations wealth.

The investment lags influence the production process and are very important in the analysis of business cycles because may cause appearing the cyclic behaviour. Tinbergen [15] and Kalecki [7] used the simple linear differential equation with delay to model the business cycle (for another example of business cycle model with delay see [9, 10, 14]).

On the other side when we analyse systems with time delay we find that there may appear both retarded and advanced differential equations. In this context we can say about anticipatory systems [4, 5, 6].

Introducing the concept of investment lags in economic growth theory leads to formulating models in terms of mixed retarded-advanced differential equations [3, 1]. In this paper we show some examples of this kind of models.

The starting of point of economic analysis of economic growth is the classical model of growth developed by Robert Solow in 1956 [13]. He proposed three main factors behind the growth, namely physical capital, labour, and the level of the technology. The analysis of the latter factor gave an impulse to develop the endogenous growth theory with different kinds of knowledge in the form of human capital, knowledge, learning processes, science development [11].

The output in this simple economy is determined by the production function

$$Y = F(A, K, L) \tag{1}$$

and the accumulation of capital is given by

$$\dot{K} = sF(A, K, L) - \delta K \tag{2}$$

where s is the constant rate of savings and δ is the capital depreciation rate.

We introduce the new variable $k = K/L$, capital per worker, and the production function can be expressed in the form

$$Y = F(A, K, L) = LF(A, K/L, 1) = Lf(k). \tag{3}$$

It is assumed that in this form the production function $f(k)$ satisfies $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$, and the so-called Inada conditions $\lim_{k\to 0} f'(k) = \infty$, $\lim_{k\to\infty} = f'(k) = 0$.

Next we assume that the labour grows at the constant rate n

$$\dot{L} = nL \tag{4}$$

and the Solow model takes the form of the one-dimensional dynamical system

$$\dot{k} = sf(k) - (n + \delta)k \tag{5}$$

In the special case if we assume the Cobb-Douglas production function

$$f(k) = Ak^{\alpha} \tag{6}$$

equation (5) has the form of the Bernoulli differencial equation and can be integrated in an exact form after the simple substitution $k \to z = k^{1-\alpha}$.

In the above considerations savings was a fixed part of output. This implies the proportional consumption during the economy development. However, households can adjust their consumption and saving decision in the infinite horizon to maximize the utility derived from consumption during their life [12]. To take into account the problem of optimal consumption we consider it as the problem of dynamic optimization [2]. Let the net investment is given by

$$\dot{k} = Ak^{\alpha}(t) - (n+\delta)k(t) - c(t) \tag{7}$$

where $c(t)$ is consumption per worker. The infinite living household choose such a level of consumption over time to maximize their utility function

$$U(c) = \int_0^{\infty} e^{-\rho t} u(c(t))\, dt \tag{8}$$

where ρ is a discount rate. We assume the constant-relative-risk-aversion (CRRA) utility function

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma} \tag{9}$$

where $1/\sigma$ is the constant elasticity of substitution between consumption in any two moments of time. The utility function fulfils the following conditions: $u'(t) > 0$ and $u''(c) < 0$.

The Pontryagin maximum principle is used to solve the maximization problem

$$\max_c U(c) \qquad \text{with respect to} \qquad \dot{k} = Ak^{\alpha}(t) - (n+\delta)k(t) - c(t) \tag{10}$$

with the initial condition $k(t) = k_0$ for $t = t_0$.

Then we obtain the system of two autonomous differential equations

$$\dot{k}(t) = Ak^{\alpha}(t) - (n+\delta)k(t) - c(t) \tag{11a}$$

$$\dot{c}(t) = \frac{c(t)}{\sigma}(\alpha A k^{\alpha-1}(t) - n - \delta - \rho), \tag{11b}$$

where $\sigma = -\frac{u''(c)c}{u'(c)}$.

The solution of this model is the saddle point. The optimization places an economy on the sepatrices (trajectories incoming to the critical point) which leads to the stationary state of the system

$$k^* = \left(\frac{n+\delta+\rho}{\alpha}\right)^{1/(\alpha-1)} \tag{12a}$$

$$c^* = \left(\frac{n+\delta+\rho}{\alpha}\right)^{\alpha/(\alpha-1)} - (n+\delta)\left(\frac{n+\delta+\rho}{\alpha}\right)^{1/(\alpha-1)}. \tag{12b}$$

2 The Growth Model with Time Delay

When consider the investment process we can see that there is some lag between the investment decision and the realization of investment. The main point is that investment decision are taken in different economic situation and the scale of investment is not necessary adequate to the situation when the investment is finished. In our consideration we assume that the time delay is constant which corresponds to the average time of investment in the economy.

To apply the idea of investment delay in the Solow type of dynamical system we assume that the output at time t depends on the capital stock available at the time $t - \tau$

$$Y(t) = F(A, K(t-\tau), L(t-\tau)) \tag{13}$$

The capital accumulation equation is

$$\dot{K}(t) = AK^{\alpha}(t-\tau)L^{1-\alpha}(t-\tau) - \delta k(t-\tau) - c(t) \tag{14}$$

where $k(t) = \varphi(t)$ for all $t \in [-\tau, 0]$ is an initial condition. $\varphi(t)$ is called an initial capital function, which is taken exogenously as given.

Additionally for simplicity we assume that the stock of labour is constant

$$\frac{\dot{L}}{L} = 0 \quad \Rightarrow \quad L = \text{const.}$$

Introducing the variables per worker $k = K/L$, $y = Y/L$ and $c = C/L$ we obtain

$$\dot{k}(t) = Ak^{\alpha}(t-\tau) - \delta k(t-\tau) - c(t). \tag{15}$$

As in previous section we assume that households maximize their utility function (8) in the constant-relative-risk-aversion (CRRA) form

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma} \tag{16}$$

where $1/\sigma$ is the constant elasticity of substitution between consumption in any two moments of time. The utility function fulfils the following conditions: $u'(t) > 0$ and $u''(c) < 0$.

Now the maximalization problem is the following

$$\max_{c} U(c) \qquad \text{with respect to} \qquad \dot{k}(t) = Ak^{\alpha}(t-\tau) - \delta k(t-\tau) - c(t) \tag{17}$$

with the initial condition $k(t) = \varphi t$ for $t \in (-\tau, 0)$.

We solve this problem using the Pontryagin maximum principle for delay differential equations (for details see [8])

$$\dot{k}(t) = Ak^{\alpha}(t-\tau) - \delta k(t-\tau) - c(t) \tag{18a}$$

$$\dot{c}(t) = \frac{c(t)}{\sigma}\left\{\left[\alpha Ak^{\alpha-1}(t) - \delta\right]\left(\frac{c(t)}{c(t+\tau)}\right)^{\sigma} e^{-\rho\tau} - \rho\right\}. \tag{18b}$$

We can see that the system is the form of retarded-advanced differential equations. The advanced argument arise in the model as the result of optimization of consumption with respect to gestation lag in the production function. Collard et al. [3] investigated this model numerically and obtain the oscillatory behaviour of consumption and capital stock.

3 The Growth Model with Endogenous Knowledge

In this section we consider the extension of the previous model. We consider that the knowledge dynamics is endogenously treated. We assume that the knowledge stock used in production is proportional to the amount of the capital stock per worker

$$A = A(k(t)) = ak(t) \tag{19}$$

where $a < 1$ is a positive constant.

Here we assume that there is a fixed delay in the production function which means that investment realized in time t are started with knowledge and capital available in time $t-\tau$. This assumption gives the equation for capital accumulation in the form

$$\dot{k}(t) = ak(t-\tau)k^{\alpha}(t-\tau) - \delta k(t-\tau) - c(t) = ak(t-\tau)^{1+\alpha} - \delta k(t-\tau) - c(t). \tag{20}$$

Adopting the analogous optimizing procedure we obtain the retarded-advanced differential equation system

$$\dot{k}(t) = ak^{1+\alpha}(t-\tau) - \delta k(t-\tau) - c(t) \tag{21a}$$

$$\dot{c}(t) = \frac{c(t)}{\sigma}\left\{[(1+\alpha)ak(t)^{\alpha} - \delta]\left[\frac{c(t)}{c(t+\tau)}\right]^{\sigma} e^{-\rho\tau} - \rho\right\}. \tag{21b}$$

The steady state of the system are the same like for the system without delay ($\tau = 0$). One can show that there at least one critical point with positive k_0 which is given by

$$k^* = \left[\frac{\rho e^{\rho\tau} + \delta}{(1+\alpha)a}\right]^{1/\alpha} \tag{22a}$$

$$c^* = a(k^*)^{1+\alpha} - \delta k^* \tag{22b}$$

We can linearize system (21) around the steady state solution (22) and then compute the corresponding characteristic equation

$$h(\lambda) = \lambda^2 - C_1 e^{-\rho\tau}\lambda + C_2 C_1^2 e^{-\rho\tau} - C_1\lambda e^{-\tau\lambda} + C_1^2 e^{-\rho\tau} e^{-\tau\lambda} + C_1 e^{-\rho\tau}\lambda e^{\tau\lambda} = 0, \tag{23}$$

where

$$C_1 = (\alpha+1)a(k^*)^{\alpha} - \delta, \qquad C_2 = \frac{\alpha(\alpha+1)a}{\sigma}(k^*)^{\alpha-1}c^* e^{-\rho\tau}.$$

Since the characteristic equation is transcendental with an infinite number of roots, the steady state point is in a generic case a saddle. The convergence to the steady state is governed by the smallest negative real eigenvalue.

4 Summary

The main aim of the paper was to point out that anticipatory dynamical systems appear naturally in the context of dynamical optimization of delay differential equation. We demonstrate it on the example of a simple model of economic growth with the endogenous knowledge. In this model the optimizing households choose the rate of consumption which depends on future (anticipated) consumption. It is characteristic that the time lag in production coincide with forward time in consumption.

We studied this model and found the solution (steady state) of this model. The consumption path satisfies the first-order Euler-type equation. After adopting the Kolmanovskii-Mishkis approach to the standard maximum principle the mixed retarded-advanced dynamical systems appears in a natural way. This analysis pointed out on the importance of retarded-advanced dynamical systems in the context of analysis of growth theory with dynamical optimization. The next step is to study the possibility of cyclic behaviour in the model.

Acknowledgments

The paper was supported in part by the Marie Curie Actions Transfer of Knowledge project COCOS (contract MTKD-CT-2004-517186) and the KBN grant no. N111 003 31.

References

[1] M. Bambi, Endogenous growth and time-to-build: The AK case. *Journal of Economic Dynamics & Control*, **32**, 1015–1040, 2008.

[2] A. C. Chiang, *Elements of Dynamic Optimization* McGraw-Hill, New York, 1992.

[3] F. Collard, O. Licandro, and L. A. Puch, The short-run dynamics of optimal growth models with delays. (2004), *European University Institute*, Economics Working Papers ECO2004/04.

[4] D. M. Dubois, Theory of computing anticipatory systems based on differential delayed-adavanced difference equations. In: *Computing Anticipatory Systems: CASYS 2001 – Fifth International Conference.* AIP Conference Proceedings 627, 3–16, 2002.

[5] D. M. Dubois, Mathematical foundations of discrete and functional systems with strong and weak anticipation. In: *Anticipatory Behavior in Adaptive Learning Systems, State-of-the-Art Survey*, ed. by M. Butz, O. Sigaud, and P. Gérard, Lecture Notes in Artificial Intelligence **2684**, 110–132, Springer, 2003.

[6] D. M. Dubois, Extension of the Kaldor-Kalecki model of business cycle with a computational ancipated capital stock. *Journal of Organisational Transformation and Social Change*, **1**, 63-80, 2004.

[7] M. Kalecki, A macrodynamic theory of business cycles, *Econometrica*, **3**, 327–344, 1935

[8] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations.* Kluwer, Dordrecht, 1999.

[9] A. Krawiec and M. Szydlowski, The Kaldor-Kalecki business cycle model. *Annals of Operations Research*, **89**, 89–100, 1999.

[10] A. Krawiec and M. Szydlowski, On nonlinear mechanics of business cycle model. *Regular and Chaotic Dynamics*, **6**, 101–118, 2001.

[11] N. G. Mankiw, D. Romer, and D. N. Weil, A contribution to the empirics of economic growth. *Quarterly Journal of Economics* **107**, 407–437 (1992).

[12] F. P. Ramsey, A mathematical theory of saving. *Economic Journal*, **38**, 543–559, 1928.

[13] R. M. Solow, A contribution to the theory of economic growth. *Quarterly Journal of Economics* **70**, 65–94 (1956).

[14] M. Szydlowski, A. Krawiec, and J. Tobola, Nonlinear oscillations in business cycle model with time lags. *Chaos, Solitons & Fractals*, **12**, 505–517, 2001.

[15] J. Tinbergen, Ein Schiffbauzyklus? *Weltwirtschaftliches Archiv*, **34**, 152–164, 1931.