Image Restoration by Multiscale Spatial Adaptive Regularization

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Abstract

An image restoration is a typically ill-posed problem. Generally, regularization scheme is used to avoid this problem. As a regularization operator, classical methods adopt one which may produce a too much smooth image. Parametric Projection Filter which has an ability to deal with colored observation noise is one of them.

On the other hand, some methods based on a spatially adaptive regularization are proposed and successful in obtaining not so smooth one. However, it is assumed that observation noise is white, and the fidelity of images is not evaluated in the space of original images in these methods.

In this paper, we propose a new restoration method by which we can evaluate the fidelity of images in the space of original images and obtain not so smooth one. We also verify the efficacy of the method by some numerical experiments.

Keywords : Restoration, Optimization, Multiscale Signal Processing, Adaptive Processing, Regularization

1 Introduction

An image degradation is modeled as follows (Oja, et al., 1986)

$$g = Af + n, \quad f \in \mathbf{R}^n, \ g, n \in \mathbf{R}^m; A \in \mathbf{R}^{m\times n}, \tag{1}$$

where f, g, A, and n denote an original image, a degraded image, a degradation matrix, and observation noise (may be colored), respectively and $\mathbf{R}^n$, $\mathbf{R}^m$, and $\mathbf{R}^{m\times n}$ denote an n-dimensional real vector space (we call it the space of original images, hereafter), an m-dimensional real vector space (we call it the space of degraded images, hereafter), and the set consisting of all $m \times n$ real matrices. An aim of the image restoration is to obtain the optimum image under some optimization criteria.

The image restoration is a typically ill-posed problem. Generally, regularization scheme (Tikhonov, et al., 1977) is used to avoid this problem. As a regularization operator, classical methods adopt one which may produce a too much smooth image. Parametric Projection Filter (Oja, et al., 1986) which has an ability to effectively deal with colored observation noise in the space of original images is one of them.

On the other hand, some methods based on a spatially adaptive regularization are proposed (You, et al., 1996) and successful in obtaining not so smooth one. However it is assumed that observation noise is white and the fidelity of images is not evaluated in the space of original images in these methods as mentioned in (Oja, et al., 1986).

In this paper, we propose a restoration method by which we are able to deal with colored observation noise in the space of original images and obtain not so smooth one by applying the multiscale spatially adaptive regularization scheme to Parametric Projection Filter. We also verify the efficacy of the proposed method by some numerical experiments.

2 Image Restoration by Parametric Projection Filter

Parametric Projection Filter (we call it PPF, hereafter) is defined as an operator $B \in \mathbf{R}^{n \times m}$ which minimizes the functional J_1 for some real parameter γ (> 0).

$$J_1(B) = tr\{(I_n - BA)(I_n - BA)'\} + \gamma E\boldsymbol{n}||B\boldsymbol{n}||^2, \tag{2}$$

where $tr\{\cdot\}$, A', I_n, $E\boldsymbol{n}$, and $||\cdot||$ denote the trace of a matrix, the transpose matrix of A, an $n \times n$ identity matrix, the expectation for $\boldsymbol{n}$, and a norm of a vector, respectively. Eq.2 is transformed as follows with the squared Schmidt norm.

$$J_1(B) = tr\{(I_n - BA)(I_n - BA)'\} + \gamma tr\{BQB'\}, \tag{3}$$

where Q denotes the covariance matrix of additive noise $\boldsymbol{n}$.

As described in (Oja, et al., 1986), the criterion eq.3 is minimized by an operator B if and only if

$$B(AA' + \gamma Q) = A'. \tag{4}$$

An operator which satisfies the condition eq.4 is obtained as

$$B_{PPF}(\gamma) = A'(AA' + \gamma Q)^{+}, \tag{5}$$

where A^+ denotes the Moore-Penrose inverse of A.

Many restoration methods adopt the term $||\boldsymbol{g} - A\boldsymbol{f}||^2$ as a part of criterion which should be minimized. However this term means the energy of noise component in the space of degraded images. Hence, it does not contribute to the fidelity of images in the space of original images, while the energy of noise components in the space of original images are controlled by an effect of the second term of eq.3 in PPF.

3 Image Restoration by Anisotropic Diffusion Scheme

As a regularization operator, classical methods usually adopt an operator which may smooth a restored image too much. Constrained Least Squares Filter which is one of them uses the Laplacian as the operator by which the noise component is effectively suppressed but the edge information of images may be lost.

As a restoration method of preserving the edge information of images, anisotropic diffusion (or regularization) scheme is proposed (You, et al., 1996). As the criterion for minimizing, You adopted

$$L(f) = \frac{1}{2}\int e^2(x,y)dxdy + \lambda \int B(|\nabla f(x,y)|)dxdy, \qquad (6)$$

subject to some conditions, where

$f(x,y)$	:	original image
$\hat{f}(x,y)$	:	restored image
$d(x,y)$	:	blur operator (shift-invariant)
$n(x,y)$	:	observation noise
$g(x,y)$	:	degraded image

$$g(x,y) = \int d(s,t)f(x-s,y-t)dsdt + n(x,y) \quad : \text{ degradation model}$$
$$e(x,y) = g(x,y) - \int d(s,t)\hat{f}(x-s,y-t)dsdt \quad : \text{ restoration residual.}$$

Minimizing the criterion eq.6 is achieved by steepest descent method. The gradient of $L(\hat{f})$ is described as follows

$$\nabla L(\hat{f}) = -\int e(u,v)d(u-x,v-y)dudv - \lambda \mathbf{div}\left(B'(|\nabla \hat{f}|)\frac{\nabla \hat{f}}{|\nabla \hat{f}|}\right), \qquad (7)$$

for some real parameter λ (> 0). The second term of eq.7 is decomposed to two components. One is that of the direction of the gradient of $\hat{f}$, the other is that of the orthogonal direction of the gradient of $\hat{f}$. Hence, an appropriate selection of the function B in eq.6 produces a restored image whose noise component is effectively suppressed on flat region while it may preserves the edge information of the image.

However, the fidelity of images in the space of original images is not insured by the criterion eq.6 as mentioned in a previous section.

4 The Proposed Method

In this paper, we propose a new restoration method which has the advantage of previous two methods as described bellow

- We construct a basic restoration form (we call it Regularized Parametric Projection Filter (RPPF), hereafter) by which we can consider observation noise in the space of original images and some features of images.
- Based on RPPF, we propose a total restoration filter in which we obtain a restored image and regularization operator step by step.

Hereafter, we describe details about these two topics.

4.1 Regularized Parametric Projection Filter

As a criterion for minimizing, we adopt the functional J_2.

$$J_2(B) = tr\{(I_n - BA)(I_n - BA)'\} + \gamma E\boldsymbol{n}||B\boldsymbol{n}||^2 + \lambda||RB\boldsymbol{g}||^2. \tag{8}$$

for some real parameters $\gamma, \lambda(> 0)$, where R denotes some regularization operator. Eq.8 is described as follows with the squared Schmidt norm in the same way as PPF.

$$J_2(B) = tr\{(I_n - BA)(I_n - BA)'\} + \gamma tr\{BQB'\} + \lambda tr\{RB\boldsymbol{g}\boldsymbol{g}'B'R'\}. \tag{9}$$

Theorem 1 The criterion J_2 is minimized by an operator B_{RPPF} if and only if

$$B_{RPPF}(AA' + \gamma Q) + \lambda R'RB_{RPPF}\boldsymbol{g}\boldsymbol{g}' = A'. \tag{10}$$

Proof:

1) Eq.9 is transformed as follows

$$\begin{aligned} J_2(B) = & \{\mathbf{vec}(I_n)\}'\mathbf{vec}(I_n) - 2\{\mathbf{vec}(B')\}'\mathbf{vec}(A) \\ & + \{\mathbf{vec}(B')\}'[I_n \otimes (AA' + \gamma Q) + \lambda(R'R) \otimes (\boldsymbol{g}\boldsymbol{g}')]\mathbf{vec}(B') \end{aligned} \tag{11}$$

by applying the relation (Magnus, et al., 1988)

$$tr\{A'B\} = \{\mathbf{vec}A\}'\mathbf{vec}B,$$

where **vec** and $(\cdot \otimes \cdot)$ denote vec operator and Kronecker product (Magnus, et al., 1988) and are defined as follows respectively,

$$\begin{cases} A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}, \\ \mathbf{vec}(A) = [\boldsymbol{a}_1{}' \ \boldsymbol{a}_2{}' \cdots \boldsymbol{a}_n{}']', \end{cases} \tag{12}$$

for

$$A = (a_{ij}) = [\boldsymbol{a}_1 \ \boldsymbol{a}_2 \cdots \boldsymbol{a}_n], \ \boldsymbol{a}_i \in \mathbf{R}^m,$$
$$B = (b_{ij}), \ a_{ij}, \ b_{ij} \in \mathbf{R}.$$

Assume that B_{RPPF} satisfies

$$\{I_n \otimes (AA' + \gamma Q) + \lambda(R'R) \otimes (\boldsymbol{g}\boldsymbol{g}')\}\mathbf{vec}(B'_{RPPF}) = \mathbf{vec}(A),$$

which is obtained by applying the relation (Magnus, et al., 1988)

$$\mathbf{vec}(ABC) = (C' \otimes A)\mathbf{vec}(B)$$

to the transpose of eq.10. Therefore $J_2(B_{RPPF})$ is described as follows

$$J_2(B_{RPPF}) = \{\mathbf{vec}(I_n)\}'\mathbf{vec}(I_n) - \{\mathbf{vec}(B'_{RPPF})\}'\mathbf{vec}(A).$$

Let $G = [I_n \otimes (AA' + \gamma Q) + \lambda(R'R) \otimes (\boldsymbol{g}\boldsymbol{g}')]$ for brevity and B be any matrix of the same dimensionality as B_{RPPF}. It holds

$$\begin{aligned} & J_2(B) - J_2(B_{RPPF}) \\ & = \{\mathbf{vec}(B')\}'G\mathbf{vec}(B') + [\mathbf{vec}(B'_{RPPF}) - 2\mathbf{vec}(B')]'\mathbf{vec}(A) \\ & = \{\mathbf{vec}(B')\}'G\mathbf{vec}(B') + \{\mathbf{vec}(B'_{RPPF})\}'G\mathbf{vec}(B'_{RPPF}) \\ & \quad -2\{\mathbf{vec}(B')\}'G\mathbf{vec}(B'_{RPPF}) \\ & = [\mathbf{vec}(B') - \mathbf{vec}(B'_{RPPF})]'G[\mathbf{vec}(B') - \mathbf{vec}(B'_{RPPF})], \end{aligned}$$

which is nonnegative since G is nonnegative definite. Hence, it is proved that $J_2(B) \geq J_2(B_{RPPF})$ for all B with the assumption eq.10.

2) Assume that $J_2(B)$ is minimized at B_{RPPF}. Then B_{RPPF} is a stationary point of the functional $J_2(B)$ and it holds

$$\frac{\partial J_2(B)}{\partial B}|_{B=B_{RPPF}} = 2[B_{RPPF}(AA' + \gamma Q) + \lambda R'RB_{RPPF}\boldsymbol{gg}' - A'] = 0.$$

This yields eq.10, which concludes the proof.

A candidate for B_{RPPF} is

$$B_{RPPF}(\gamma, \lambda) = \mathbf{unvec}[\{(AA' + \gamma Q) \otimes I_n + \lambda(\boldsymbol{gg}') \otimes (R'R)\}^{+}\mathbf{vec}(A')], \quad (13)$$

where **unvec** denotes the inverse operator of **vec**.

4.2 Multiscale Adaptive Restoration

We treat the restoration problem in the wavelet domain (n stages MRA (Mallat, 1989)). Image signals are decomposed to a quad-tree structure by wavelet transform as shown in Fig.1.

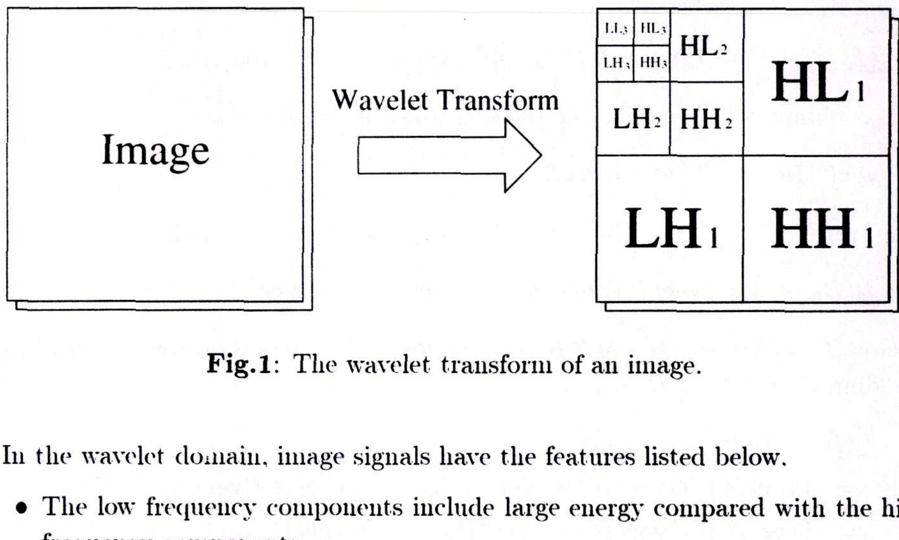

Fig.1: The wavelet transform of an image.

In the wavelet domain, image signals have the features listed below.

- The low frequency components include large energy compared with the high frequency components.
- There exists some correlation between two successive components. (For instance, LH_1 and LH_2.)

On the other hand, degradation operators are usually low-pass filters in the practical situation. Thus it is expected that the low frequency components are restored more accurately than the high frequency components and that some information about already estimated components contribute to getting others. We construct a new restoration algorithm based on the issues mentioned above.

4.2.1 Image Degradation in the Wavelet Domain

Let $W_{n...1}$ be an orthogonal wavelet transform operator (n stages MRA (Mallat, 1989)). $W_{n...1}$ is written as below

$$W_{n...1} = W_n W_{n-1} \cdots W_1. \tag{14}$$

where W_i ($i = 1, \ldots, n$) denotes a wavelet transform operator by which we obtain the i-th stage components from the $(i-1)$-th stage components. Therefore we have eq.1 as follows

$$(W_{n...1}\boldsymbol{g}) = (W_{n...1}AW'_{n...1})(W_{n...1}\boldsymbol{f}) + (W_{n...1}\boldsymbol{n}). \tag{15}$$

in the wavelet domain, since $W_{n...1}$ is an orthogonal matrix. In this formulation, covariance matrix of additive noise $W_{n...1}\boldsymbol{n}$ is written as follows

$$Q_W = W_{n...1}QW'_{n...1}.$$

and also the matrix expression of an original image $W_{n...1}\boldsymbol{f}$ is as follows

$$F_W^n = \begin{bmatrix} F_W^{LL,n} & F_W^{HL,n} & \cdots & \\ F_W^{LH,n} & F_W^{HH,n} & & F_W^{HL,1} \\ & \vdots & \ddots & \\ & F_W^{LH,1} & & F_W^{HH,1} \end{bmatrix}.$$

where $F_W^{LL,i}$, $F_W^{LH,i}$, $F_W^{HL,i}$, and $F_W^{HH,i}$ denote LL, LH, HL, and HH-component of i-th stage in the wavelet domain. And we define a vector expression of $W_{n...1}\boldsymbol{f}$ as follows

$$W_{n...1}\boldsymbol{f} = \mathbf{vec}(F_W^n).$$

4.2.2 Image Restoration by Multiscale RPPF

We propose a restoration algorithm described below based on RPPF.

Step.1 Transform the degradation model to the wavelet domain as eq.15.

Step.2 Obtain a restored image by $B_{RPPF}(\gamma, 0)$ which is equal to PPF in the wavelet domain (n-th stage).

Step.3 Let $k = n$.

Step.4 Make a regularization operator R_{k-1} from restored component $F_W^{LL,k}$. (discussed in a later section)

Step.5 Construct $(k-1)$-th stage degradation model by premultiplying by W_k^t to eq.15.

Step.6 Obtain a restored image by $B_{RPPF}(\gamma, \lambda)$ in the wavelet domain ($(k-1)$-th stage) and let $k \leftarrow k-1$.

Step.7 If $k > 0$ then repeat Steps.4 $\sim$ 6.

Thus we obtain a final restored image. The outline of this algorithm is shown in Fig.2.

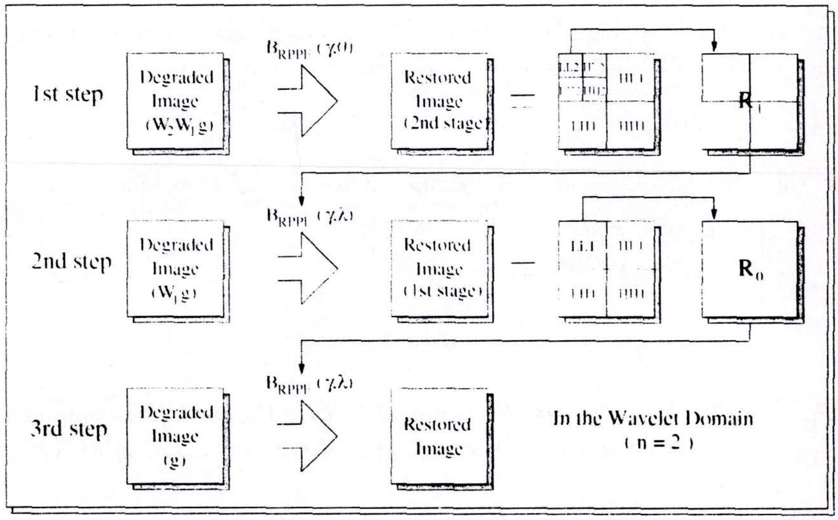

Fig.2: Outline of the Multiscale RPPF.

4.2.3 Construction of Regularization Operator

Our objective to use a regularization operator is to reduce the influence of noise on flat regions while preserving the edge information of images. In order to realize this, we adopt one which consists of a weighted sum of the second differential of LL-component.

Let $\nabla F_W^{LL,k}(i,j)$ be the gradient of $F_W^{LL,k}(i,j)$,which is described as follows

$$\nabla F_W^{LL,k}(i,j) = \begin{bmatrix} F_W^{LL,k}(i,j_n) - F_W^{LL,k}(i,j) \\ F_W^{LL,k}(i_n,j) - F_W^{LL,k}(i,j) \end{bmatrix}.$$

where $i_n = \min(i+1, r_k)$, $j_n = \min(j+1, c_k)$ for $F_W^{LL,k} \in R^{r_k \times c_k}$. A vector field consisting of the orthogonal vector of the gradient $\nabla F_W^{LL,k}(i,j)$ is

$$O_W^{LL,k}(i,j) = \begin{bmatrix} O_W^{LL,k}(i,j)_h \\ O_W^{LL,k}(i,j)_v \end{bmatrix} = \begin{bmatrix} -F_W^{LL,k}(i_n,j) + F_W^{LL,k}(i,j) \\ F_W^{LL,k}(i,j_n) - F_W^{LL,k}(i,j) \end{bmatrix},$$

and that of $(k-1)$-th stage ($O_W^{LL,k-1}(i,j)$) is obtained by the linearly interpolating. We define the weight matrices as follows

$$W_h^{k-1}(i,j) = \begin{cases} |O_W^{LL,k-1}(i,j)_h| / \|O_W^{LL,k-1}(i,j)\| & : \ if\ \|O_W^{LL,k-1}(i,j)\| > T, \\ 1/\sqrt{2} & : \ otherwise, \end{cases}$$
$$W_v^{k-1}(i,j) = \begin{cases} |O_W^{LL,k-1}(i,j)_v| / \|O_W^{LL,k-1}(i,j)\| & : \ if\ \|O_W^{LL,k-1}(i,j)\| > T, \\ 1/\sqrt{2} & : \ otherwise, \end{cases}$$

where T denotes some real number for the threshold. Finally, we construct the operator R_{k-1} consisting of the weighted sum of the second differential based on $W_x(i,j)$ and $W_y(i,j)$ as follows

$$\begin{aligned} &\mathbf{unvec}[R_{k-1}\mathbf{vec}(F_W^{k-1})](i,j) \\ &= \begin{cases} \begin{array}{l} W_h^{k-1}(i,j)(\ F_W^{LL,k-1}(i,j_p) \\ \quad -2F_W^{LL,k-1}(i,j) + F_W^{LL,k-1}(i,j_n)\ ) \\ +W_v^{k-1}(i,j)(\ F_W^{LL,k-1}(i_p,j) \\ \quad -2F_W^{LL,k-1}(i,j) + F_W^{LL,k-1}(i_n,j)\ ) \end{array} & : \ if \begin{cases} 1 \le i \le r_{k-1} \\ 1 \le j \le c_{k-1}, \end{cases} \\ 0 & : \ otherwise, \end{cases} \end{aligned}$$

where $i_n = \min(i+1, r_{k-1})$, $j_n = \min(j+1, c_{k-1})$, $i_p = \max(i-1, 1)$, $j_p = \max(j-1, 1)$ for $F_W^{LL,k-1} \in R^{(r_{k-1} \times c_{k-1})}$.

5 Numerical Examples

In this section, we show some numerical examples to confirm the efficacy of the proposed method.

5.1 Example 1

Figure 3 is an original image(16 × 16 pixels, 256 gray scale) and Fig.4 is the degraded image which is obtained by averaging 7 pixels vertically and adding noise which is horizontally white and whose vertical covariance matrix (16 × 16) is described as follows

$$Q = \begin{bmatrix} 12.5 & 6.25 & 0 & \cdots & 0 & 6.25 \\ 6.25 & 12.5 & 6.25 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 6.25 & 12.5 & 6.25 \\ 6.25 & 0 & \cdots & 0 & 6.25 & 12.5 \end{bmatrix}.$$

Figures 5, 6, and 7 are the restored images by Moore-Penrose inverse (MPI), PPF, and the proposed method (MRPPF). The parameters for PPF and MRPPF are those by which SNR of restored images are maximized. γ(for PPF) is 0.0011. γ, λ, and T (for MRPPF) are 0.00041, 3.9×10^{-8}, and 1, respectively. We adopt Haar wavelet for MRPPF. The number of stage in MRPPF is 2. We show SNR of restored images in Table 1.

5.2 Example 2

Figure 8 is an original image (32×32 pixels, 256 gray scale) and Fig.9 is the degraded image which is obtained by averaging 15 pixels vertically and the nature of noise is same as Example 1, except the size of covariance matrix (32×32).

Figures 10, 11, and 12 are the restored images by MPI, PPF, and MRPPF. We use same parameters in Example 1. We show SNR of restored images in Table 2.

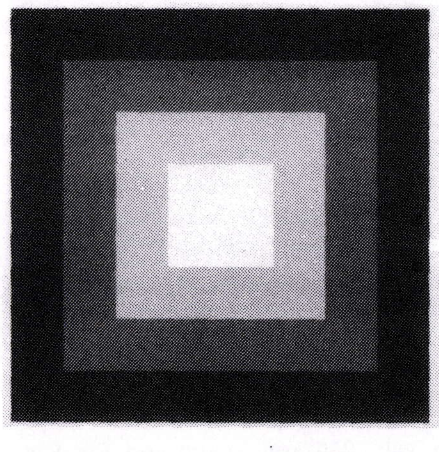

Fig.3: An original image.

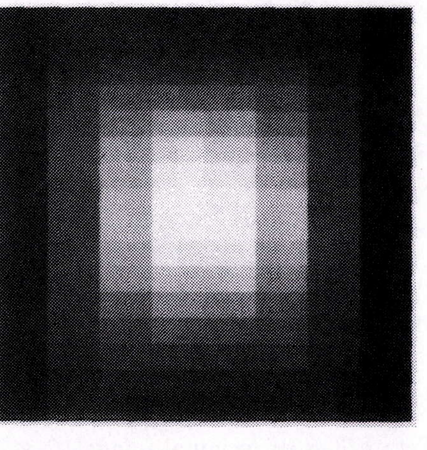

Fig.4: A degraded image.

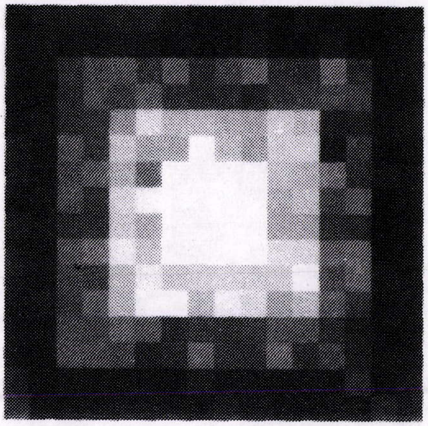

Fig.5: A restored image by A^+.

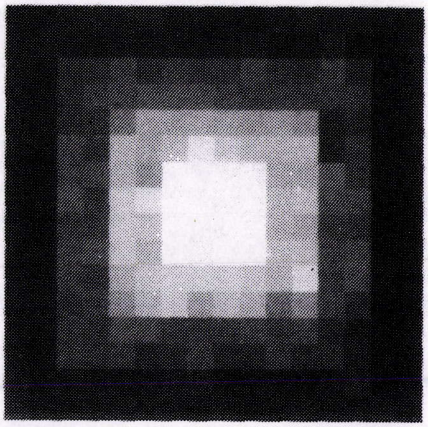

Fig.6: A restored image by PPF.

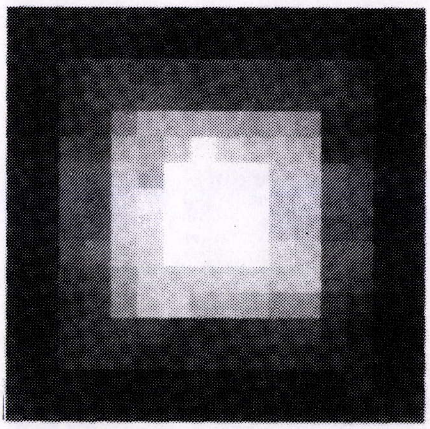

Fig.7: A restored image by MRPPF.

Fig.8: An original image.

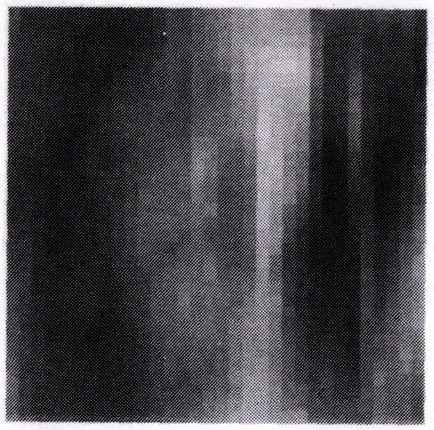

Fig.9: A degraded image.

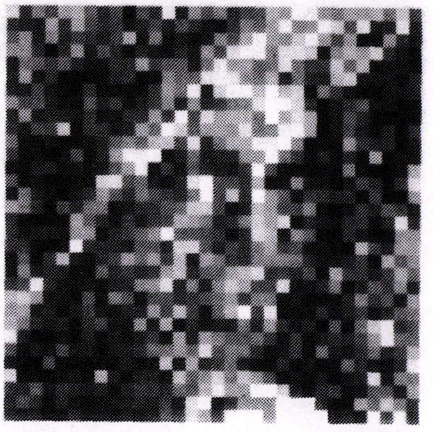

Fig.10: A restored image by A^+.

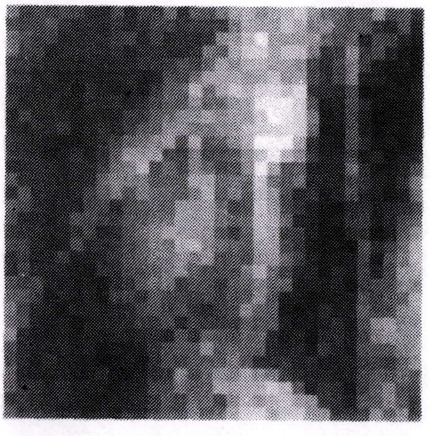

Fig.11: A restored image by PPF.

Fig.12: A restored image by MRPPF.

Table 1: SNR of restored images(Example 1).

Method	SNR(dB)
MPI	13.6
PPF	16.1
MRPPF	16.6

Table 2: SNR of restored images(Example 2).

Method	SNR(dB)
MPI	6.4
PPF	12.1
MRPPF	13.0

6 Conclusion

In this paper, we constructed a new basic restoration form by which we can effectively deal with observation noise in the space of original images and use some regularization scheme simultaneously by extending Parametric Projection Filter. And based on it, we proposed a restoration filter in which we estimate an effective regularization operator step by step with multiscale adaptive estimation approach and we also verified that the proposed method is effective by some numerical experiments.

References

J. R. Magnus and H. Neudecker (1988). *Matrix Differential Calculus with Applications in Statistics and Econometrics.* Wiley.

S. Mallat (1989). A theory for multiresolution signal decomposition: The wavelet representation. *IEEE Trans. Pattern Anal.& Mach. Intell.*, Vol. 11, No. 7, pp. 674–693.

E. Oja and H. Ogawa (1986). Parametric projection filter for image and signal restoration. *IEEE Trans. Acoustics, Speech & Signal Process*, Vol. ASSP-34, No. 6, pp. 1643–1653.

A. Tikhonov and V. Arsenin (1977). *Solutions of Ill-posed Problems.* Wiley.

Y. You and M. Kaveh (1996). Ringing Reduction in Image Restoration by Orientation-Selective Regularization. *IEEE Signal Processing Letters*, Vol.3, No. 2, pp. 29–31.