Anticipation at the Juncture of Geometry and Calculus

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Abstract
The structure "Finslerian teleparallelism" might have been anticipated through a deeper implementation of the ideas that led to great progress in differential geometry in the 20th century. That structure's significance is manifested through the Kähler calculus of differential forms. Based on Clifford algebra, this calculus supersedes Élie Cartan's. It revolves around Kähler's equation, a generalization of Dirac's. The juncture of geometry and the calculus is to be understood in the sense that, through the aforementioned implementation, one can create a Kaluza-Klein type structure where the torsion part of the structural equations is given by a fully geometric Kähler equation. Its input is the differential form whose exterior covariant derivative is precisely the torsion in its role as output differential form, thus yielding a closed geometric system of structural equations.
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1. Introduction

This paper connects a particular geometry and a version of the calculus of Kähler (1960-1962) developed by the present authors. Our goal here is to show that one can obtain in an anticipatory way the key equation for the tasks of identifying geometry and physics and joining the gravitational and non-gravitational sectors of the physics. In view of modern mathematical knowledge, the existence of such junction, which takes the form of an equation, can be anticipated from the physics itself. Indeed, Einstein's gravitation is related to the second equation of structure, i.e. the specification of the curvature(s). Metric-compatible teleparallelism (TP) specifies the metric curvature in terms of torsion and derivatives thereof. The torsion still remains to be specified. An anticipated geometric unification of gravitation and quantum physics then leads one to anticipate that the specification of the torsion will have to take place through a quantum mechanical equation, namely the Kähler equation since this is the "Dirac equation in the calculus of differential forms, which is the calculus of geometry. In this paper, we deal with the development of the aforementioned KK space, and of the Kähler equation which will replace its first equation of structure.

In the accompanying paper, we proceed with the emergence of physics from the resulting junction of geometry and calculus. In the same way as there are the anticipations of Rosen (1985) and of Dubois (2000), respectively for the theories of physical and living systems, the "natural science" that emerges as a step between mathematics and physics appears to be a mother of the physics, namely one more step in the ladder of natural sciences. We leave it for history to decide whether it is "the mother", while remarking that, in order to be useful in connection with the issues of emergence and anticipation, this theory need not be the holy grail sought by physicists. We refer to it as demiurgism, for reasons that will become clear in the second paper. Its roots are in the attempt at physical unification with TP by Einstein (1930). In simple words, TP means that the affine connection or rule to compare vectors from tangent spaces at different points (of spacetime in our case) yields results that are independent of the path used for their comparison. More precisely, TP is the requirement of true equality of tangent vectors at different points of a differentiable manifold. Equality, being an equivalence relation, requires the transitive property and is thus incompatible with making an affine connection with the Christoffel symbols. Some progress in that incipient effort was made, however, by Cartan in trying to help Einstein, who did neither make any progress himself nor appreciate Cartan's (Debever, 1979).

The highly reductionist term "mother of the physics" emerges in our case from simply adopting another Einstein thesis: the logical homogeneity of physics and geometry (Einstein, 1934). Reductionism complements here emergence, as they are to each other like the two sides of a coin. Every physical concept, law, structure or even physical theory (electrodynamics, gravitation, quantum mechanics, classical mechanics) has to emerge from it. Hence, emergence is here far more encompassing than in collective organizational phenomena. Of course, this is a program rather than a theory, but it has already delivered much, relative to manpower invested.

The contents of the paper reflects the fact that, some of the topics are treated in greater detail in preprints containing our presentations in other conferences (Vargas-Torr, 2005a,b,c,d). In section 2, we deal with that most relevant Cartan perspective of the evolution of geometry, starting with Riemann and up to the late 1920's. We do not know of any presentation of this perspective (expanded in Vargas-Torr, 2005c) other than in Cartan's own papers, of renown difficulty. See in and by Gardner (1989, Introduction). Section 3 summarizes the scarcely known formulation by Clifton (Vargas-Torr, 1993) of the theory of Finsler bundles. It makes Finsler geometry look like pre-Finsler geometry, specially if one uses differential forms rather than their components (say $\omega_\mu{}^\nu$ rather than $\Gamma_\mu{}^\nu{}_\lambda$ in $\omega_\mu{}^\nu=\Gamma_\mu{}^\nu{}_\lambda\omega^\lambda$, as the latter expression is not sufficiently general for Finsler geometry; the ω^λ are non-holonomic in the Cartan calculus, i.e. $d\omega^\lambda\neq 0$, except when dealing with coordinate basis fields, $\omega^\nu=dx^\nu$). The first part of section 4 is for the benefit of readers who are not familiar with Clifford algebra. The second part deals with the basics of the Kähler calculus, which is based on Clifford algebra. Not available in English, this calculus is virtually unknown. In section 5, we summarize the KK structure of the authors (Vargas-Torr, 1997b), whose very definition requires some familiarity with the Kähler calculus. More comprehensive knowledge of that calculus is, however, needed to fully grasp the power and potential implications of equation (31), key to the emerging physics of the accompanying paper.

2. The Evolution of Geometry

In this section, we discuss the Cartan perspective of the evolution of the geometry that is common ground to all differential geometers, which thus excludes Finsler geometry. For completeness, let us state that synthetic (i.e. by drawn figures) and analytic geometry are most of the time chapters in the study of what we shall define as elementary geometries, specially, the Euclidean and projective ones. Before we become more precise, elementary geometries are the geometry of flat spaces. When dealing with their generalizations, the nature of the problems that make sense and worth considering makes their study go by the name of theory of connections.

Our aim in this section is to show how the theory of connections by Cartan relates to the ideas by Felix Klein on groups and geometric equality through the theory of integrability of differential systems. One thus anticipates a geometry that implements those ideas to a still greater extent. To better understand the "Klein-Cartan" perspective, one has to understand how the basic ideas of Klein differ from those of Riemann.

The paradigm of differential geometry is due to Cartan, through his theory of connections, and specially of affine connections in the early 1920's (Cartan, 1923-25; we refer the reader to his *Complete Works*, identifying the paper by the year and some more information when necessary). He reconciled what appeared to be the incompatible concepts of geometry by Riemann and Klein. Riemann's generalizations of geometry were based on the concept of distances more general than Euclid's. His first geometric paper, a conceptual one without equations, was his public lecture for a professorship at Göttingen. The title of his second and last paper on geometry (other than on minimal surfaces) speaks of the casual development of Riemannian geometry: "*Mathematical comments which attempt to answer the question posed by the very illustrious Parisian Academy: Find what must the calorific state of an indefinite homogeneous solid body so that a system of isotherm curves at a given instant remain isotherms at any instant of time and the temperature of a point may be expressed as a function of time and another two independent variables*" (Riemann, 1861, see his Complete Works). Riemann derives there the curvature that now bears his name. Cartan (1936) characterized the early Riemannian geometry as follows: "*In a first stage up to 1917, year of the almost simultaneous introduction by Levi-Civita and Schoutten of the notion of parallelism, Riemannian geometry has simply been regarded as the theory of invariants of a quadratic differential form* $g_{ij}dx^i dx^j$ *of n variables with respect to the infinite group of analytic transformations of these variables*". The quotation that follows speaks of the beginning of a second stage: "*With the introduction of his definition of parallelism, Levi-Civita was the first one to make the* false *metric* spaces *of Riemann become (not true Euclidean spaces, which is impossible, but at least)* spaces with Euclidean connection ..." (emphasis in original) (Cartan, 1924, in his paper on Recent Generalizations of the Notion of Space).

Another 19th century development of geometry is Klein's Erlangen program (Klein, 1893). According to it, a geometry is the study of properties (of figures) that remain invariant under the transformations of a group. Cartan credits H. Poincaré with making evident that the axioms of geometric equality just express the group property of sets of transformations that leave unchanged the properties of corresponding geometric figures. Speaking in particular of Euclidean geometry, Cartan (1924, in his paper Group Theory and the Recent Researches ...) said: "*...the statement that the displacements constitute a group is precisely the expression in exact language of the axiom according to which two* *figures equal to a third are equal to each other*". Cartan (1936) associated Klein's view with geometric equality so deeply that the concept of geometric equality itself rather than that of group played for him the preeminent role, as in the following characterization: "*Grossly speaking, Klein's point of view consists in retaining from Euclid's geometry as fundamental notion the equality of figures*" .

The two viewpoints or programs were unified by Cartan. But, as Dieudonne explained [See Gardner, 1989], this was possible by replacing in Klein's Erlangen program groups with principal fiber bundles. In the bundles, the concept of group still plays a very important role, though in a more sophisticated way. Thus Cartan's unification was based on extending Klein's program, not Riemann's. If geometric equality was so crucial, why not implement this concept to the maximum possible (other than trivial) extent? One must be aware, however, of some Klein errors, one of which consists in assigning an infinite Lie group to Riemannian geometry. This error, pointed out by Cartan (1936), still infects physical thinking (see Vargas-Torr (2005c)).

From the above, three concepts are seen to be essential in Cartan's interpretation and generalization of Klein's program: groups, geometric equality and principal fiber bundles. We thus anticipate that the maximum implementation of these concepts will result in a teleological geometry, i.e. as if it had a purpose. Although any development may be viewed retrospectively (and incorrectly in most cases) as if it had a purpose, exceptional excellence, when achieved through a natural course of action, justifies the practical use of the term.

Cartan's approach consisted in introducing the structure of generalized spaces as a problem of integrability of types of systems of differential equations which arise in the study of the flat or elementary cases, and of Minkowski space in particular. Euclidean and Minkowski spaces are specific cases of affine spaces. Consider thus the action of the affine group on tangent tensor bases in those spaces. If we arbitrarily make one of its points represent the zero vector, affine space can be assimilated to a vector space. A pair $(\boldsymbol{P}, \boldsymbol{e}_\mu)$ constituted by arbitrary point and arbitrary basis attached to it can be given as

$$\boldsymbol{P} = \boldsymbol{Q} + A^\mu \boldsymbol{a}_\mu, \qquad \boldsymbol{e}_\mu = A_\mu{}^\nu \boldsymbol{a}_\nu, \tag{1}$$

where $\det A_\mu{}^\nu \neq 0$, and where $\boldsymbol{Q}$ denotes a fixed point and $\boldsymbol{a}_\nu$ denotes a fixed basis (anything fixed is assigned zero differential). Indices run from zero to $n-1$ (hence from zero to 3 in spacetime). We can represent Eq. (1) in matrix form, where the column matrix with entries $(\boldsymbol{Q}, \boldsymbol{a}_\mu)$ is transformed into the column matrix with entries $(\boldsymbol{P}, \boldsymbol{e}_\mu)$ by the square matrix g acting on the left and whose first row and first column are $(1, A^0, ..., A^{n-1})$ and $(1,0,0,0,...)$ respectively. The other entries are constituted by the $A_\mu{}^\nu$'s.

At each point $\boldsymbol{P}$, the linear group (present in the equations $\boldsymbol{e}_\mu = A_\mu{}^\nu \boldsymbol{a}_\nu$) repeats itself. Let us just say that this family of isomorphic groups constitutes the essence of a principal fiber space. Now as in the following, readers should keep in mind that we are dealing here with a $(n+n^2)$-dimensional manifold, i.e. 20 in spacetime viewed as an affine space and before we notice that it is endowed with a metric. The equations that we obtain differentiating (1) and replacing the fixed basis with the "moving basis" can be written in abbreviated form as

$$d\boldsymbol{P} = \omega^\mu \boldsymbol{e}_\mu, \qquad d\boldsymbol{e}_\mu = \omega_\mu{}^\nu \boldsymbol{e}_\nu. \tag{2}$$

where the omegas constitute a set of 20 independent differential 1-forms. These omegas are clearly well defined functions of the A^μ, $A_\lambda{}^\nu$ and their differentials. Easy computations show that the $\omega_\mu{}^\nu$'s are not closed, i.e. $d\omega_\mu{}^\nu \neq 0$. They are the so call invariant forms of the linear group. This is the system generated by the elementary affine geometry and which constitutes the basis of its generalization.

Suppose now that we are given a set of 20 differential forms (ω^λ, $\omega_\nu{}^\mu$) which satisfies some restricting conditions (Vargas-Torr, 1993) but is otherwise largely general. These ω^λ and $\omega_\nu{}^\mu$ will be just expressions (in terms of some other system of coordinates and some other basis field) of the differential forms in eq. (2) if and only if the equations .

$$d\omega^\nu - \omega^\lambda \wedge \omega_\lambda{}^\nu = 0, \qquad d\omega_\mu{}^\nu - \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\nu = 0, \tag{3}$$

are satisfied (As we said in the introduction, $\omega_\mu{}^\nu = \Gamma_\mu{}^\nu{}_\lambda \omega^\lambda$, but the $\Gamma_\mu{}^\nu{}_\lambda$ are not defined by the metric in general, since a metric -thus a Riemannian structure- may not exist in any affine space. The system (2) will be integrable in spite of the aforementioned generality and become the system (1) through integration. The system (3) constitutes the equations of structure of affine space. All the foregoing applies mutis-mutandis to the Euclidean and Poincaré groups, which are restrictions of the affine group by $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \delta_{\mu\nu}$ and $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \eta_{\mu\nu}$ respectively. The $\omega_\nu{}^\mu$'s are no longer independent but restricted by the condition

$$\omega_{\nu\mu} + \omega_{\mu\nu} = 0. \tag{4}$$

This restriction, whether in finite or differential form, reduces the dimensionality of the bundle to $n+[n(n-1)/2]$, i.e. 10 for the Poincaré group of flat spacetime.

The generalized affine spaces are given by any set of differential 1-forms (ω^λ, $\omega_\nu{}^\mu$) which do not satisfy equations (3) in general, but are quadratic differential forms in the ω^ρ only. In other words:

$$d\omega^\nu - \omega^\lambda \wedge \omega_\lambda{}^\nu = R^\nu{}_{\mu\lambda} \omega^\mu \wedge \omega^\lambda \equiv \Omega^\nu, \qquad d\omega_\mu{}^\nu - \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\nu = R_\mu{}^\nu{}_{\sigma\lambda} \omega^\sigma \wedge \omega^\lambda \equiv \Omega_\mu{}^\nu, \tag{5}$$

where d is the exterior derivative and where $\wedge$ is the familiar exterior product. In other words, this is the product in the Grassmann algebra, i.e. the antisymmetrized tensor product of completely antisymmetric tensors (Lichnerowicz, 1962). More elegant is the definition of Grassmann algebra as a quotient algebra of the general tensor algebra (MacLane-Birkhoff, 1970). See also our simplified summary (Vargas-Torr, 2005d).

The differential forms Ω^ν and $\Omega_\mu{}^\nu$, known as torsion and affine curvature, live in the bundle. Excluding for simplicity valuedness indices, general differential 2-forms in the bundle are in principle more general, namely as in

$$R_{\mu\lambda}\, \omega^\mu \wedge \omega^\lambda + U_\mu{}^\nu{}_\lambda\, \omega^\mu \wedge \omega_\nu{}^\lambda + V^\mu{}_\nu{}^\lambda{}_\sigma\, \omega_\mu{}^\nu \wedge \omega_\lambda{}^\sigma. \tag{6}$$

A Euclidean (also Lorentzian) connection on a manifold is an affine connection where the $\omega_\lambda{}^\nu$ satisfy in addition the relation (4), the expression (6) for general differential 2-forms remaining valid. The difference between being Euclidean and being Lorentzian shows up when raising or lowering indexes, in equation (4) in particular.

In Riemannian (and pseudo-Riemannian) geometry, which are particular cases of geometry involving Euclidean and pseudo-Euclidean connections, the $R^\nu{}_{\mu\lambda}$ are all zero. No consequences regarding the differential system (2) follow. On the other hand, if $R_\mu{}^\nu{}_{\sigma\lambda}=0$, the system $d\boldsymbol{e}_\mu = \omega_\mu{}^\nu \boldsymbol{e}_\nu$ is integrable and the integration gives of course the rotation (respectively Lorentz transformation) or the linear group, depending on which case (read bundle) we are dealing with. We then have TP or path independent equality of vectors at a distance. The distance is implicit in the metric restriction, since $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = g_{\mu\nu}$

immediately yields

$$dP\cdot dP = dx^{\mu}e_{\mu}\cdot dx^{\nu}e_{\nu} = g_{\mu\nu}\, dx^{\mu}dx^{\nu} \qquad (7)$$

in terms of arbitrary coordinate basis fields.

The differentiation of eqs. (5) together with substitution of (5) on the differentiated equations yields the Bianchi identities. For present purposes, we are simply interested in the first one. In TP, it states that the torsion's exterior covariant derivative is zero.

3. Anticipation and Teleological Geometry

The goal of this section is to develop generalized affine geometry further while preserving the metric contents of (pseudo-)Riemannian geometry. We proceed by analyzing the progress that geometry has made, and identifying what that progress is due to. The implementation to a greater extent of what has already been successful there will constitute the anticipated "state of geometry" for unification. In viewing our KK structure as a state of geometry, we may say that this state appears to have a purpose, i.e. it is teleological. It is the purpose of canonical (from the horizontal invariants of Finslerian TP) maximum implementation of geometric equality and of the use of complexes of groups and concomitant bundles. Details follow.

As mentioned, a first step towards *geometric equality* in the original Riemannian geometry was Levi-Civita's rule to compare vectors in different tangent spaces. As a simple alternative in the plane to the standard Euclidean connection, one can have, for instance, the TP connection with torsion where the polar coordinate lines are viewed as having constant direction. In the punctured sphere and the torus, we have the TP connections where the vertical and horizontal circles (which are not extremals) are lines of constant direction.

The theory of connections on Finsler bundles achieves a more profound implementation of the role of *groups* and *bundles* than the theory of manifolds where general lengths of curves are defined. The Lorentzian signature then emerges as the canonical one (Vargas-Torr, 2005c). In the old view, Lorentzian geometry is simply pseudo-Riemannian. The geometry of the spacetime of Special Relativity (SR) is then represented by the pair constituted by the Poincaré group and its Lorentz subgroup In the new perspective, flat Lorentzian geometry is represented by a triple consisting of a group, a subgroup and a sub-subgroup. It is not a matter of fancy. It is a matter of analyzing Finsler connections and seeing what groups participate in its construction. Instead of asking what is the geometry of Minkowski spacetime, one should ask what is the elementary geometry of metric compatible Finsler connections on pseudo-Riemannian metrics with Lorentzian metrics. One reaches the new perspective of SR, but now viewed as a necessity. The corresponding bundle results from rearranging of the inertial frames of spacetime, M, over the so called space of elements (or bundle of directions or sphere bundle), S(M), of spacetime. In this case, S(M) is a space comprising 3-space, time and velocity.

Understanding the evolution of Riemannian geometry helps to understand the difference between those two views. In the first stage in Cartan's characterization of the geometry of Riemannian, the latter's curvature lacks geometric meaning; it is just a set of symbols. It was only in 1917, two years after the birth of General Relativity (GR), that Levi-Civita (LC) provided them with a geometric meaning through its namesake connection $\alpha_\mu{}^\nu$ ($=\Gamma_\mu{}^\nu{}_\lambda dx^\lambda$), where the $\Gamma_\mu{}^\nu{}_\lambda$ are the Christoffel symbols. But $\alpha_\mu{}^\nu$ is an affine connection only if used to connect bases at different points, i.e. $de_\mu=\alpha_\mu{}^\nu e_\nu$.

Because of its relevance for the accompanying paper, let us point out that the early GR, based on the Riemannian geometry of the first stage, was roughly consistent with other connections and, therefore, with other affine curvatures. It has not realized generally that there are two concepts of curvature when a manifold is endowed with a metric and compatible affine connection other than the LC connection, as the $\alpha_\mu{}^\nu$'s still retain their significance as differential forms from which to obtain the Riemannian or metric curvature. Hence, had Cartan produced his theory of affine connections in 1916, and had he been immediately and thoroughly understood, GR might have adopted some metric-compatible connection other than LC's to take care of the affine properties of the space, while leaving unchanged its metric properties, like the separation of extremals and the original purpose of the Riemannian curvature. When, in the late 1920's, Einstein used TP connections to push forward the geometrization of the physics, he failed to realize that his Einstein equations might still emerge, through the relation between affine and metric curvatures that Cartan pointed to him and which Einstein chose to ignore (Debever, 1930). To be precise, Cartan gave without explanation the Ricci tensor in terms of the torsion when the connection is TP and the affine curvature is zero.

We return to Finsler geometry. If Cartan had every reason to refer to the original spaces of Riemann as false spaces, the same qualification applies to the original Finsler geometry. In other words, one should avoid seeing Finsler geometry only as manifolds where the concept of length of curves is defined. It was Y. H. Clifton who, upon our prodding (Vargas-Torr, 1993), defined affine-Finsler connections for geometry on Finsler bundles. In the latter case, one takes the set $B(M_4)$ of all tangent vector bases to a manifold M_4 and refibrates it over the 7-dimensional manifold $S(M_4)$ constituted by all directions at all points (Two tangent vectors at a point represent the same direction when they differ by a positive factor). One then introduces a connection in this (re)fibration.

In the fibers of $B(M_4)\to M_4$, the linear group acts freely and transitively. In the Finsler bundle, $B(M_4)\to S(M_4)$, on the other hand, the fibers are similarly acted upon the group that leaves a direction unchanged. The refibration is achieved by establishing an isomorphism between tangent bases to M and bases of reduced tangent vectors to S(M) (Vargas-Torr, 1993). In establishing the isomorphism, one takes a direction as preferred. When one adds a metric structure through a positive definite metric, the set $B(M_4)$ of bases is replaced by its restriction $B'(M_4)$ to the frames (orthonormal bases). The group in the fibers then becomes the rotation group O(n-1) if the metric is positive definite. If the signature of the metric is Lorentzian, the group in the fibers is O(n-1) if the preferred direction is time-like. Otherwise it is the subgroup O(1,n-2) of O(1,n-1), in which case one is arbitrarily singling out a subgroup of O(1,n-1) which is not distinguished. In other words, we have found that the elementary geometry of metric-Finsler connections with signature (1,3) is a triple of already familiar groups (Poincaré, Lorentz and O(3)), the bundle being $B'(M_4)\to S(M_4)$ with O(3) acting on the fibers. The Finsler fibration thus makes a larger use of the group structure than the usual fibration. The arbitrariness in choosing a preferred direction in affine-Finsler geometry is removed on manifolds endowed with a metric of Lorentzian signature. There is no reason to look into the subgroups of O(3) and try to further extend the concept of elementary geometry by involving still another level of (sub)groups, since there is no intrinsic consideration to look into the subgroup O(2) of O(3). To conclude, the geometric structure of spacetime physics should be Finslerian. Pseudo-Riemannicity should be viewed as a historical accident at a time when one did not know of other options.

In sections of the Finsler bundle, and because the frames are adapted, the soldering forms, ω^μ, (i.e. "the square root of the metric") are

$$\omega^0 = l dt + A_m(dx^m - u^m\, dt), \tag{8a}$$

$$\omega^r = A^r{}_m\,(dx^m - u^m\, dt) \tag{8b}$$

(Notice that the ω^r are of a special form, since they depend on three rather than four independent elements of a basis of differential 1-forms). It is to be noticed that these expressions depend explicitly on the u^m, even if the metric is Riemannian. This dependence, however, cancels out for the latter metric,

$$ds^2 = d\boldsymbol{P}\cdot d\boldsymbol{P} = \Sigma_\mu\, \varepsilon_\mu\, (\omega^\mu)^2, \tag{9}$$

where $\varepsilon_\mu = \pm 1$, depending on signature and specific value of μ.

The equation of the autoparallels is given by $\omega^i = 0$, $i=1,2,3$ and $du=0$. Since, in the Finsler bundle, $\boldsymbol{e}_0 = \boldsymbol{u}$ (which is to say that the frames are adapted), we have $d\boldsymbol{u} = d\boldsymbol{e}_0 = \omega_0{}^i \boldsymbol{e}_i$. The condition $\omega^j=0$ expresses that the curves of interest are "natural liftings" of spacetime curves (all the differential 1-forms are multiples of just one of them on curves, and $\omega^m = dx^m - u^m dt = 0$ simply means that $u^m = dx^m/dt$). Hence, the equations of the autoparallels are given by $\omega^j = \omega_0{}^i = 0$.

The equations of structure in Finsler bundles have the usual left hand sides (Vargas-Torr, 1993), but the right hand sides have greater richness. The first one reads:

$$d\omega^\nu - \omega^\lambda \wedge \omega_\lambda{}^\nu = R^\nu{}_{\mu\lambda}\,\omega^\mu \wedge \omega^\lambda + S^\nu{}_{\lambda i}\,\omega^\lambda \wedge \omega_0{}^i \equiv \Omega^\nu \tag{10}$$

Because of the assumption of TP, the curvature equation reduces to

$$d\omega_\mu{}^\nu - \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\nu = 0, \tag{11}$$

and the first Bianchi identity to

$$d\Omega^\nu + \Omega^\lambda \wedge \omega_\lambda{}^\nu = 0. \tag{12}$$

In metric compatible TP, we rewrite the second Bianchi identity as

$$d\alpha_\mu{}^\nu - \alpha_\mu{}^\lambda \wedge \alpha_\lambda{}^\nu = d(\omega_\mu{}^\nu - \beta_\mu{}^\nu) - (\omega_\mu{}^\lambda - \beta_\mu{}^\lambda) \wedge (\omega_\lambda{}^\nu - \beta_\lambda{}^\nu), \tag{13}$$

and set $d\omega_\mu{}^\nu - \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\nu$ (affine curvature) equal to zero. $\beta_\mu{}^\nu$ is the difference between the affine connection of spacetime and its LC connection (we shall be interested in Finslerian connections on pseudo-Riemannian metrics) . The contraction of the left hand side gives the Einstein or Ricci tensor, which is automatically matched with a similar contraction on the right hand side. In other words, eq. (13) contains a version of the Einstein system where the right hand side is geometric, as he sought. The identification of actual energy-momentum tensors is given briefly in our accompanying paper. For more detail see (Vargas-Torr, 2005b).

We have given elsewhere examples of TP Finslerian connections (Vargas-Torr, 1997a). Further progress towards the anticipated structure comes from the calculus and from the physics.

4. The Evolution of the Calculus

By the term calculus, we mean the formalized extension of the ordinary calculus that replaces everywhere functions with differential forms of different grades, even mixed grades. Before Kähler, it had been overlooked that the calculus of differential forms need not be just an *exterior* calculus, even though integration and geometry requires antisymmetry. But antisymmetry is not enough for quantum mechanics, not even enough for a theory of harmonicity for differentials. The Laplacian fuses the exterior ("curl") and interior (divergence) derivatives, both of which are lost in the fusion. The "Kähler" calculus gives us a derivative with exterior (antisymmetric) and interior (symmetric) parts, which do not loose their identities in their union. The annulment of this derivative defines strict harmonic differentials. The annulment of its square gives the lesser concept of harmonic differentials. Ignore the symmetric part of his derivative and one gets the Cartan calculus, like ignoring the symmetric part of two quantities of grade one in the underlying algebra of the Kähler calculus yields the exterior product. And yet, this calculus remains one about antisymmetric differential forms, as geometry requires, but meeting the challenges of quantum mechanics with a vengeance.

The underlying algebra is Clifford algebra. It might as well have been called Euclidean algebra. Indeed, the Clifford product of two vectors or two differential 1-forms integrates the dot and vector products of Euclidean geometry, after the spurious element in the vector product has been removed (The vector product is an artifact of the vector product in three dimensions; only for n=3, there is one and only one direction perpendicular to the plane defined by two vectors).

Let us imagine that some mathematician had written in the mid twenties the pertinent papers by Kähler (1960, 61, 62). The fine structure of the hydrogen atom was solved in a dedicated paper (1961), and also, even more elegantly, as an exercise following his presentation of strict harmonic differentials (Kähler 62). Such an achievement in physics, coming from the natural development of the exterior calculus, would have blown Dirac out from second place in the physics rankings of the twentieth century, as his instincts would have told him that he could not compete with such beauty. The "Kähler equation" would be today the foundation of quantum physics. It can be anticipated that physics will one day remove any spuriousness due to historical accident, and that the Kähler calculus and equation will prevail. The right mathematics always enriches the physics.

We proceed to illustrate the naturalness of Clifford algebra. Suppose a space of dimension n>3 with a product of vectors other than the dot product, and which we shall designate by juxtaposition. Until further notice, or until we add further properties, this product could be a tensor product, or what we shall later define as Clifford product or many other products. Suffice to notice how little structure is needed for the validity of the first few equations that follow. We assume that our product has the distributive property but not commutativity or anticommutativity. The identity

$$\boldsymbol{ab} \equiv (1/2)\,(\boldsymbol{ab}+\boldsymbol{ba}) + (1/2)\,(\boldsymbol{ab}-\boldsymbol{ba}), \qquad (14)$$

gives rise to

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b} \qquad (15)$$

where the two terms of the right are defined as

$$\boldsymbol{a}\cdot\boldsymbol{b} \equiv (1/2)\,(\boldsymbol{ab} + \boldsymbol{ba}), \qquad \boldsymbol{a}\wedge\boldsymbol{b} \equiv (1/2)\,(\boldsymbol{ab} - \boldsymbol{ba}) \tag{16}$$

and are respectively symmetric and antisymmetric. Under the assumption that the product is distributive (the tensor product is known to be so, and such is also the case with the Clifford product), we readily get

$$(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4 + \dots) \wedge (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 + b_4\boldsymbol{e}_4 + \dots) =$$
$$= \sum_{i<j}(a_i b_j - a_j b_i)\, \boldsymbol{e}_i\wedge\boldsymbol{e}_j \tag{17}$$

One recognizes that when the indices run from one to three, the right hand side is like the vector product, except that $\boldsymbol{e}_i\wedge\boldsymbol{e}_j$ now plays the role of $\boldsymbol{e}_k$, where (i,j,k) is a cyclic permutation of (1,2,3). Thus the wedge product so defined makes reference to the oriented plane of the vectors in the product, not to the vectors perpendicular to those planes. The vector product of linear algebra in 3 dimensions is a wedge product followed by another operation called Hodge duality, but only in three dimensions does one get back a vector in this way. Because of the point we wanted to make, we ended up with a basis of antisymmetric products on the right hand side of Eq. (17). If $\boldsymbol{ab}$ were specifically the tensor product, $\boldsymbol{a}\otimes\boldsymbol{b}$, we could have also given the right hand side in terms of $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$, since the antisymmetry of the coefficients $(a_i b_j - a_j b_i)$ picks the antisymmetric part of $\boldsymbol{a}\otimes\boldsymbol{b}$.

A Clifford algebra, like a Grassmann algebra, is an algebra whose elements are sums of totally antisymmetric tensors of different grades. Both algebras are quotients by ideals of the general tensor algebra constructed over a vector space or, more generally, over a module. With one ideal, we get exterior algebra, and we get Clifford algebra with another ideal. The difference is in the multiplication rules (Lounesto, 2002). See also our simplified summary (Vargas-Torr, 2005d). In both cases, one is performing tensor products and then reducing the expressions in different ways. Of course, there are rules to operate without going through the tensor products. The symbol used indicates what rules apply. Clifford multiplication is performed most easily by using orthonormal bases of vectors. The rules are then the same as for matrix multiplication of linear combinations of products of gamma matrices. Except for what would be just a minor subtlety here, the algebras of the sigma and Dirac matrices are Clifford algebras. The complex numbers can also be viewed as constituting an algebra (Lounesto, 2002), and not just as a field. The complex numbers, the quaternions, the Pauli matrices and their products, and the gamma matrices and their products constitute examples of Clifford algebras corresponding to vector spaces of respective dimensions 1, 2, 3 and 4. The multiplication rules are the same in Clifford algebra, but one does not need to have matrices. These may simply constitute a particular representation for some Clifford algebra.

Let us give some more of the flavor of these algebras. An r-multivector, A_r, is an antisymmetric tensor of grade r, usually referred to as an r-vector. It can be viewed as an antisymmetric tensor product of a number r of vectors. An equation of the type (15) also holds in Clifford algebra (the Clifford product then being denoted by the symbol $\vee$) when one of the factors is a vector and the other one is an r-vector, A_r:

$$\boldsymbol{a}\vee\boldsymbol{A}_r = \boldsymbol{a}\cdot\boldsymbol{A}_r + \boldsymbol{a}\wedge\boldsymbol{A}_r \tag{18}$$

where juxtaposition now denotes specifically the Clifford product and where

$$a\wedge A_r = (1/2)[a\vee A_r + (-1)^r A_r\vee a], \quad a\cdot A_r = (1/2)[a\vee A_r - (-1)^r A_r\vee a]. \tag{19}$$

The Clifford quantities $a\cdot A_r$ and $a\wedge A_r$ are of respective grades $r-1$ and $r+1$. A decomposition like in (14) is not of significance when at least one of the two factors is not a vector. The product $A_r\vee B_s$ will rather have parts of grades $r+s$, $r+s-2$, $r+s-4$, ..., $|r-s|$. The part of grade $r+s$ is $A_r\wedge B_s$. In the calculus of gamma matrices, A_r and B_s would be represented by sums of products involving r gamma matrices each (representing A_r) and by sums of products involving s gamma matrices each (B_s), products which cannot each be reduced into products of smaller numbers of factors. After the appropriate reductions in the product of A_r and B_s by the usual rules of gamma matrices, we would have a sum of terms which are (when not zero) irreducible products of $r+s$, $r+s-2$, $r+s-4$, ... and $|r-s|$ gamma matrices.

Kähler constructed a Clifford algebra of differential forms defined by the relation

$$\omega^\mu\vee\omega^\nu + \omega^\nu\vee\omega^\mu = 2\eta^{\mu\nu}, \tag{20}$$

which contains the equations $\omega^\mu\wedge\omega^\nu=-\omega^\nu\wedge\omega^\mu$ and $\omega^\mu\cdot\omega^\nu=\eta^{\mu\nu}$, where the $\eta^{\mu\nu}$ are the elements of the diagonal matrix with diagonal (1, -1,-1,-1) in the spacetime case. To be precise, Kähler used coordinate bases, namely

$$dx^\mu\vee dx^\nu + dx^\nu\vee dx^\mu = 2g^{\mu\nu}. \tag{21}$$

This is a serious drawback far beyond the obvious, both for computations (Vargas-Torr, 2005a) and for further evolution of the Kähler calculus (Vargas-Torr, 2005b). Equation (21) has to be seen in the context that $dx^\mu\wedge dx^\nu$ is an element of a basis for an integrand, normally written as $dx^\mu\, dx^\nu$, where this could be the $dxdy$, or $d\rho d\theta$, etc. There is an accompanying tangent tensor algebra in his first paper in the subject (Kähler, 1960). An example for illustration would be a differential r-form $u^\mu{}_\nu$ of (1,1) valuedness, which is compact notation for his more detailed notation $u^\mu{}_{\nu\, 1\ldots r}dx^1\wedge\ldots\wedge dx^r$ for the same quantities.

Kähler defined (his) "covariant derivatives", $d_\sigma u^\mu{}_\nu$, with which he used in turn to introduce other "derivatives"

$$\underline{\partial} u^\mu{}_\nu = dx^\sigma\vee d_\sigma u^\mu{}_\nu = dx^\sigma\wedge d_\sigma u^\mu{}_\nu + dx^\sigma\cdot d_\sigma u^\mu{}_\nu = du^\mu{}_\nu + \delta u^\mu{}_\nu. \tag{22}$$

Our first use of inverted commas has to do with the fact that $\underline{\partial} u^\mu{}_\nu$, $du^\mu{}_\nu$ and $\delta u^\mu{}_\nu$ also are covariant expressions. The second use has to do with the fact that, although $d_\sigma u^\mu{}_\nu$ and $du^\mu{}_\nu$ satisfy the Leibnitz rule, $\underline{\partial} u^\mu{}_\nu$ and $\delta u^\mu{}_\nu$ do not. The derivative $du^\mu{}_\nu$, i.e.$dx^\sigma\wedge d_\sigma u^\mu{}_\nu$, recovers the exterior covariant derivative, i.e. the $d(u^\mu{}_\nu \mathbf{e}_\mu\otimes\mathbf{e}^\nu)$ of the theory of moving frames, and becomes the exterior derivative if there are no tensor indices (scalar-valuedness). In order to avoid details, let us simply state that δ is the generalization of the co-derivative. However, in spite of the Leibnitz rule issue, we prefer to use the term interior derivative, as it helps realize that δ is not the interior derivative when the connection is not the LC connection. Readers are warned that we use the symbol $\underline{\partial}$ where Kähler uses the symbol δ. He does not introduce a symbol that would correspond to our δ. We shall not reproduce here Kähler's cumbersome formulas defining $d_\sigma u^\mu{}_\nu$, which he introduces as an ansatz. Some details about our natural alternative approach to $\underline{\partial} u^\mu{}_\nu$ will be provided in the next section.

The equation

$$\partial u = a \vee u \tag{23}$$

is known as the Kähler (1960) equation. Both a (input) and u (output) are inhomogeneous differential forms and may be tensor-valued. As in the original, we drop indices to avoid clutter. The symbol $\vee$ refers only to the algebra of differential forms. Up to constants, a is simply $m+eA$ for a particle in an electromagnetic field (m is mass, e is charge and A is the 4-potential differential form). Specializing to the Coulomb case, Kähler (1961) solved the hydrogen atom's fine structure in a paper dedicated to this topic. An improved presentation of his calculus later led to an improved solving of that atom in the same paper (Kähler, 1962).

In the case $a=0$, we have $\partial u=0$, which defines the so called strict harmonic differential forms. Less special are the harmonic differential forms, satisfying $\partial\partial u=0$. The solutions of the equation $d_h c=0$ are particular strict harmonic differential forms known as constant differentials. They have the property that, if $\partial\psi = a \vee \psi$, then

$$\partial(\psi \vee c) = a \vee (\psi \vee c). \tag{24}$$

Examples of important constant differentials, related to the rotational and time translation symmetries are

$$\tau^{\pm} = \frac{1 \pm i dx^1 \vee dx^2}{2}, \qquad \varepsilon^{\pm} = \frac{1 \mp i dt}{2}, \tag{25}$$

They are instrumental in relating the solutions of the Kähler and Dirac equations.

The equation $\partial u = a \vee u$ in itself, and through its special case $\partial u=0$, constitutes the cornerstone of the Kähler calculus. It supersedes the Dirac equation (Vargas-Torr, 2005a), which is a pillar equation of the physics.

5. The Juncture of Geometry and Calculus

One is led to anticipate some confluence of geometry and calculus from the foregoing considerations about how their key equations relate to two pillar equations of the physics, combined with the widespread expectation for physical unification. It will certainly be justified to speak of unification if the structure of some geometric space is given by a KD equation. Retrospectively, this looks particularly feasible in TP, since the affine curvature is then zero and so is the exterior (covariant) derivative of the torsion. It seems that only the interior (covariant) derivative remains to be determined. Unlike the exterior derivative, there is not an "interior structure" where the interior derivative could be defined. This is so because grading, which is of the essence of exterior and interior derivatives, is inimical to algebras of symmetric products. Even in Clifford algebra, the usual grading is induced: any Clifford product can be written as a sum of terms which are exterior products, each with a definite grading but which may vary from one term to another. Fortunately, one does not need more than that. It is then a matter of either constructing a Clifford algebra in our TP Finsler space, or in some other space directly related to it. A KD equation for the torsion would constitute an equation of structure of that space. There are a few steps in getting to the sought KD equation.

The original Kähler calculus, developed for the LC connection, has to be adapted to arbitrary metric compatible affine connection. This requires deriving the expression for the covariant derivative. The process is very clear for scalar-valued differential 1-forms. Using the first equation of structure, we get

$$d(a_\mu \omega^\mu) = a_{\mu,\nu}\omega^\nu \vee \omega^\mu + a_\lambda d\omega^\lambda = \omega^\nu \wedge [a_{\mu,\nu} + a_\lambda(R^\lambda{}_{\nu\mu} - \Gamma_\mu{}^\lambda{}_\nu)]\ \omega^\mu, \tag{26}$$

(comma represents coefficients in the expansion of a differential form with respect to the ω^ν, and with respect to the dx^ν in particular). It is to be noticed that the factor ω^ν at the front of the last expression comes, in the case of the term $da_\mu\ \omega^\mu$, from da_μ (the alternative, wrong, would be $-\omega^\nu \wedge da_\nu$). Similar considerations apply to the other factor, under similar argument (Vargas-Torr, 2005a). From eq. (26), we get:

$$d_\nu(a_\mu \omega^\mu) = [a_{\mu,\nu} + a_\lambda(R^\lambda{}_{\nu\mu} - \Gamma_\mu{}^\lambda{}_\nu)]\ \omega^\mu, \tag{27}$$

Dot multiplication by ω^ν yields:

$$\delta(a_\mu \omega^\mu) = a^\nu{}_{,\nu} - a_\lambda \Gamma^{\nu\lambda}{}_\nu. \tag{28}$$

Consider next the same issue of defining an interior derivative in the Finsler bundle. The differentiation produces now differential forms dx^ν and du^i. We will need a dot product in the 7-dimensional module of differential forms spanned by dx^ν and du^i. No such dot product exists. Thus, there is no way to implement the Kähler rules on the manifold S(M). TP, however, suggests a way out. In this case, ω_l^j can be chosen to be zero on sections of the Finsler bundle, B(M)→ S(M). Equivalently ω_l^j can be chosen to be the invariant forms of SO(3) (in the spacetime case). Hence, all the information that is specific to individual TP Lorentz-Finsler connections is in (ω^μ, ω_0^j). In addition, the Finsler bundle fits the Lorentzian signature, but it does not seem that physical fields need depend on the velocity coordinates. This suggests the following course of action (Vargas-Torr, 1997b).

Given that ω_0^j determines $d\boldsymbol{e}_0$ ($d\boldsymbol{e}_0 = \omega_0^j \boldsymbol{e}_j$), it determines $d\boldsymbol{u}$, since $\boldsymbol{e}_0 = \boldsymbol{u}$ (the frames are adapted in the Finsler bundle). Furthermore, $\boldsymbol{u}$ is dual to proper time. The form

$$d\wp = d\boldsymbol{P} + d\tau \boldsymbol{u} = \omega^\mu \boldsymbol{e}_\mu + d\tau \boldsymbol{u}. \tag{29}$$

spans a 5-dimensional manifold. It is a kind of Kaluza-Klein (KK) space that may be viewed as a sum of spacetime and a 1-dimensional manifold spanned by the unit vector $\boldsymbol{u}$ representing the particles. At this point, the spacetime part of the connection $d\boldsymbol{e}_\mu$ can be chosen to be zero (constant spacetime frame field). The torsion $d(d\wp)$ then involves the $d\omega^\mu$ and the differential forms $\omega_4{}^A$ $(A=0,1,...4)$ that, grossly speaking, now play the role of the ω_0^j. We shall ignore interesting issues that arise here, like the meaning of $\omega_4{}^A$.

This KK space is endowed with a dot product of differential forms. We have

$$\omega^0 \cdot \omega^0 - \omega^1 \cdot \omega^1 - \omega^2 \cdot \omega^2 - \omega^3 \cdot \omega^3 - d\tau \cdot d\tau = 0, \tag{30}$$

where the meaning of the different terms in Eq. (31) can be inferred by reference to the fact that this equation is the Clifford form of $ds^2 = d\tau^2$. Furthermore, $\omega^A \cdot \omega^B = 0$, if $A \neq B$.

It may be said that the spacetime manifold interacts with a 1-dimensional manifold, representing individual particles. Needless to say that different particles may be seen as interacting through their direct interactions with the 4-dimensional manifold at different parts of the latter. One is now able to define an interior covariant derivative, the details not being needed here. This KK space is the teleological structure to which we have been referring. We use the circumflex to refer to quantities in the KK space, as in $\underline{\partial}\hat{u} = \hat{a} \vee \hat{u}$. It is not needed over $\underline{\partial}$ since the fields do not depend on the coordinates of the particle, i.e. on τ (The traditional KK space also has characteristics along these lines).

We have shown, both in the Finsler bundle (Vargas-Torr, 1999) and in the KK space (Vargas-Torr, 2005b), that the torsion which makes the autoparallels become the equations of the motion of SR with Lorentz force is $\boldsymbol{\Omega} = -F\boldsymbol{u}$ ($= -F\boldsymbol{e}_0$ in the Finsler bundle). Since $\boldsymbol{u}$ spans the fifth dimension (for which we use the subscript 4), we have $\Omega^4 = -F$. When this torsion is used in the contracted curvature equation, the energy-momentum tensor also comes to be what one expects it to be, except for an interesting discrepancy. It is then clear that, in the unification of geometry and calculus, the KD equation should have as input differential form something like $d\wp$, or $d\boldsymbol{P} \vee d\tau \boldsymbol{u}$, etc.

We now change the approach to structure in the KK space canonically associated with Finslerian TP. Instead of specifying a torsion that will satisfy the first Bianchi identity (i.e. that its exterior covariant derivative is zero), we view that specification as taking place through a KD equation playing the role of a field equation for the torsion (actually for the connection through the torsion). Hence, apart from annulment of the affine curvature, viewed as a statement relating metric curvature and torsion, the structure of the KK space is constituted by

$$\underline{\partial}\boldsymbol{\aleph} = \hat{\boldsymbol{a}} \vee \boldsymbol{\aleph}, \qquad (31)$$

where $\boldsymbol{\aleph}$ is the torsion of the KK space and where $\hat{\boldsymbol{a}}$ is $d\wp$ or $d\boldsymbol{P} \vee d\tau \boldsymbol{u}$, or even $\boldsymbol{u} \vee d\boldsymbol{P}$. Of great interest is the intimate relation between $\boldsymbol{\aleph}$ and any of these $\hat{\boldsymbol{a}}$'s. For eq. (31) to work, one should recover that the exterior covariant derivative of (at least) the spacetime part of the torsion is zero. But this is for the accompanying paper.

6. Conclusions

This paper has been a tour the force taking us from the original Riemannian concept to the junction of geometry and calculus, both at extremely specialized points of development. Consequently it may be difficult for many readers to realize what has been achieved. In the heart of the geometry, whose upper and lower halves are the two equations of structure, we have replaced the ventricles with the pump of the calculus of differential forms, namely the Kähler equation with torsion as output differential form. Lack of rejection is guaranteed by the fact that the input differential is constituted by the same differential forms in terms of which (and their derivatives) torsion and curvature are defined. Thus the curvature specification, (13), now considered in the KK space, together with the new torsion equation, (31), constitute a very sophisticated closed geometric system, comprising gravitation and quantum physics. Not surprisingly, the front interaction in the quantum half is the electromagnetic one. The strong and weak ones appear to be disguised in the technical details.

We have shown that realistic anticipation by physicists with the required knowledge of the evolution of mathematics and mathematical ability could have got them very fast from point Riemann through the two Einstein passes to point Cartan-Kähler in KK space. Our journey between those two points was largely guided by anticipation, but it is possible to do even better than we did. Anticipate that our mathematical tool kit will become better and that the different problems of unification will one day be solved, solving in the process fundamental problems like the impossibility at present of integrating vector-valued differential forms in GR. As explained in the introduction, one is then lead to anticipate a junction of geometry and calculus. Let us start enumerate some of those anticipations, in case they have escaped the readers attention.

There is no path-independent equality of vectors at a distance in GR based on the LC connection. Hence, *anticipate* that there is, which is the same as to postulate TP (first Einstein pass). Since the theory of Finsler bundles tells us that the Lorentzian signature is canonical, *anticipate* the Finslerian perspective for the underlying geometry of the anticipated gravitation, hence postulate Finslerian TP. Express that the affine curvature is zero in terms of metric and torsion and discover Einstein equations with torsion as source. Obtain the equations of the autoparallels in Finsler geometry and discover in it the gravitational and Lorentz contributions. Thus *anticipate* the logical homogeneity of physics and geometry (second Einstein pass), i.e. that the equations of structure become the field equations. The first Bianchi identity of TP already specifies that the exterior covariant derivative of the torsion is zero. Thus, *anticipate* a geometric formulation of quantum mechanics and that the specification of the torsion will be given by a Kähler equation in a formulation of geometry. *Anticipate*, further and therefore, that there is an alternative way of interpreting how one has to view the differential invariants that define a TP Finsler bundle, required for a Kähler equation. The alternative is easily found. It is our KK space. In the process, one identifies what specific torsion has to be identified with the electromagnetic field at the level of linear (Maxwell) electrodynamics. Solve the hydrogen atom to verify that the Kähler equation works. That permits one to *anticipate* the input differential form for the Kähler equation that constitutes the first equation of what we called our teleological geometry (If there was any doubt as to the appropriateness of this term, there should not be any more). Check that the exterior part of the Kähler equation in our KK space indeed contains the homogeneous Maxwell's equations and *anticipate* that the full Maxwell system is a degraded version of the Kähler equation for the torsion. The proof is in the pudding, to be prepared in the accompanying paper. The authors beg for empathy of the readers, given the extreme youth of the art of in implementing anticipation in the formulation of physical theory.

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