Determination of Feasible set of Solutions for Mixed Integer Nonlinear Optimization Problem

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Abstract
Mixed Integer Nonlinear Problems (referred as MINLP) is a nonlinear optimization problem, where two types of variables are present, namely integer variables and continuous ones. The presence of integer variables extends fundamentally the areas of MINLP applications. There is a linear goal function subject to linear and nonlinear constraints (quadratic forms).Two dimensional case of integer variables as well as continuous ones is analyzed. Main subject of interest is construction of feasible set of variables. Some numerical results will be given, where water distribution network will be interesting application area.
Keywords: mixed integer variables, optimization algorithm, feasible set.

1 Introduction

Mixed Integer Nonlinear Programming (MINLP) Problem [1], [4], [7], [8] one can describes as follows:

Minimize $f(y) + D\,x;$
Subject to

$g(y) + H\,x \leq 0$
$L \leq y \leq U$
$x = \{0,1,2,..\};$

where **y** is a vector of variables that are continuous real numbers.
The expression

$f(y) + D\,x;$

is the objective function and expression

$g(y) + H\,x;$

represents the set of constraints. L and U are vectors of lower and upper bounds on the variables. Expressions $D\,x$ and $H\,x$ are inner products.

The optimization problem under consideration is formulated as follows:

$$\sum_i \alpha_i n_i + \sum_j \beta_j \cdot y_j = (\alpha^T n + \beta^T \cdot y) \to \min$$

Where: n_i, y_j are integer variables and continuous correspondingly;

$n = (n_1, n_2, ,..., n_p)^T, y = (y_1, y_2, ,..., y_{p'})^T$;

and

$\alpha^T = [\alpha_1, \alpha_2, ..., \alpha_p]$; $\beta^T = [\beta_1, \beta_2, ..., \beta_{p'}]$

are known constants. Coefficients α_i, $\beta_j > 0$; $i = 1, 2, \ldots, p$.
Some linear and nonlinear constraints should be also $j = 1, \ldots, p'$ taken into account.
In this paper the detailed analysis will be carried on in the case, when $p = p' = 2$.

2 Problem Description

The optimization problem is formulated as follows:

$$\alpha n + \alpha_0 n_0 + \beta \cdot y(n) + \beta_0 y_0(n_0) \to \min \tag{1}$$

Subject to

$$\begin{Bmatrix} H_0 - G_0\left(\dfrac{y_0}{n_0}\right)^2 - (k_0 + z_0)y_0^2 - d_0 \geq 0 \\ H - G\left(\dfrac{y}{n}\right)^2 - (k + z)y^2 - d \geq 0 \end{Bmatrix} \tag{2}$$

$$y(n) + y_0(n_0) = \sigma$$

All the variables $y_0(n_0)$, $y(n)$ are subject to lower upper bounds constraints:

$$n\underline{y} \leq y \leq n\overline{y}, \quad n_0 \underline{y}_0 \leq y_0 \leq n_0 \overline{y}_0 \tag{3}$$

Where:
$\underline{y}, \overline{y}, \underline{y}_0, \overline{y}_0$ – denote respectively, upper and lower bounds of the variables y, y_0.
The coefficients $\alpha, \alpha_0, \beta, \beta_0 > 0$; n, n_0, y, y_0 are variables and the rests of elements in (2), (3) are known parameters.

The algorithm for problem (1) subject to constraints (2), (3) is composed on the following parts:

(i) the determination of the functions:

$$y = y(n); [y_0 = y_0(n_0)] \tag{4}$$

or

$$n = n(y); [n_0 = n_0(y_0)] \tag{5}$$

(ii) the determination of the sets of feasible solutions;
(iii) the determination of the optimal solution of the MINLP.
In this paper the subject of interest concerns point (ii) only.

3 An algorithm for the determination of feasible set solutions

We are looking for a solution of (1) subject to (2) and (3.) Taking into account already presented algorithms outlines it is necessary to determine (4) or (5) in explicit forms and finally optimal solution. From (2) and (3) one obtains the relations between continuous variables y, y_0 and integer variables n, n_0 by introducing following parameters:

$$0 \le p - p^* = z \cdot y^2, \quad 0 \le p_0 - p^* = z_0 y_0^2, \tag{6}$$

Let's look at the problem (1) under the constraints (2), (3). The set of feasible variables (n, $y(n)$, $y_0(n_0)$, n_0) should be determined first. There are one-to one relations between "y" an "n" an d "y_0" an "n_0"as follow:

$$\left\{ \begin{array}{lll} y = \sqrt{\dfrac{H-d-p}{G+kn^2}} \cdot n & \text{or} & n = \dfrac{\sqrt{G} \cdot y}{\sqrt{H-d-p-ky^2}} \\ y_0 = \sqrt{\dfrac{H_0-d_0-p_0}{G+k_0 n_0^2}} \cdot n_0 & \text{or} & n_0 = \dfrac{\sqrt{G_0} \cdot y_0}{\sqrt{H_0-d_0-p_0-k_0 y_0^2}} \end{array} \right\} \tag{7}$$

Hence the intersection of the sets in (3) is empty, if

$$\tilde{n}\bar{y} \ge \bar{y}(\tilde{n}+1)\underline{y} \Rightarrow \left\{ \begin{array}{l} \tilde{n} \ge \dfrac{\underline{y}}{\bar{y}-\underline{y}} \\ \tilde{n}_0 \ge \dfrac{\bar{y}_0}{\bar{y}_0-\underline{y}_0} \end{array} \right. \tag{8}$$

From (8) the feasible sets for p and p_0 parameters are obtained, when n and n_0 is fixed. Taking into account (6) one obtains:

$$\begin{Bmatrix} n\underline{y} \leq y(n) \quad \text{and} \quad n_0 \underline{y}_0(n_0) \\ 0 \leq p \leq \overline{H} - d - \underline{y}^2 \cdot kn^2 = \overline{p}; \qquad 0 \leq p_0 \leq \overline{H} - d_0 - k_0 n_0^2 \underline{y}_0 = \overline{p}_0 \end{Bmatrix} \tag{9}$$

Where:

$$\overline{H} = H - G\underline{y}^2; \qquad \overline{H}_0 = H_0 - G\underline{y}_0^2;$$

When p is chosen between 0 and $\overline{p}$, the values y is element of the following set:

$$y \in \left[n\underline{y};\ n\sqrt{\frac{H-d}{G+kn^2}} \right]$$
$$\left(\text{correspondingly } p_0 \in [0, \overline{p}] \Rightarrow y_0 \in \left[n_0 \underline{y}_0; n_0 \sqrt{\frac{H-d}{G_0 + k_0 n_0^2}} \right] \right); \tag{10}$$

Under the assumption that:

$$\sqrt{\frac{H-d}{G_0 + k_0 n_0^2}} \leq \overline{y}_0 . \tag{11}$$

In opposite case:

$$y_0 \in \left[n_0 \underline{y}_0, n_0 \overline{y}_0 \right]. \tag{12}$$

From the formula (9) the maximal values of variables n i n_0 are obtained. Because $\overline{p}$ i $\overline{p}_0 \geq 0$, hence

$$\overline{H} - d - \underline{y}^2 kn^2 \geq 0 \quad \text{and} \quad n \leq \overline{n} = \frac{1}{\underline{y}} \sqrt{\frac{\overline{H} - d}{k}}; \tag{13}$$

and

$$n_0 \leq \overline{n}_0 = \frac{1}{\overline{y}_0} \sqrt{\frac{\overline{H}_0 - d_0}{k_0}}; \tag{14}$$

Hence the maximal feasible values of y and y_0 are bounded, as below:

$$0 \le y(n) \le y(\bar{n}) \quad \text{and} \quad 0 \le y_0(n_0) \le y_0(\bar{n}_0)$$

$$y \le \sqrt{\frac{\bar{H}-d}{k}} \quad \text{and} \quad y_0 \le \sqrt{\frac{\bar{H}_0-d_0}{k_0}}\,. \tag{15}$$

Thus the estimated value of σ_{max} can be determined as:

$$\sigma_{\max} = \bar{\sigma} \ge \sigma = y + y_0$$

$$\sigma_{\max} = \sqrt{\frac{\bar{H}-d}{k}} + \sqrt{\frac{\bar{H}_0-d_0}{k_0}} = \bar{\sigma} \ge y(n) + y_0(n_0)\,. \tag{16}$$

Now we should take into account fact, that σ is a linear function of y and y_0 variables:

$$\sigma = y + y_0 \tag{17}$$

So, the additional constraints are obtained now, namely lower bounds of n i n_0

$$\underline{n} \le n \text{ and } \underline{n}_0 \le n_0 \tag{18}$$

and constraints modification of y_0.
Finally, following inequality (10), is obtained:

$$\left\{\begin{array}{l} \underline{n} = \dfrac{\sqrt{G}\left(\sigma - \sqrt{\dfrac{\bar{H}_0-d_0}{k_0}}\right)}{\sqrt{H-d-k\left(\sigma-\sqrt{\dfrac{\bar{H}_0-d_0}{k_0}}\right)^2}}; \\ \underline{n}_0 = \dfrac{\sqrt{G_0}\left(\sigma - \sqrt{\dfrac{\bar{H}-d}{k}}\right)}{\sqrt{H_0-d_0-k_0\left(\sigma-\sqrt{\dfrac{\bar{H}-d}{k}}\right)^2}}; \end{array}\right\}. \tag{19}$$

The following feasible sets occur for n and n_0 variables as follows:

$$\begin{Bmatrix} n \in \{0\} \cup [\underline{n}, \bar{n}] \\ n_0 \in \{0\} \cup [\underline{n}_0, \bar{n}_0] \end{Bmatrix} \tag{20}$$

For each feasible $n \neq 0$, under the assumption, that $\sqrt{\frac{H-d}{G_0 + k_0 n_0^2}} \leq \bar{y}$, one obtains from (12) feasible set of $y(n)$ values :

$$y(n) \in A_n \triangleq \left[n\underline{y}, n\sqrt{\frac{H-d}{G+kn^2}} \right]. \tag{21}$$

Hence, for each feasible pair of $(n, y(n))$ exists y_0, which should belong the following set:

$$y_0 \in A_n^0 \triangleq \left[\sigma - n\sqrt{\frac{H-d}{G+kn^2}}; \quad \sigma - n\underline{y} \right] \neq \varnothing. \tag{22}$$

For each feasible triple of elements $(n, y(n), y_0(n))$ exists nonempty set in the form of:

$$n_0 \in N_n^0 \triangleq \left[\frac{\sqrt{G_0}(\sigma - n\underline{y})}{\sqrt{H_0 - d_0 - k_0(\sigma - n\underline{y})^2}}; \quad \frac{\sigma - n\sqrt{\frac{H-d}{G+kn^2}}}{\underline{y}_0} \right]. \tag{23}$$

The feasible set of elements in the form of quadruples (n, y, y_0, n_0) is created in the following way:

$$D = \bigcup_{n \in [\underline{n}, \bar{n}]} \{(n, y, y_0, n_0); \quad \exists y \in A_n \neq 0; \quad \exists y_0 \in A_n^0, \exists n_0 \in Nn^0; \quad \underline{n} \leq n_0 \leq \bar{n}\} \tag{24}$$

The set of constraints could be in general incoherent set. The quadratic forms in (2) can not be neither (semi -) positive nor (semi -) negative ones and the feasible set of constraints can be non-convex one.

4 Numerical example for the proposed method

Water distribution network system seems to be a very good illustration for the already presented algorithm. System consists of two pumping station, with known maximal number of pump identical units (see Fig 1). The aggregated network is composed of two arcs with known parameters and known receiver's demands. The goal function in such a system represents the cost of electrical energy consumed by the pumps units when receiver's demands are fulfilled [2], [3], [5].

The mathematical model for this system is described as follow (25). It differs slightly from previously used models in the chapter **2**, **3**, because the quadratics forms (2) are "centred" by y^*, y_0^*.

$$\begin{cases} H_0 - G_0\left(\frac{y_0}{n_0} - y_0^*\right)^2 - (k_0 + z_0)y_0^2 - d_0 - p^* = 0 \\ H - G_0\left(\frac{y}{n} - y^*\right)^2 - (k + z)y^2 - d - p^* = 0 \end{cases} \tag{25}$$

$$y(n) + y_0(n_0) = \sigma$$

The systems parameters [2]:

Pump station P:
Number of pumps units: 3
Parameter H = 52,5776647
Parameter G = 0,0004761
Parameter y^* = -30,5363336
Parameter $\bar{y}$ = 192
Parameter $\underline{y}$ = 132
Parameter y_{max} = 431,64
Parameter $\tilde{n}$ =3,22
Parameter α = 1599,183
Parameter β = 19,775
Networks parameters:
Arc's pipeline resistant coefficient k = 0,00006058

Geodesic height d = 28,685

Pump station P_0:
Number of pumps units: 2
Parameter H_0 = 58,13348
Parameter G_0 = 0,000313388
Parameter y_0^* = 218,07342
Parameter $\bar{y}_0$ = 292
Parameter $\underline{y}_0$ = 245,4
Parameter y_{max} = 609,61
Parameter $\tilde{n}_0$ = 2,48
Parameter α_0 = 7992,195
Parameter β_0 = 10,798
Networks parameters:
Arc's pipeline resistant coefficient k_0 = 0,00003782
Geodesic height d_0 = 41,735

Aggregated water distribution models have been described by many authors [2], [3], [5]. The cost of electricity consumed by pump stations in the system is our goal function. The solution determines the number of pumps which should be turn on in the pump stations and their actual yields. It can be easy find that feasible set can be incoherent for all considered variables.

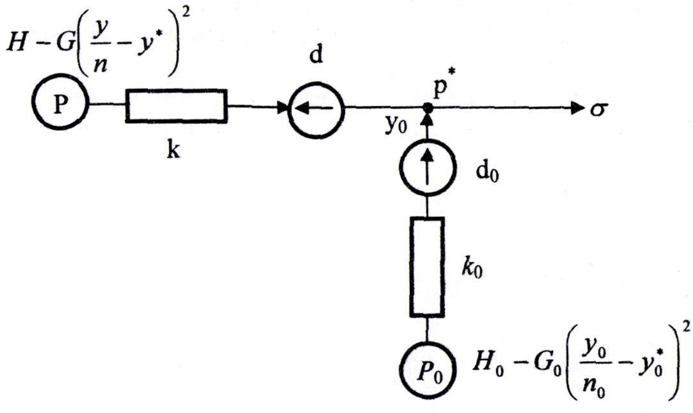

Figure 1: Aggregated water distribution system.

Parameters $\{\underline{n}, \overline{n}, \overline{n}_0, \underline{n}_0, \underline{y}(n), ..., y_0(\overline{n}_0), \overline{p}(n)\}$ determination.

On the Fig. 2, 3 are shown described beforehand parameter and functions. The feasible sets are denoted by A_1, A_2, A_3 and A_1, A_2.

For the first pump's station there were following parameters calculated:

- The one–to–one function $y(n)$ is given as follows:

$$y(n) = n\frac{-0{,}0144 + \sqrt{24{,}315(0{,}00048 + 0{,}00006498n^2) - 0{,}000028 \cdot n^2}}{0{,}00048 + 0{,}00006498n^2} \tag{26}$$

- The sets of feasible y according to the formula (21) is given below:

$A_1 = [132; 183{,}778]$
$A_2 = [264; 323{,}130]$
$A_3 = [396; 414{,}248]$

For the second pump's station one obtains:

- The one–to–one function $y_0(n_0)$ is given as follows:

$$y_0(n_0) = n_0 = \frac{3{,}66 + \sqrt{(16{,}265)(0{,}0031 + 0{,}00003782n_0^2) - n_0^2 \cdot 0{,}0070567}}{0{,}0031 + 0{,}00003782n_0^2} \tag{27}$$

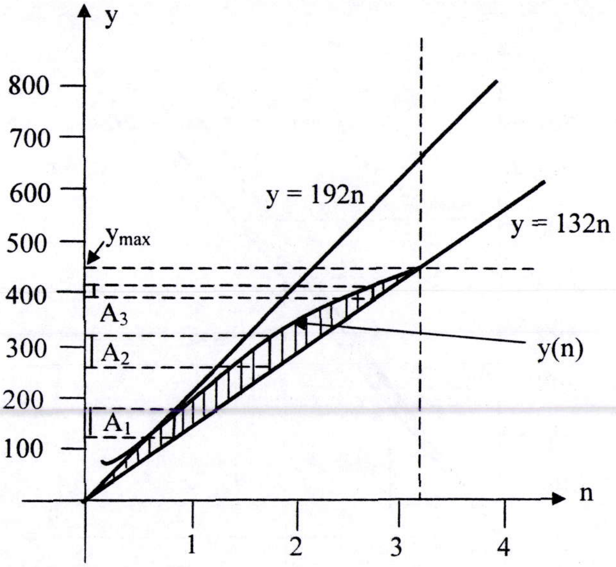

Figure 2: Feasible set for variable $y = y(n)$

- The sets of feasible variable y, according to the formula (22):

$A_1 = [243; 283{,}44]$
$A_2 = [490; 523{,}85]$

Declared consumers demand and maximal system outflow are given as follow:

$\sigma = 1000$
$\sigma_{\max} = \bar{y}_0(\bar{n}_0) + y(\bar{n}) = 1044{,}793$

The main goal of water distribution networks is to fulfill the receiver's demand. It is necessary to deliver the appropriate volume of water in the specified time intervals. Since the operation and maintenance of such system can be included in the capital costs, the most used optimization criterion is that of electrical energy costs minimization. The goal function (1) has its interpretation as the energy costs used by pumping stations in the analyzed water distribution network [2], [3].
On the Fig. 4 are shown the optimal strategy for all feasible σ.
There are some gaps on the σ (receiver's demands) axis. It is that for some σ values the optimal solution doesn't exist.

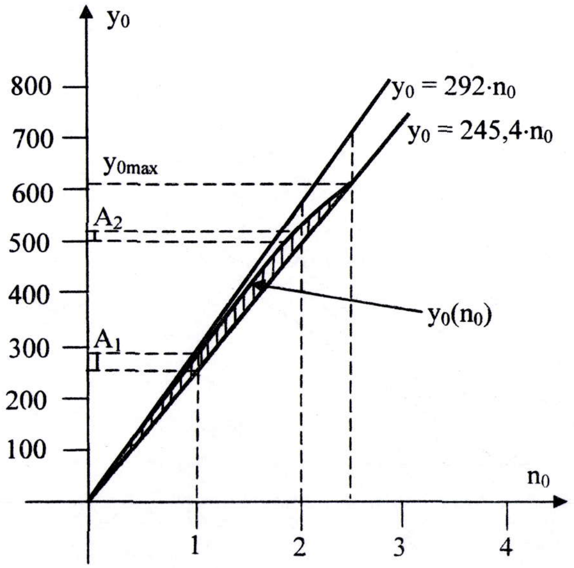

Figure 3: Feasible set of $y_0 = y_0\,(n_0)$

The one segment on the Fig. 4 denotes:

- The relationship between $\sigma = y + y_0$ and minimal energy necessary for the realization of such demand.
- The natural numbers above the each segment line denote the number of pump section which should be turn on at suitable pump station, where demanded value of σ should be achieved.

Our optimization problem is related to water distribution network system and involves both nonlinear constraints to model physical phenomena and integer variables to model control decisions. In the mentioned before case nonlinear constraints are connected with water distribution network. The discrete variables enable to turn on pumping sections in each pump station so they are control variables.

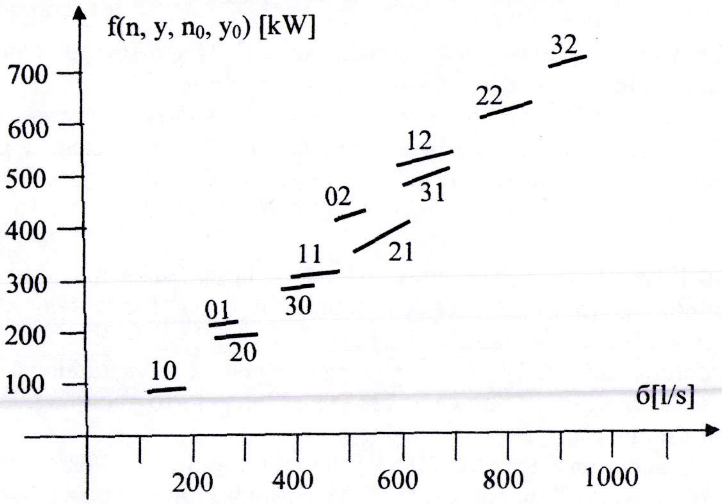

Figure 4: The relationship between goal function and receivers demands.

5 Remarks

The usual way to find the solutions for Mixed Integer Nonlinear Optimization Problem (MINLP) is by using a branch-and-bound method [1], [4]. [8], [9]. First necessary step relays on searching the feasible set of variables. The construction of such a set was presented here for the dimension of both variables equal two. Before, a very good illustration of mentioned method was given for water distribution network. Although, the dimension of continuous and discrete variables seems to be small, but obtained results looks very promising. The proposed attempt is a very good basis for more complicated and larger systems with higher variables dimensions. In the next study the algorithm for solving MINLP in the form of (1) subject to (2), (3) constraints will be presented. Of course, the dimension of both variables, continuous and integer will be greater than two.

The featured here algorithm is connected with the sequential optimization strategy [1]. It consists of solving a sequence of sub-problems where the level of difficulties is decreasing. The major motivation is to solve a sequence of simpler problems to avoid solving a large single and more difficult MINLP. The procedure allows reducing number of problems and obtained results confirm the usefulness of the proposed attempt.

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