Optimal Multiperiod Investment Strategy for Project Portfolio

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Abstract
Project portfolio investment is a crucial decision in many organizations, which must make informed decisions on investment, where the appropriate distribution of investment is complex, due to varying levels of risk, resource requirements, and interaction among the proposed projects. In this paper, we discuss an analytical optimal solution to the mean-variance formulation of the problem of optimization of multiperiod investment strategy for project portfolio. Specifically, analytical optimal multiperiod investment strategy for project portfolio is derived. An efficient algorithm is proposed in order to maximize the expected value of the terminal wealth under constraint that the variance of the terminal wealth is not greater than a preassigned risk level or to minimize the variance of the terminal wealth under constraint that the expected terminal wealth is not smaller than a preassigned level. A numerical example is given.
Keywords: Project portfolio; Multiperiod investment strategy; Optimization.

1 Introduction

Investment of project portfolio is to seek the best allocation of wealth among a basket of projects. Project portfolio investment is the periodic activity involved in investing a portfolio of projects, that meets an organization's stated objectives without exceeding available resources or violating other constraints. Some of the issues that have to be addressed in this process are the organization's objectives and priorities, financial benefits, intangible benefits, availability of resources, and risk level of the project portfolio. Investment strategy for a set of proposed projects is a crucial decision in many organizations, which must make informed decisions on investment, where the appropriate distribution of investment is complex, due to varying levels of risk, resource requirements, and interaction among the proposed projects.

There is a long stream of research in the field of project portfolio investment. It begins with early engineering management work using mathematical programming models, usually maximizing portfolio returns subject to budget and other constraints. Souder (1973, 1978) gives the earliest reviews of this initial research. Later, Czajkowski and Jones (1986) demonstrated a mixed integer linear programming model, which accounted for interactions between projects. Still later Graves and Ringuest (1992) showed that multiple objective mathematical programming models could be used in this context.

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Another stream of literature germane to our paper is related to financial portfolio models. The mean–variance methodology for the portfolio selection problem, posed originally by Markowitz (1959, 1989), has played an important role in the development of modern portfolio selection theory. It combines probability and optimization techniques to model the behavior investment under uncertainty. The return is measured by mean, and the risk is measured by variance, of a portfolio of assets. The Markowitz's mean–variance model for portfolio selection can be formulated mathematically in two ways: minimizing risk when a level return is given, maximizing return when a level risk is given. In the mean–variance portfolio selection problem, previous research includes Sharpe (1970), Szegö (1980), Perold (1984), Pang (1980), Elton and Gruber (1991), etc.

The mean-variance formulation by Markowitz (1959, 1989) provides a fundamental basis for project portfolio investment in a single period. The problem of multiperiod portfolio investment has been studied by Smith (1967), Mossin (1968), Merton (1990), Samuelson (1969), Fama (1970), Hakansson (1971), Elton and Gruber (1975), Winkler and Barry (1975), Francis (1976), Dumas and Luciano (1991), Grauer and Hakansson (1993), and Pliska (1997). The literature in multiperiod portfolio investment has been dominated by the results of maximizing expected utility functions of the terminal wealth and/or multiperiod consumption. Specially, investment situations where the utility functions are of power form, logarithm function, exponential function, or quadratic form have been extensively investigated in the literature.

To our knowledge, no analytical or efficient numerical method for finding the optimal strategy for the mean-variance formulation of the problem of multiperiod project portfolio investment has been reported in the literature. In this sense, the concept of the Markowitz's mean-variance formulation has not been fully utilized in multiperiod project portfolio investment. This paper considers an analytical optimal solution to the mean-variance formulation of the problem of multiperiod project portfolio investment. Specifically, analytical optimal multiperiod investment strategy for project portfolio is derived. An efficient algorithm is proposed in order to maximize the expected value of the terminal wealth under constraint that the variance of the terminal wealth is not greater than a preassigned risk level or to minimize the variance of the terminal wealth under constraint that the expected terminal wealth is not smaller than a preassigned level.

2 Problem Statement

We consider a portfolio with $(m+1)$ risky projects, with random rates of returns. Let w_0 be an initial wealth of an investor at time 0. The investor can allocate his wealth among the $(m+1)$ projects. The wealth can be reallocated among the $(m+1)$ projects at the beginning of each of the following $(T-1)$ consecutive time periods. The rates of return of the risky projects at time period τ within the planning horizon are denoted by a vector $\mathbf{r}_\tau=[r_{\tau(0)}, r_{\tau(1)}, \ldots, r_{\tau(m)}]'$, where $r_{\tau(j)}$ is the random return for project j at time period τ . It is assumed in this paper that vectors $\mathbf{r}_\tau$, $\tau = 0, 1, \ldots, T-1$, are statistically independent and return $\mathbf{r}_\tau$ has a known mean $\mathrm{E}\{\mathbf{r}_\tau\}=[\mathrm{E}\{r_{\tau(0)}\}, \mathrm{E}\{r_{\tau(1)}\}, \ldots, \mathrm{E}\{r_{\tau(m)}\}]'$ and a known covariance

$$\mathrm{Cov}\{\mathbf{r}_\tau\} = \begin{bmatrix} \sigma_{\tau(00)} & \cdots & \sigma_{\tau(0m)} \\ \vdots & \ddots & \vdots \\ \sigma_{\tau(0m)} & \cdots & \sigma_{\tau(mm)} \end{bmatrix}. \tag{1}$$

Let w_τ be the wealth of the investor at the beginning of the τth period, and let $s_{\tau(j)}, j \in \{1, \ldots, m\}$, be the amount invested in the jth risky project at the beginning of the τth time period. The amount investigated in the 0th risky project at the beginning of the τth time period is equal to

$$w_\tau - \sum_{j=1}^{m} s_{\tau(j)}. \tag{2}$$

An investor is seeking a best multiperiod investment strategy, $\mathbf{s}_\tau=[s_{\tau(0)}, s_{\tau(1)}, \ldots, s_{\tau(m)}]'$ for $\tau = 0, 1, 2, \ldots, T-1$, such that either (i) the expected value of the terminal wealth w_T, $\mathrm{E}\{w_T\}$, is maximized if the variance of the terminal wealth, $\mathrm{Var}\{w_T\}$, is not greater than a preassigned risk level $v^\bullet$, or (ii) the variance of the terminal wealth, $\mathrm{Var}\{w_T\}$, is minimized if the expected terminal wealth, $\mathrm{E}\{w_T\}$, is not smaller than a preassigned level $e^\bullet$. Mathematically, a mean-variance formulation for multiperiod project portfolio investment can be posed as one of the following two forms:

(i) Maximize

$$\mathrm{E}\{w_T\}$$

subject to

$$\mathrm{Var}\{w_T\} \le v^\bullet,$$

$$w_{\tau+1} = \sum_{j=1}^{m} r_{\tau(j)} s_{\tau(j)} + \left(w_\tau - \sum_{j=1}^{m} s_{\tau(j)} \right) r_{\tau(0)} = r_{\tau(0)} w_\tau + \mathbf{R}_\tau' \mathbf{s}_\tau, \quad \tau = 0, 1, 2, \ldots, T-1, \tag{3}$$

and

(ii) Minimize

$$\mathrm{Var}\{w_T\}$$

subject to

$$\mathrm{E}\{w_T\}\geq e^{\bullet},$$

$$w_{\tau+1}=\sum_{j=1}^{m} r_{\tau(j)} s_{\tau(j)}+\left(w_{\tau}-\sum_{j=1}^{m} s_{\tau(j)}\right) r_{\tau(0)}=r_{\tau(0)} w_{\tau}+\mathbf{R}_{\tau}' \mathbf{s}_{\tau}, \quad \tau=0, 1, 2, \ldots, T-1, \tag{4}$$

where

$$\mathbf{R}_{\tau}=[r_{\tau(1)}-r_{\tau(0)}, r_{\tau(2)}-r_{\tau(0)}, \ldots, r_{\tau(m)}-r_{\tau(0)}]'. \tag{5}$$

Notice that

$$\mathrm{E}\{\mathbf{r}_{\tau}\mathbf{r}_{\tau}'\}=\mathrm{Cov}\{\mathbf{r}_{\tau}\}+\mathrm{E}\{\mathbf{r}_{\tau}\}\mathrm{E}\{\mathbf{r}_{\tau}'\}. \tag{6}$$

It is assumed in this paper that $\mathrm{E}\{\mathbf{r}_{\tau}\mathbf{r}_{\tau}'\}$ is positive definite for all time periods, that is, $\mathrm{E}\{\mathbf{r}_{\tau}\mathbf{r}_{\tau}'\}>0$, $\forall\,\tau=0(1)T-1$.

Formulation (i) or (ii) enables an investor to specify a risk level he can afford when he is seeking to maximize his expected terminal wealth or specify an expected terminal wealth he would like to achieve when he is seeking to minimize the corresponding risk.

A strategy of multiperiod project portfolio investment is an investment sequence,

$$\mathbf{S}_T=[\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{T-1}], \tag{7}$$

where

$$\mathbf{s}_{\tau}=[s_{\tau(1)}, s_{\tau(2)}, \ldots, s_{\tau(m)}]', \quad \forall\,\tau=0(1)T-1, \tag{8}$$

More specifically, $\mathbf{S}_T$ is a feedback policy and $\mathbf{s}_{\tau}$ maps the wealth at the beginning of the τth period, w_{τ}, into a project portfolio decision in the τth period, i.e., $\mathbf{s}_{\tau}\equiv\mathbf{s}_{\tau}(w_{\tau})$. A multiperiod project portfolio investment strategy, $\mathbf{S}_T^*$, is said to be efficient if there exists no other multiperiod portfolio policy, $\mathbf{S}_T$, such that $\mathrm{E}\{w_T;\mathbf{S}_T\}\geq\mathrm{E}\{w_T;\mathbf{S}_T^*\}$ and

$\mathrm{Var}\{w_T;\mathbf{S}_T\}\leq\mathrm{Var}\{w_T;\mathbf{S}_T^*\}$ with at least one equality strictly. By varying the value of $v^\bullet$ in (i) or the value of $e^\bullet$ in (ii), the set of efficient multiperiod project portfolio investment strategies can be generated.

3 Optimal Multiperiod Investment Strategy

Theorem 1. The optimal strategy for multiperiod project portfolio investment for problems (i) and (ii) is specified by the following analytical expression:

$$\mathbf{s}_\tau^* = -\mathrm{E}^{-1}\{\mathbf{R}_\tau\mathbf{R}_\tau'\}\mathrm{E}\{r_{\tau(0)}\mathbf{R}_\tau\}w_\tau$$

$$+\left(\frac{gw_0+b_1(2ch)^{-1}}{2}\right)\left(\prod_{t=\tau+1}^{T-1}\frac{a_{1t}}{a_{2t}}\right)\mathrm{E}^{-1}\{\mathbf{R}_\tau\mathbf{R}_\tau'\}\mathrm{E}\{\mathbf{R}_\tau\}.\quad \tau=0(1)T-2, \tag{9}$$

$$\mathbf{s}_{T-1}^* = -\mathrm{E}^{-1}\{\mathbf{R}_{T-1}\mathbf{R}_{T-1}'\}\mathrm{E}\{r_{T-1(0)}\mathbf{R}_{T-1}\}w_{T-1}$$

$$+\left(\frac{gw_0+b_1(2ch)^{-1}}{2}\right)\mathrm{E}^{-1}\{\mathbf{R}_{T-1}\mathbf{R}_{T-1}'\}\mathrm{E}\{\mathbf{R}_{T-1}\}, \tag{10}$$

where

$$h=\begin{cases}\dfrac{b_1}{2\sqrt{c(v^\bullet-qw_0^2)}}, & \text{when (i) is solved,}\\[2ex] \dfrac{b_1^2}{2c[e^\bullet-(a_1+gb_1)w_0]}, & \text{when (ii) is solved,}\end{cases} \tag{11}$$

$$a_{1\tau}=\mathrm{E}\{r_{\tau(0)}\}-\mathrm{E}\{\mathbf{R}_\tau'\}\mathrm{E}^{-1}\{\mathbf{R}_\tau\mathbf{R}_\tau'\}\mathrm{E}\{r_{\tau(0)}\mathbf{R}_\tau\}, \tag{12}$$

$$a_{2\tau}=\mathrm{E}\{r_{\tau(0)}^2\}-\mathrm{E}\{r_{\tau(0)}\mathbf{R}_\tau'\}\mathrm{E}^{-1}\{\mathbf{R}_\tau\mathbf{R}_\tau'\}\mathrm{E}\{r_{\tau(0)}\mathbf{R}_\tau\}, \tag{13}$$

$$b_\tau=\mathrm{E}\{\mathbf{R}_\tau'\}\mathrm{E}^{-1}\{\mathbf{R}_\tau\mathbf{R}_\tau'\}\mathrm{E}\{\mathbf{R}_\tau\}, \tag{14}$$

$$b_{1\tau} = b_\tau \left(\prod_{t=\tau+1}^{T-1} a_{1t} \right) \left(2 \prod_{t=\tau+1}^{T-1} a_{2t} \right)^{-1}, \tag{15}$$

with

$$\prod_{t=T}^{T-1} a_{1t} = \prod_{t=T}^{T-1} a_{2t} = 1, \tag{16}$$

$$a_1 = \prod_{\tau=0}^{T-1} a_{1\tau}, \tag{17}$$

$$a_2 = \prod_{\tau=0}^{T-1} a_{2\tau}, \tag{18}$$

$$b_1 = \sum_{\tau=0}^{T-1} \left(\prod_{t=\tau+1}^{T-1} a_{1t} \right) b_{1\tau}, \tag{19}$$

$$c = \frac{b_1}{2} - b_1^2, \tag{20}$$

$$g = \frac{a_1 b_1}{c}, \tag{21}$$

$$q = a_2 - a_1^2 - cg^2. \tag{22}$$

In this case, the expected value and the variance of the terminal wealth are given by

$$\mathrm{E}\{w_T\} = (a_1 + gb_1)w_0 + \frac{b_1^2}{2ch}, \tag{23}$$

and

$$\mathrm{Var}\{w_T\} = \frac{b_1^2}{4ch^2} + qw_0^2 . \tag{24}$$

Proof. The proof is carried out by means of the method of Lagrange multipliers, optimality principle, and the method of mathematical induction. It is omitted here and will appear elsewhere. □

Thus, the optimal multiperiod investment strategy for project portfolio consists of two terms and exhibits a decomposition property between the investor's risk attitude and his current wealth. The second term in $\mathbf{s}_\tau^*$ is dependent on the investor's risk attitude and is independent of his current wealth. It can be calculated off-line before the real investment process starts. The first term in $\mathbf{s}_\tau^*$ is dependent on the current wealth and is independent of the investor's risk attitude. It is calculated on-line at every time period when the current wealth is observed.

4 Numerical Example

Consider the case of a stationary multiperiod process with $T = 4$. An investor has one unit of wealth at the very beginning of the planning horizon, i.e., w_0=1 The investor is trying to find the best allocation of his wealth among three risky projects, 0, 1, and 2 in order to maximize $\mathrm{E}\{w_4\}$ while keeping his risk not exceeding 1; that is, $v^* = 1$. The expected returns for risky projects, 0, 1, and 2 are $\mathrm{E}\{r_{\tau(0)}\}$=1.189, $\mathrm{E}\{r_{\tau(1)}\}$=1.314, and $\mathrm{E}\{r_{\tau(2)}\}$=1.335, τ= 0, 1, 2, 3. The covariance of $\mathbf{r}_\tau=[r_{\tau(0)}, r_{\tau(1)}, r_{\tau(2)}]'$ is

$$\mathrm{Cov}\{\mathbf{r}_\tau\} = \begin{bmatrix} 0.0099 & 0.0114 & 0.0120 \\ 0.0114 & 0.0535 & 0.0508 \\ 0.0120 & 0.0508 & 0.0864 \end{bmatrix}, \ \tau = 0,\ 1,\ 2,\ 3. \tag{25}$$

Thus,

$$\mathrm{E}\{\mathbf{R}_\tau\} = \mathrm{E}\{[r_{\tau(1)} - r_{\tau(0)}, r_{\tau(2)} - r_{\tau(0)}]'\} = [0.125,\ 0.146]', \tag{26}$$

$$\mathrm{E}\{\mathbf{R}_\tau\mathbf{R}'_\tau\} = \mathrm{E}\left\{\begin{bmatrix} r_{\tau(0)}^2 - 2r_{\tau(0)}r_{\tau(1)} + r_{\tau(1)}^2 & r_{\tau(0)}^2 - r_{\tau(0)}r_{\tau(1)} - r_{\tau(0)}r_{\tau(2)} + r_{\tau(1)}r_{\tau(2)} \\ r_{\tau(0)}^2 - r_{\tau(0)}r_{\tau(1)} - r_{\tau(0)}r_{\tau(2)} + r_{\tau(1)}r_{\tau(2)} & r_{\tau(0)}^2 - 2r_{\tau(0)}r_{\tau(2)} + r_{\tau(2)}^2 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0.056225 & 0.05555 \\ 0.05555 & 0.093616 \end{bmatrix}, \tag{27}$$

$$\mathrm{E}\{r_{\tau(0)}\mathbf{R}'_\tau\} = [\mathrm{E}\{r_{\tau(0)}r_{\tau(1)}\} - \mathrm{E}\{r_{\tau(0)}^2\}, \mathrm{E}\{r_{\tau(0)}r_{\tau(2)}\} - \mathrm{E}\{r_{\tau(0)}^2\}] = [0.150125, 0.175694].$$

$$\tau = 0, 1, 2, 3. \tag{28}$$

Furthermore, we have

$$b_\tau = 0.290972. \tag{29}$$

$$a_{1\tau} = 0.839340049. \tag{30}$$

$$a_{2\tau} = 1.003433478, \quad \tau = 0, 1, 2, 3, \tag{31}$$

$$a_1 = 0.496308579, \tag{32}$$

$$a_2 = 1.013804806, \tag{33}$$

$$b_1 = 0.36969, \tag{34}$$

$$c = 0.048174261, \tag{35}$$

$$g = 3.808681367, \tag{36}$$

and

$$q = 0.068664187. \tag{37}$$

From (11), the corresponding

$$h = 0.87266436. \tag{38}$$

The associated optimal strategy for multiperiod project portfolio investment is given, using equations (9) and (10), as follows:

$$\mathbf{s}_0^* = -\begin{bmatrix} 1.97189131 \\ 0.706668067 \end{bmatrix} w_0 + \begin{bmatrix} 3.960204 \\ 1.394898 \end{bmatrix}, \tag{39}$$

$$\mathbf{s}_1^* = -\begin{bmatrix} 1.97189131 \\ 0.706668067 \end{bmatrix} w_1 + \begin{bmatrix} 4.734435 \\ 1.667604 \end{bmatrix}, \tag{40}$$

$$\mathbf{s}_2^* = -\begin{bmatrix} 1.97189131 \\ 0.706668067 \end{bmatrix} w_2 + \begin{bmatrix} 5.660031 \\ 1.993626 \end{bmatrix}, \tag{41}$$

$$\mathbf{s}_3^* = -\begin{bmatrix} 1.97189131 \\ 0.706668067 \end{bmatrix} w_3 + \begin{bmatrix} 6.766584 \\ 2.383385 \end{bmatrix}. \tag{42}$$

The investment in the 0th project at period τ is equal to

$$s_{\tau(0)} = w_\tau - \sum_{j=1}^{m} s_{\tau(j)}, \quad \forall \tau = 0(1)3. \tag{43}$$

Since it is known that w_0=1, we obtain from (39) a percent investment (on w_0) in three risky projects, 0, 1, and 2, at period τ=0 as follows: $s_{0(0)}$= 97.323%, $s_{0(1)}$= 1.988%, $s_{0(2)}$= 0.689%.

The corresponding expected terminal wealth and the risk level are given by $E\{x_4\}$ = 3.529827827 and $Var\{x_4\}$=1, respectively, using equations (23) and (24).

5 Conclusions

The Markowitz mean-variance approach has been extended in this paper to multiperiod project portfolio investment problems.

The derived analytical optimal multiperiod project portfolio investment strategy provides investors with the best strategy to follow in a dynamic investment environment.

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