Examining Stability of Second-Order Slave Systems

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Abstract
Dynamical behavior of any second-order linear system is described by a couple of stable and unstable attractors. The same attractors describe behavior of the linearized part of the piecewise-linear (PWL) systems. Using PWL analyses we examine each region separately. In paper there are considered stable and unstable trajectories and Poincare maps.

Keywords: stability, attractor, PWL system, Poincare map, synchronization of chaos

1 Introduction

Considering the Pecora-Carroll drive concept of synchronization of chaos [1], we have analyzed second-order PWL circuits as basic synchronized subsystems. The second-order PWL system is described by two ordinary differential piecewise-linear equations which general form is given by

$$\dot{\mathbf{x}} = \mathbf{A}.\mathbf{x} + \mathbf{b}.h\left(\mathbf{w}^{\mathrm{T}}.\mathbf{x}\right), \tag{1.1}$$

where $\mathbf{A} \in \Re^{2x2}$, $\mathbf{b} \in \Re^2$, $\mathbf{w} \in \Re^2$ and the function $h(\mathbf{w}^{\mathrm{T}}.\mathbf{x})$ is given by

$$h(\mathbf{w}^{\mathrm{T}}.\mathbf{x}) = \frac{1}{2}\left(\left|\mathbf{w}^{\mathrm{T}}.\mathbf{x}+1\right| - \left|\mathbf{w}^{\mathrm{T}}\mathbf{x}-1\right|\right). \tag{1.2}$$

The function $h(\mathbf{w}^{\mathrm{T}}.\mathbf{x})$ is the continuous and odd-symmetric memoryless PWL feedback function partitioning $\Re^2$ by two parallel lines U_{+1} and U_{-1} into an inner region D_0 and two outer regions

2 Linear Analysis

The piecewise-linear analysis is a mean by which the state space of a nonlinear dynamical system is divided into a set of separate affine regions that may be studied in isolation and then „glued together" along their boundaries. The state equations (1.1) consist of the linear part that characterizes the inner region and the affine part that characterizes the outer regions. The dynamical behavior of this PWL system is determined by two sets of eigenvalues representing two characteristic polynomials associated with the corresponding regions. The characteristic polynomials are contained in state matrices $\mathbf{A}_0$ and $\mathbf{A}$, where $\mathbf{A}_0$ belongs to the inner region and $\mathbf{A}$ belongs to the outer region. The matrix $\mathbf{A}_0$ is defined to be $\mathbf{A}_0 = \mathbf{A} + \mathbf{b}.\mathbf{w}^{\mathrm{T}}$.

Considering the vector $\mathbf{w}$ as $\mathbf{w} = [1,0]$, the state matrices are defined to be

$$\begin{aligned} \mathbf{A}_0 &= \begin{bmatrix} a_{11} + b_1 & a_{12} \\ a_{21} + b_2 & a_{22} \end{bmatrix} \\ \mathbf{A} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{aligned} \tag{2.1}$$

The dynamical behavior of any second-order linear system is described by a couple of stable and unstable attractors [2]. The same attractors describe the behavior of the linear and affine parts of the PWL system as well. Using the piecewise-linear analysis, we examine each region separately and then glue the pieces together.

We have observed that the second-order PWL circuit synchronizes if it spirals toward either the origin or the limit cycle. The behavior of this circuit is described by either the stable focus or the stable limit cycle. Due to analysis of the state matrices $\mathbf{A}_0$ and $\mathbf{A}$, the stable PWL circuit synchronizes if eigenvalues of some part are of complex conjugate values. It ensures a spiral motion.

Consider the state matrices (2.1), the preposition of a spiral motion is mathematically expressed in this way that coefficients of the linear part are given by

$$(a_{11} + a_{22})^2 < 4(a_{11}.a_{22} + a_{12}.a_{21}) \tag{2.2}$$

and coefficients of the affine part are given by

$$((a_{11} + b_1) + a_{22})^2 < 4((a_{11} + b_1).a_{22} + a_{12}.(a_{21} + b_2)). \tag{2.3}$$

Because the vector field is separated into three regions, a general solution of the second-order PWL system consists of particular solutions which are determined into each parts separately.

A solution of the linear vector field is

$$\begin{aligned} x_1 &= C_1^{in}.{}^1x_1 + C_2^{in}.{}^2x_1 \\ x_2 &= C_1^{in}.{}^1x_2 + C_2^{in}.{}^2x_2, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} {}^1x_j &= e^{\mathrm{Re}(\upsilon).t}\left({}^1l_j.\cos(\mathrm{Im}(\upsilon).t) - {}^2l_j.\sin(\mathrm{Im}(\upsilon).t)\right) \quad (j = 1,2) \\ {}^2x_j &= e^{\mathrm{Re}(\upsilon).t}\left({}^1l_j.\cos(\mathrm{Im}(\upsilon).t) + {}^2l_j.\sin(\mathrm{Im}(\upsilon).t)\right) \quad (j = 1,2) \end{aligned} \tag{2.5}$$

are generic solutions of a second-order linear system. Parameters ${}^1l_j$ and ${}^2l_j$ are equal to ${}^1k_j = {}^1l_j + i.{}^2l_j$, or ${}^2k_j = {}^1l_j - i.{}^2l_j$ and parameters ${}^ik_j$ are solutions of the matrix equation $(\mathbf{A} - \upsilon_1.\mathbf{I}).\mathbf{k} = \mathbf{0}$. Constants C_1^{in}, C_2^{in} are set up by a point, where the trajectory of the PWL system crosses the lines U_{+1} or U_{-1} from the outer region to the inner region.

A general solution of the affine vector field is

$$x_1 = C_2^{out}.{}^1x_1 + C_2^{out}.{}^2x_1 + x_{q,1}$$
$$x_2 = C_1^{out}.{}^1x_2 + C_2^{out}.{}^2x_2 + x_{q,2}, \quad (2.6)$$

where ${}^ix_j$ are equal to Equations (2.5). Constants C_1^{out}, C_2^{out} are set up by a point where the trajectory crosses the lines U_{+1} or U_{-1} from the inner region to the outer region. An equilibrium point $\mathbf{X_q}$ in the outer region is given by

$$\mathbf{X_q} = -\mathbf{A}^{-1}.\mathbf{b} \quad (2.7)$$

if $\mathbf{A}^{-1}$ exists.

It is known that the second-order PWL circuit can synchronize if it is stable. The PWL system will be stable if real parts of eigenvalues of the linear and affine parts are negative. If all real parts are positive, the system will be unstable. In the case of different signs of real parts' values, the second-order PWL circuits can exhibit stable or unstable behavior and a decision about the stability must be made numerically.

3 Examining Stability with Using Characteristic Multipliers and Poincare Maps

The very common method of examining the stability of PWL and any other nonlinear systems is the method based on characteristic multipliers. The characteristic multipliers are determined by eigenvalues of a fundamental matrix of variational equations. Consider the nonlinear dynamical system described by n first-order nonlinear differential equations

$$\frac{dx_1}{dt} = f_1(x_1,\ldots,x_n)$$
$$\ldots \quad (3.1)$$
$$\frac{dx_n}{dt} = f_n(x_1,\ldots,x_n),$$

where $\boldsymbol{f}(\boldsymbol{x})$ are $\mathbf{C}^1$ functions. We can write $\frac{d\mathbf{x}}{dt} = \boldsymbol{f}(\boldsymbol{x})$ in short.

Variational equations of the system (3.1) are defined to be

$$\frac{d\mathbf{r}}{dt} = \frac{\partial f(x)}{\partial \mathbf{x}} . \boldsymbol{r} , \tag{3.2}$$

where the matrix $\frac{\partial f(x)}{\partial \mathbf{x}}$ is evaluated in points of a trajectory $\boldsymbol{x}(t)$ of the system (3.1). The fundamental matrix of variational equations is defined to be

$$\boldsymbol{U}_{x(0)}(t) = \left[r_{ij}(t)\right]. \tag{3.3}$$

and its initial conditions are $r_{ij}(0) = \delta_{ij}$.

Because eigenvalues of the fundamental matrix $\boldsymbol{U}_{x(0)}(\omega)$, where ω is a period, are just characteristic multipliers $\sigma_1,...,\sigma_n$, we try to get solutions $\boldsymbol{r}_j(t); j = 1,..,n$ of variational equations (3.2).

Theorem: Let us denote $\sigma_1,...,\sigma$ as the characteristic multipliers of the system (3.1). The nonlinear dynamical system will be Lyapunov stable if the characteristic multipliers satisfy conditions $|\sigma_i| \le 1;\ \mathrm{i} = 1,...,\mathrm{n}$. The nonlinear dynamical system will be asymptotically stable if the characteristic multipliers satisfy conditions $|\sigma_i| < 1;\ \mathrm{i} = 1,...,\mathrm{n}$ [4].

To obtain characteristic multipliers and points $\boldsymbol{x}(t)$ of a trajectory, the period ω and the fundamental matrix $\boldsymbol{U}_{x(0)}(\omega)$ must be known. There have been proposed a lot of methods for determining trajectories and their periods. In our research we have used the Poincare map method. The Poincare map method allows finding a point of a trajectory in the transversal plane [4].

In theory, if the second-order system is stable and starts from any point in the state space, its trajectory will go to either an equilibrium point or a limit cycle. You can get a closed set of crossing points in the Poincare map. If the system is unstable, the trajectory can go away in infinity and you can get an open set of crossing points.

3.1 The Algorithm for Computing Crossing Points in the Poincare Map

Consider the upper-plane

$$S(x_1,...,x_n) = 0 \tag{3.4}$$

which defines a Poincare map and a new variable

$$x_{n+1} = S(x_1,...,x_n) \tag{3.5}$$

that introduces a new differential equation

$$\frac{dx_{n+1}}{dt} = f_{n+1}(x_1,\ldots,x_n), \tag{3.6}$$

where $f_{n+1} = \sum_{i=1}^{n} f_i \frac{\partial S}{\partial x_i}$.

Then the system (3.1) is enlarged by the equation (3.6) and the new system is given by

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1,\ldots,x_n) \\ &\cdots\cdots \\ \frac{dx_{n+1}}{dt} &= f_{n+1}(x_1,\ldots,x_n). \end{aligned} \tag{3.7}$$

To obtain a trajectory of the system (3.7), we have to use the one-step integration method [4]. While the system (3.7) is being evaluated the variable x_{n+1} is being traced. If the variable x_{n+1} changes its sign, the evaluation will be interrupted and we will solve new equations given by

$$\begin{aligned} \frac{dx_1}{dx_{n+1}} &= \frac{f_1}{f_{n+1}} \\ &\cdots\cdots \\ \frac{dx_n}{dx_{n+1}} &= \frac{f_n}{f_{n+1}} \\ \frac{dt}{dx_{n+1}} &= \frac{1}{f_{n+1}} \end{aligned} \tag{3.8}$$

that determine a point in the Poincare map. Differential equations (3.8) are evaluated in a reverse direction by one numerical step $x_{n+1} = -S$ [4]. The solution of Equations (3.8) is the set $(x_1,\ldots,x_n,t_{present})$ and the point $(x_1,\ldots,x_n)$ fits into the Poincare map. The period of the trajectory is equal to $\omega = t_{present} - t_{last}$, where t_{last} has been stored.

Equations (3.7) and variational equations (3.2) must be computed numerically in common. After getting the crossing points, the characteristic multipliers are obtained from the matrix (3.3) with linear algebra rules.

4 Conclusion

Second-order PWL systems play a key role in designing synchronizing systems and synchronized chaotic signals. It is known a controlled system has to be stable. In the Pecora-Carroll drive concept the second-order PWL system should be a part of the third-order PWL system as well as it should be stable. We have used the theory in this paper in examining stability of second-order PWL systems. Algorithms in this paper follow designing rules of the synchronizing chaotic systems.

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