Precision and Stability Analysis of Euler, Runge-Kutta and Incursive Algorithms for the Harmonic Oscillator

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Abstract
This paper deals with a comparison from the precision and stability point of view of different discrete algorithms for simulating differential equation systems, applied in the case of a simple differential system: the harmonic oscillator. It points out the relation between the classical and incursive algorithms and shows the effect of incursion on the precision and stability.
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1 Introduction

The numerical simulation of continuous systems modelled by differential equations implies discrete transformation that leads to discrete equations, in which numerical integration formulas appear instead of differential derivatives.

The main issues in numerical simulation of differential systems are the stability and the precision of the simulation results, compared to the exact solution of the considered differential equations.

There is a wide range of numerical integration formulas (methods), of different complexity, from the simplest one - Euler, to more complex, such as high-order Runge-Kutta methods. The formulas complexity influences the results precision and stability, but also the simulation time, which could become critical in case of very complex systems, such as those encountered in hydrodynamics. This is why the analysis of discrete systems from the precision and stability point of view is very important in the numerical simulation process.

In section 2 we present precision and stability considerations for several classical numerical integration methods applied in the case of a simple differential system: the harmonic oscillator. In section 3 we show how incursion can improve the precision and stability, and the relation between the classical and (hyper)incursive discrete systems.

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2 Differential and Discrete Equation Systems: the Harmonic Oscillator Case

Let us consider the example of the harmonic oscillator represented by the following ordinary differential equations

$$dx(t)/dt = v(t) \tag{1a}$$
$$dv(t)/dt = -\omega^2 x(t) \tag{1b}$$

where x(t) is the position and v(t) the velocity as functions of the time t. The pulsation is given by $\omega = 2\pi/T$, where T is the period of oscillations.
The solution is given by

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \tag{2}$$

The parameters A and B are defined by the initial conditions x(0) and v(0).
This solution is stable: the amplitudes of the oscillations (represented by A and B) are fixed. In the phase space, given by (x(t), v(t)), the solutions are given by closed curves (orbital stability).

2.1 The Discrete Derivatives and the Concept of Time

It exists two discrete derivatives from the definition of the differential derivative dx(t)/dt : the forward and the backward derivatives defined as

$$dx(t)/dt = \lim_{h \to +0} [x(t + h) - x(t)]/h \tag{3a}$$

and

$$dx(t)/dt = \lim_{h \to -0} [x(t) - x(t - h]/h \tag{3b}$$

where $\lim_{h \to \pm 0}$ means the limit for *h* tending to zero by positive or negative values, *h* being an interval of time.
The differential derivative corresponds to these limits (which are equal for continuous derivable functions). Let us notice that these two discrete derivatives tend to different derivative for non-derivable functions as fractal curves, for example. Moreover, the discrete derivatives are always different for discrete equation systems !
The simulation of pure differential equations is impossible. This is only the discrete transformation which is computable (recursive function). Thus, there is already a difference in the representation of a system view form: differential equations or discrete equations. In differential equations, there is no time interval, there is only the current time. In discrete systems, there are the current time and the interval of time *h*.

Let us show now how to use these derivatives for transforming the differential equations (1a,b) to discrete equations.

2.2 The Harmonic Oscillator Discrete Equations

In computer science and discrete mathematics, it is common to have a special notation for representing the discrete equation system.

The discrete time is defined as

$$t_k = t_0 + kh \qquad \text{with } k = 0,1,2,... \tag{4}$$

where t_0 is the initial value of the time and k is the counter of the number of interval of time h.

The discrete variables are defined as

$$x_k = x(t_k) \tag{5a}$$
$$y_k = y(t_k) \tag{5b}$$

The discrete equations used in the harmonic oscillator case for computing the position and the velocity at consecutive moments have the general form

$$x_{k+1} = A\ x_k + B\ v_k \tag{6a}$$
$$v_{k+1} = C\ v_k - D\omega^2 x_k \tag{6b}$$

where A, B, C and D are coefficients with values specific to the numerical integration methods applied.

2.3 Precision and Stability Considerations

Since the exact solution for (1a,b) is

$$x_{k+1} = \cos(h\omega)\ x_k + \sin(h\omega)/\omega\ v_k \tag{7a}$$
$$v_{k+1} = \cos(h\omega)\ v_k - \sin(h\omega)\,\omega\, x_k \tag{7b}$$

it follows that the precision of different numerical solutions can be evaluated by comparing the values of their coefficients with values of the exact ones. Ideally

$$A = C = \cos(h\omega) \text{ and } B = D = \sin(h\omega)/\omega$$

Obviously, when numerical integration methods are applied, the precision is influenced by the values of the time interval h, also called integration step.

Usually $h = T/n$, with $n \geq 20$. Since $\omega = 2\pi/T$, where T is the period of oscillations, it follows that

$$h\omega = 2\pi / n \leq 0.1\pi.$$

Let us consider a few well-known integration methods.

The simplest one is the Euler method, with the use of the discrete equations

$$x_{k+1} = x_k + h\, v_k \tag{8a}$$
$$v_{k+1} = v_k - h\omega^2 x_k \tag{8b}$$

that have the form of (6a,b) with $A_e = C_e = 1$ and $B_e = D_e = h$.

More elaborated methods are the Runge-Kutta methods of different orders ($R = 2,3,4,\ldots$). The general expressions for the Runge-Kutta methods applied to the equations of the harmonic oscillator are

$$x_{k+1} = x_k + (K_{R,1}/c_1 + \ldots + K_{R,R}/c_R) \tag{9a}$$
$$v_{k+1} = v_k + (L_{R,1}/c_1 + \ldots + L_{R,R}/c_R) \tag{9b}$$

where $(1/c_1 + \ldots + 1/c_R) = 1$.

The formulas for coefficients in (9a,b) are given in Table 1.

Table 1: Coefficients used in Runge-Kutta equations.

Order	$K_{R,i}$	$L_{R,i}$	c_i
R = 2	$K_{2,1}=h\, v_k$ $K_{2,2}=h\,(v_k + L_{2,1})$	$L_{2,1} = -h\,\omega^2 x_k$ $L_{2,2} = -h\,\omega^2 (x_k + K_{2,1})$	$c_1 = c_2 = 2$
R = 3	$K_{3,1}=h\, v_k$ $K_{3,2}=h\,(v_k + L_{3,1}/2)$ $K_{3,3}=h\,(v_k - L_{3,1} + 2\,L_{3,2})$	$L_{3,1} = -h\,\omega^2 x_k$ $L_{3,2} = -h\,\omega^2 (x_k + K_{3,1}/2)$ $L_{3,3} = -h\,\omega^2 (x_k - K_{3,1} + 2K_{3,2})$	$c_1 = 6$ $c_2 = 6/4$ $c_3 = 6$
R = 4	$K_{4,1}=h\, v_k$ $K_{4,2}=h\,(v_k + L_{4,1}/2)$ $K_{4,3}=h\,(v_k + L_{4,2}/2)$ $K_{4,4}=h\,(v_k + L_{4,3})$	$L_{4,1} = -h\,\omega^2 x_k$ $L_{4,2} = -h\,\omega^2 (x_k + K_{4,1}/2)$ $L_{4,3} = -h\,\omega^2 (x_k + K_{4,2}/2)$ $L_{4,4} = -h\,\omega^2 (x_k + K_{4,3})$	$c_1 = c_4 = 6$ $c_2 = c_3 = 3$

By making the appropriate substitutions in equations (8a,b), we obtain the specific discrete equations for:

RK2 - second order Runge-Kutta oscillator

$$\begin{aligned} x_{k+1} &= x_k + h\,(v_k + v_k + L_{2,1})/2 = x_k + h\, v_k - h^2\omega^2 x_k/2 \\ &= (1 - h^2\omega^2/2)\, x_k + h\, v_k \end{aligned} \tag{11a}$$
$$\begin{aligned} v_{k+1} &= v_k - h\,\omega^2\,(x_k + x_k + K_{2,1})/2 = v_k - h\omega^2 x_k - h^2\omega^2 v_k/2 \\ &= (1 - h^2\omega^2/2)\, v_k - h\omega^2 x_k \end{aligned} \tag{11b}$$

RK3 - third order Runge-Kutta oscillator

$$
\begin{aligned}
x_{k+1} &= x_k + h\,(v_k + 4\,v_k + 4\,L_{3,1}/2 + v_k - L_{3,1} + 2\,L_{3,2})/6 \\
&= x_k + h\,v_k - h^2\omega^2\,(x_k + 2(x_k + K_{3,1}/2))/6 \\
&= x_k + h\,v_k - h^2\omega^2\,(3\,x_k + h\,v_k)/6 \\
&= (1 - h^2\omega^2/2)\,x_k + (1 - h^2\omega^2/6)\,h\,v_k \qquad (12a) \\
v_{k+1} &= v_k - h\,\omega^2\,(x_k + x_k + 4K_{3,1}/2 + x_k - K_{3,1} + 2K_{3,2})/6 \\
&= v_k - h\omega^2\,x_k - h^2\omega^2\,(v_k + 2(v_k + L_{3,1}/2))/6 \\
&= v_k - h\omega^2\,x_k - h^2\omega^2\,(3\,v_k - h\,\omega^2\,x_k)/6 \\
&= (1 - h^2\omega^2/2)\,v_k - (1 - h^2\omega^2/6)\,h\omega^2\,x_k \qquad (12b)
\end{aligned}
$$

RK4 - fourth order Runge-Kutta oscillator

$$
\begin{aligned}
x_{k+1} &= x_k + h\,(v_k/6 + (v_k + L_{4,1}/2)/3 + (v_k + L_{4,2}/2)/3 + (v_k + L_{4,3})/6) \\
&= x_k + h\,v_k - h^2\omega^2\,(x_k/6 + (x_k + K_{4,1}/2)/6 + (x_k + K_{4,2}/2)/6) \\
&= x_k + h\,v_k - h^2\omega^2\,(x_k/2 + h\,v_k/12 + h\,(v_k + L_{4,1}/2)/12) \\
&= (1 - h^2\omega^2/2)\,x_k + h\,v_k - h^2\omega^2(h\,v_k/6 - h^2\omega^2\,x_k/24) \\
&= (1 - h^2\omega^2/2 + h^4\omega^4/24)\,x_k + (1 - h^2\omega^2/6)\,h\,v_k \qquad (13a) \\
v_{k+1} &= v_k - h\omega^2\,(x_k/6 + (x_k + K_{4,1}/2)/3 + (x_k + K_{4,2}/2)/3 + (x_k + K_{4,3})/6) \\
&= v_k - h\omega^2\,x_k - h^2\omega^2\,(v_k/6 + (v_k \quad L_{4,1}/2)/6 + (v_k + L_{4,2}/2)/6) \\
&= (1 - h^2\omega^2/2)\,v_k - h\omega^2\,x_k - h^2\omega^2\,(-h\,\omega^2\,x_k/12 - h\,\omega^2(x_k + K_{4,1}/2)/12) \\
&= (1 - h^2\omega^2/2)\,v_k - (1 - h^2\omega^2/6)\,h\omega^2\,x_k - h^2\omega^2\,(-h\,\omega^2(h\,v_k)/24) \\
&= (1 - h^2\omega^2/2 + h^4\omega^4/24)\,v_k - (1 - h^2\omega^2/6)\,h\omega^2\,x_k \qquad (13b)
\end{aligned}
$$

These equations have the same form as (6a,b), with the coefficients given in table 2.

Table 2: Coefficients used in harmonic oscillator Runge-Kutta discrete equations.

Method	$\mathbf{A_R = C_R}$	$\mathbf{B_R = D_R}$
RK2	$(1 - h^2\omega^2/2)$	h
RK3	$(1 - h^2\omega^2/2)$	$(1 - h^2\omega^2/6)\,h$
RK4	$(1 - h^2\omega^2/2 + h^4\omega^4/24)$	$(1 - h^2\omega^2/6)\,h$

If we compare these coefficients with the exact ones, it is obvious that they represent approximations obtained by retaining 1 to 3 terms of the Taylor series for the sin and cos functions:

$$
\begin{aligned}
\sin(h\omega) &= h\omega - h^3\omega^3/3! - \ldots = h\omega\,(1 - h^2\omega^2/6 + \ldots) \\
\cos(h\omega) &= 1 - h^2\omega^2/2! + h^4\omega^4/4! - \ldots
\end{aligned}
$$

The stability analysis for discrete systems of the general form (6a,b) can be performed by using the unilateral Z-transform, defined as

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \qquad (14)$$

where n is an integer and z is a complex variable.

For obtaining the Z-transform of the equations (6a,b), we apply its shift property
$Z\{x(n+1)\} = z\,X(z) - z\,x_0$
and obtain:
$z\,X(z) - z\,x_0 = A\,X(z) + BV(z) \quad => \quad (z - A)\,X(z) = z\,x_0 + BV(z)$
$z\,V(z) - z\,v_0 = C\,V(z) - D\,\omega^2 X(z) \quad => \quad (z - C)\,V(z) = z\,v_0 - D\,\omega^2 X(z)$
$=> \quad (z - A)(z - C)\,X(z) = z\,(z-A)\,x_0 + B\,z\,v_0 - B\,D\,\omega^2 X(z)$
$=> \quad X(z) = P_1(z) / (z^2 - (A + C)z + (A\,C + B\,D\,\omega^2))$
This Z-transform has two poles:
$z_{1,2} = ((A + C) \pm \sqrt{((A + C)^2 - 4(AC + BD\,\omega^2))} / 2$
$\quad = ((A + C) \pm \boldsymbol{i}\,\sqrt{-(A + C)^2 + 4(AC + BD\,\omega^2)}) / 2$ (15)
which are complex when
$(A + C)^2 < 4(AC + BD\,\omega^2)$ (16a)
The position of the Z-transform poles relative to the unit circle defines the system stability: a system is stable if the poles lie inside the unit circle, is unstable if the poles lie outside the unit circle and at the limit of stability if the poles lie on the unit circle.
It follows that the condition for stability is:
$((A + C)^2 - (A + C)^2 + 4(AC + BD\,\omega^2)) / 4 \le 1$
$AC + BD\,\omega^2 \le 1$ (16b)
and the orbital stability must satisfy the strict equality
$AC + BD\,\omega^2 = 1$ (16c)

So, for the harmonic oscillator, the conditions for obtaining an orbital stability are given by relations 16a and 16c, rewritten as

$\boldsymbol{(A + C)^2 < 4}$ (17a)
$\boldsymbol{AC + BD\,\omega^2 = 1}$ (17b)

in using the equality (from the relation 16c), $BD\,\omega^2 = 1 - AC$, in the relation 16a.

Let us verify the stability condition for Euler and Runge-Kutta oscillators:

- Euler:

$AC + BD\,\omega^2 = 1 + h^2\omega^2 > 1$

Consequently this discrete system is **unstable**.

- RK2:

$AC + BD\,\omega^2 = (1 - h^2\omega^2 / 2)^2 + h^2\omega^2 = 1 - h^2\omega^2 + h^4\omega^4 / 4 + h^2\omega^2 = 1 + h^4\omega^4 > 1$

This system is also **unstable**.

- RK3:

$AC + BD\,\omega^2 = (1 - h^2\omega^2 / 2)^2 + (1 - h^2\omega^2 / 6)^2 h^2\omega^2$
$= 1 - h^2\omega^2 + h^4\omega^4 / 4 + (1 - h^2\omega^2 / 3 + h^4\omega^4 / 36)h^2\omega^2$
$= 1 + h^2\omega^2\,(-1 + h^2\omega^2 / 4 + 1 - h^2\omega^2 / 3 + h^4\omega^4 / 36)$
$= 1 + h^4\omega^4\,(h^2\omega^2 - 3) / 36$

The stability condition for this case becomes : $(h^2\omega^2 - 3) \leq 0 \Rightarrow \boldsymbol{h\omega \leq \sqrt{3}}$ condition that can be satisfied by an appropriate choice of the integration step h.
The orbital stability is obtained for $\boldsymbol{h\omega} = \sqrt{3}$, but this gives a very small number integration steps, $n = 2\pi / h\omega$, by period.

- RK4:

$$
\begin{aligned}
AC + BD\,\omega^2 &= (1 - h^2\omega^2/2 + h^4\omega^4/24)^2 + (1 - h^2\omega^2/6)^2 h^2\omega^2 \\
&= 1 + h^4\omega^4/4 + h^8\omega^8/24^2 - h^2\omega^2 + h^4\omega^4/12 - h^6\omega^6/24 + \\
&\qquad + h^2\omega^2 - h^4\omega^4/3 + h^6\omega^6/36 \\
&= 1 + h^6\omega^6(h^2\omega^2 - 8)
\end{aligned}
$$

In this case the stability condition is: $(h^2\omega^2 - 8) \leq 0 \Rightarrow \boldsymbol{h\omega \leq 2\sqrt{2}}$
and imposes the upper limit of the integration step h.
The orbital stability is obtained for $\boldsymbol{h\omega = 2\sqrt{2}}$, but the condition 17a for an oscillatory solution is no more satisfied. Indeed, we must have $h\omega < 2$ (for $h\omega = 2$, $(A + C)^2 > 1$).

3 The Incursive Oscillators

Rather recently, Daniel Dubois proposed a new schema for simulating differential equation systems in using both the forward and backward discrete derivatives (3a,b).

The following discrete equation system uses the forward derivative for the first equation and the backward derivative for the second equation.

The order in which the equations are computed is important (if no transformation is made as in the implicit schema of Euler).

Compute firstly the first equation to obtain x_{k+1}, and then compute the second equation in using the just computed x_{k+1}.

$$x_{k+1} = x_k + h\, v_k \tag{18a}$$
$$v_{k+1} = v_k - h\,\omega^2\, \boldsymbol{x_{k+1}} \tag{18b}$$

Daniel Dubois called such a system, an **incursive system**, for inclusive or implicit recursive system.

A second possibility occurs if we use the backward derivative for the first equation and the forward derivative for the second equation as follows

$$x_{k+1} = x_k + h\, \boldsymbol{v_{k+1}} \tag{19b}$$
$$v_{k+1} = v_k - h\,\omega^2\, x_k \tag{19a}$$

In this second incursive schema, it is necessary to compute firstly the second equation to obtain v_{k+1} which is then transmitted to the first equation.

In both cases the Z-transform poles lie on the unit circle, which means that by applying incursion the system shows an orbital stability.

An important difference between the classical schemas and the incursive schema is the fact that in the incursive schema, the order in which the computations are made is important, this is a sequential computation of equations. At the contrary, in the classical schemas, the order in which the computations are made is without importance: this is a parallel computation of equations.

The order of computations is not relevant if we modify equations 18a and 19b as follows:

$$v_{k+1} = v_k - h\,\omega^2 (x_k + h\,v_k) = (1 - h^2\omega^2)\,v_k - h\,\omega^2\,x_k \tag{18b'}$$
$$x_{k+1} = x_k + h\,(v_k - h\,\omega^2\,x_k) = (1 - h^2\omega^2)\,x_k + h\,v_k \tag{19b'}$$

Thus we obtain the **Incursive on v (Incv)** and the **Incursive on x (Incx)** algorithms

$$x_{k+1} = x_k + h\,v_k \tag{20a}$$
$$v_{k+1} = (1 - h^2\omega^2)\,v_k - h\,\omega^2\,x_k \tag{20b}$$

$$x_{k+1} = (1 - h^2\omega^2)\,x_k + h\,v_k \tag{21a}$$
$$v_{k+1} = v_k - h\,\omega^2\,x_k \tag{21b}$$

Table 3 gives the coefficients of these incursive algorithms.

Table 3: Coefficients of incursive equations.

Algorithms	A	B = D	C
Incv	1	h	$1-h^2\omega^2$
Incx	$1-h^2\omega^2$	h	1

From the orbital stability conditions 17ab we have, in both cases,

$$(A + C)^2 = (2 - h^2\omega^2)^2 < 4, \quad \textbf{for} \quad h\omega < 2$$
$$AC + BD\,\omega^2 = 1 - h^2\omega^2 + h^2\omega^2 = 1$$

which gives to the discrete harmonic oscillator an orbital stability for $h\omega < 2$.
Both incursive algorithms insure better than the Euler and Rung-Kutta algorithms.

It is interesting to remark that by mediating the equations of the two incursive systems [$(20a + 21a)/2$ and $(20b + 21b)/2$]

$$x_{k+1} = (x_k + h\,v_k + (1 - h^2\omega^2)\,x_k + h\,v_k))/2 = (1 - h^2\omega^2/2)\,x_k + h\,v_k \tag{22a}$$
$$v_{k+1} = ((1 - h^2\omega^2)\,v_k - h\,\omega^2\,x_k + v_k - h\,\omega^2\,x_k)/2 = (1 - h^2\omega^2/2)\,v_k - h\,\omega^2\,x_k \tag{22b}$$

we obtain the RK2 system, which looks more precise, but is unstable.
Nevertheless, stability can be achieved if "half-step incursion" is applied on (22a,b) rewritten as follows:

$$x_{k+1} = x_k + h\,(v_k - h/2\,\omega^2\,x_k) = x_k + h\,v_{k+1/2}$$
$$v_{k+1} = v_k - h\omega^2\,(x_k + h/2\,v_k) = v_k - h\omega^2\,x_{k+1/2}$$

where

$$x_{k+1/2} = x_k + h/2\,v_k$$
$$v_{k+1/2} = v_k - h/2\,\omega^2\,x_k$$

Applying "half-step incursion" implies the use of the backward derivative either for the first or for the second half-step equation and transform again the full-step equations and leads to:

HSIx.

$$x_{k+1/2} = x_k + h/2\,v_{k+1/2} \doteq x_k + h/2\,(v_k - h/2\,\omega^2\,x_k) = (1 - h^2\omega^2/4)\,x_k + h/2\,v_k$$

$$x_{k+1} = (1 - h^2\omega^2/2)\,x_k + h\,v_k \tag{23a}$$
$$v_{k+1} = v_k - h\omega^2\,x_{k+1/2} = v_k - h\omega^2((1 - h^2\omega^2/4)\,x_k + h/2\,v_k)$$
$$= (1 - h^2\omega^2/2)\,v_k - h\,(1 - h^2\omega^2/4)\,\omega^2\,x_k \tag{23b}$$

HSIv.

$$v_{k+1/2} = v_k - h/2\,\omega^2\,x_{k+1/2} = v_k - h/2\,\omega^2(x_k + h/2\,v_k)$$
$$= (1 - h^2\omega^2/4)\,v_k - h/2\,\omega^2\,x_k$$

$$x_{k+1} = x_k + h\,v_{k+1/2} = x_k + h((1 - h^2\omega^2/4)\,v_k - h/2\,\omega^2\,x_k)$$
$$= (1 - h^2\omega^2/2)\,x_k + h(1 - h^2\omega^2/4)\,v_k \tag{24a}$$
$$v_{k+1} = (1 - h^2\omega^2/2)\,v_k - h\,\omega^2\,x_k \tag{24b}$$

Table 4: Coefficients of half-step incursion equations

Equations	**A = C**	**B**	**D**
HSIx	$1 - h^2\omega^2/2$	h	$h\,(1 - h^2\omega^2/4)$
HSIv	$1 - h^2\omega^2/2$	$h\,(1 - h^2\omega^2/4)$	h

Let us compare the coefficients in table 4 with those in RK2 and RK3 (table 2). While A and C are identical, the situation is different for B and D.

$$B_{HSIx} = B_{RK2} = h - h^3\omega^2/6 + h^3\omega^2/6 = B_{RK3} + h^3\omega^2/6$$
$$D_{HSIx} = h - h^3\omega^2/4 = D_{RK2} - h^3\omega^2/4 = D_{RK3} - h^3\omega^2/12$$

Similarly

$$B_{HSIv} = B_{RK2} - h^3\omega^2/4 = B_{RK3} - h^3\omega^2/12$$
$$D_{HSIv} = D_{RK2} = D_{RK3} + h^3\omega^2/6$$

From the conditions 17ab we have, in both cases,

$$(A+C)^2 = (2-h^2\omega^2)^2 < 4, \textbf{ for } h\omega < 2$$
$$AC + BD\,\omega^2 = (1-h^2\omega^2/2)^2 + h^2(1-h^2\omega^2/4)\omega^2$$
$$= 1 - h^2\omega^2 + h^4\omega^4/4 + h^2\omega^2 - h^4\omega^4/4 = 1$$

which means that both systems show an orbital stability for $h\omega < 2$.

If we combine in the same way the equations (18a,b) with (19a,b) the result is

$$x_{k+1} = (x_k + h\,v_k + x_k + h\,v_{k+1})/2 = x_k + h\,(v_k + v_{k+1})/2$$
$$v_{k+1} = (v_k - h\,\omega^2 x_{k+1} + v_k - h\,\omega^2 x_k)/2 = v_k - h\,\omega^2 (x_k + x_{k+1})/2$$

which is in fact the classical Trapeze integration method.

There are two ways to transform these equations in order to obtain executable variants: either replace v_{k+1} in the first equation (Tv) or replace x_{k+1} in the second equation (Tx), and then rewrite both equations, as follows.

Tv.

$$x_{k+1} = x_k + h\,(v_k + (v_k - h\,\omega^2 (x_k + x_{k+1})/2))/2$$
$$= (1-h^2\omega^2/4)\,x_k + h\,v_k - h^2\omega^2/4\,x_{k+1}$$

With the notation, E = $(1 + h^2\omega^2/4)$, the discrete equations become

$$x_{k+1} = (1-h^2\omega^2/4)/E\,x_k + h/E\,v_k$$
$$v_{k+1} = v_k - h\,\omega^2 (x_k + (1-h^2\omega^2/4)/E\,x_k + h/E\,v_k)/2$$
$$= (E - h^2\omega^2/2)/E\,v_k - h\omega^2 x_k (E + (1-h^2\omega^2/4)/E/2$$
$$= (1+h^2\omega^2/4 - h^2\omega^2/2)/E\,v_k - h\omega^2 x_k (1+h^2\omega^2/4 + 1 - h^2\omega^2/4)/E/2$$
$$= (1-h^2\omega^2/4)/E\,v_k - h/E\,\omega^2 x_k$$

For **Tx** the resulting equations are exactly the same, with

A = C = $(1-h^2\omega^2/4)/E$
B = D = h/E

and the conditions 17ab give

$$(A+C)^2 = (2(1-h^2\omega^2/4)/E)^2 < 4, \textbf{ for } h\omega > 0$$
$$AC + BD\,\omega^2 = (1-h^2\omega^2/4)^2/E^2 + h^2\omega^2/E^2 = (1+h^2\omega^2/4)^2/(1+h^2\omega^2/4)^2 = 1$$

which means that both systems show an orbital stability for any values of $h\omega > 0$.

From the above it follows that incursion is very useful approach, since by applying it the unstable systems (Euler and Runge-Kutta) were brought at the limit of stability.

4 Simulations with the Euler, Runge-Kutta and Incursive Algorithms

Let us give a few simulations with the Euler, Runge-Kutta and Incursive algorithms.

Figure 1 gives the simulation in the phase space ($x(t)$, $v(t)$) of the exact algorithm 7ab, for $\omega = 1$, $h = 0.3$, with the initial conditions $x(0) = 2$ and $v(0) = 0$.
The orbital stability is well-shown in the phase space as a closed curve.

Figure 2 gives the simulation in the phase space ($x(t)$, $v(t)$) of the Euler algorithm 8ab, for $\omega = 1$, $h = 0.3$, with the initial conditions $x(0) = 0.2$ and $v(0) = 0$.
The simulation shows the strong instability of this algorithm: the amplitude increases drastically with the time steps.

Figures 3abcd give the simulation in the phase space ($x(t)$, $v(t)$) of the Runge-Kutta algorithms 9ab, for $\omega = 1$, $h = 0.3$.

Figure 3a gives the simulation of the Runge-Kutta-2 algorithm, with the initial conditions $x(0) = 1.5$ and $v(0) = 0$, with 500 time steps.
The simulation shows the instability of this algorithm: the amplitude increases

Figure 3b gives the simulation of the Runge-Kutta-3 algorithm with the initial conditions $x(0) = 2$ and $v(0) = 0$, with 500 time steps.
The simulation shows the instability of this algorithm: the amplitude decreases.

Figure 3c gives the simulation of the Runge-Kutta-4 algorithm with the initial conditions $x(0) = 2$ and $v(0) = 0$, with 500 time steps.
The simulation seems to show an orbital stability, but with more time steps (see figure 3d), this algorithm is also unstable: the amplitude decreases more slowly than in the Runge-Kutta-3 algorithm.

Figure 3d gives the simulation of the Runge-Kutta-4 algorithm with the initial conditions $x(0) = 2$ and $v(0) = 0$, with 20000 time steps.
The simulation shows clearly that this algorithm is also unstable: the amplitude decreases with the time steps.

Figures 4ab give the simulation in the phase space ($x(t)$, $v(t)$) of the Incursive algorithms 18ab and 19ab, for $\omega = 1$, $h = 0.3$, with the initial conditions $x(0) = 2$ and $v(0) = 0$, with 20000 time steps.
The simulations show a perfect orbital stability for both the two incursive algorithms.

Remark 1: The integration step, $h = 0.3$, is chosen with a rather large value in view of seeing more clearly the stability of each algorithms. This gives the simulation of one period with about 21 steps. In practice, smaller integrations steps are chosen for assuring a better stability of the algorithms, but this increases the number of time steps.

Remark 2: For the Euler and the Runge-Kutta-2 algorithms, the initial value is taken smaller than in the other algorithms, because they are unstable with an increases of the amplitude. So the simulation results remain within the same coordinates scale.

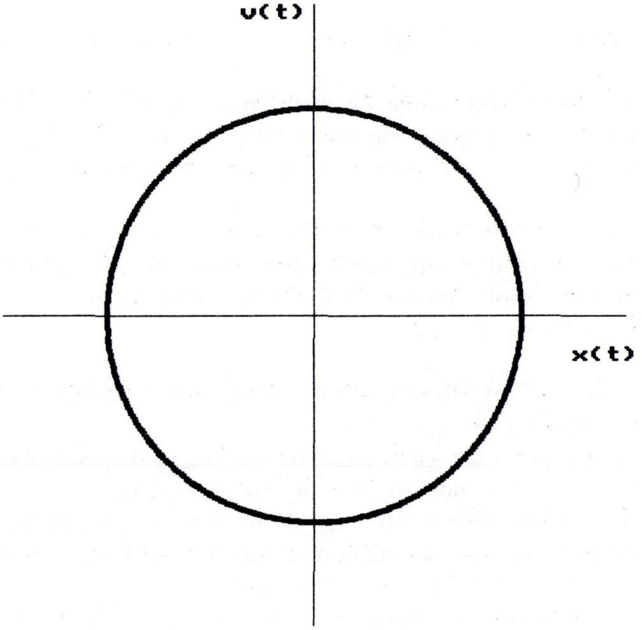

Figure 1: Simulation of the exact algorithm for the harmonic oscillator.

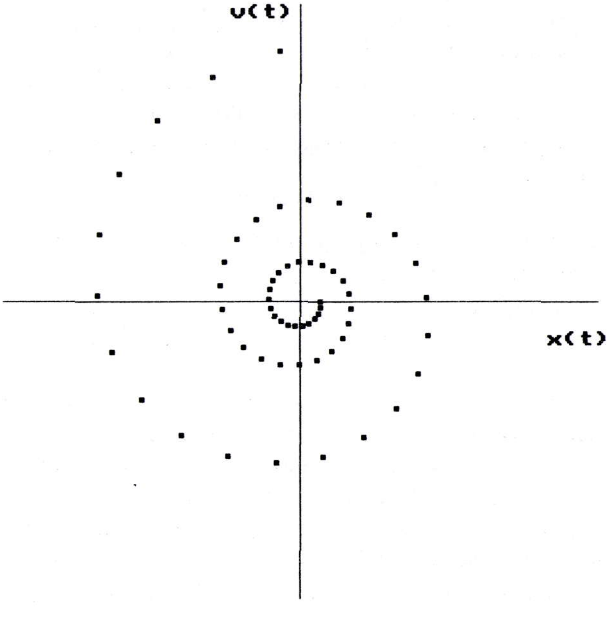

Figure 2: Simulation of the Euler algorithm for the harmonic oscillator.

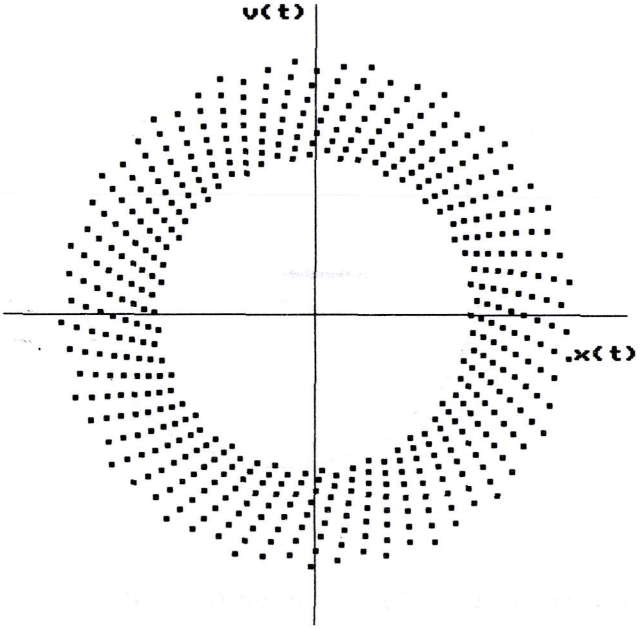

Figure 3a: Simulation of the Runge-Kutta-2 algorithm for the harmonic oscillator.

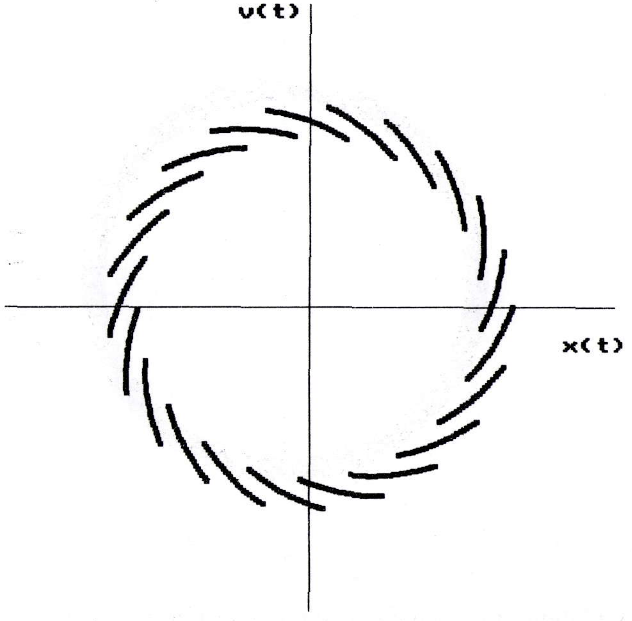

Figure 3b: Simulation of the Runge-Kutta-3 algorithm for the harmonic oscillator.

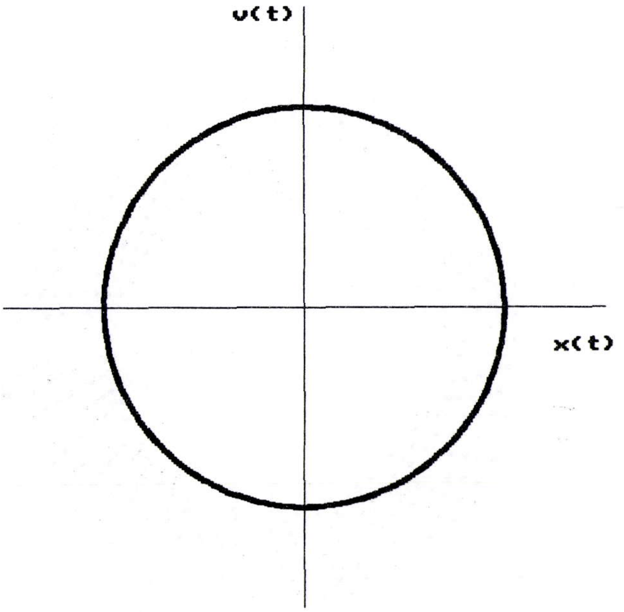

Figure 3c: Simulation of the Runge-Kutta-4 algorithm for the harmonic oscillator, with 500 time steps.

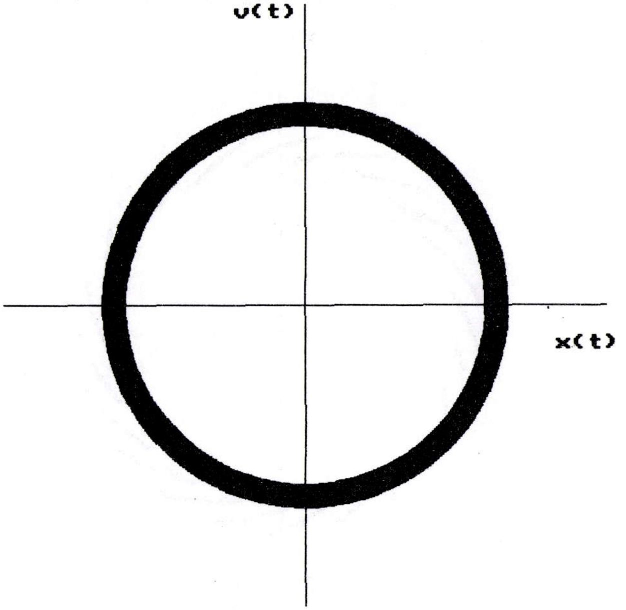

Figure 3d: Simulation of the Runge-Kutta-4 algorithm for the harmonic oscillator, with 20000 time steps.

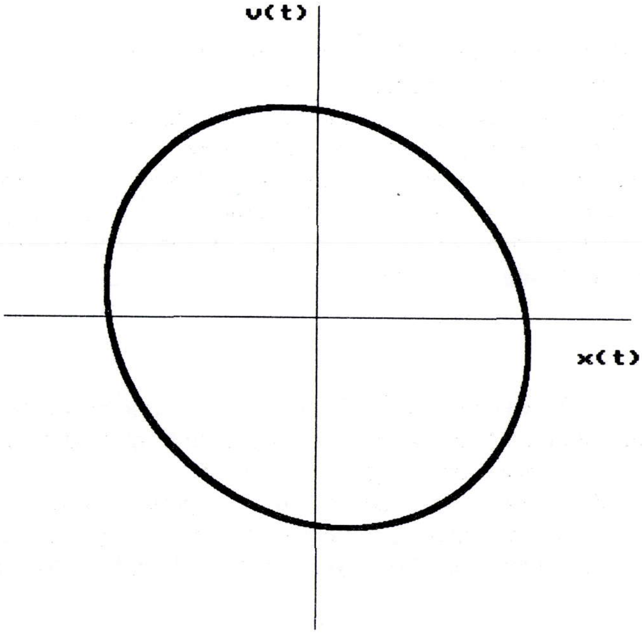

Figure 4a: Simulation of the Incursive-v algorithm for the harmonic oscillator, with 20000 time steps.

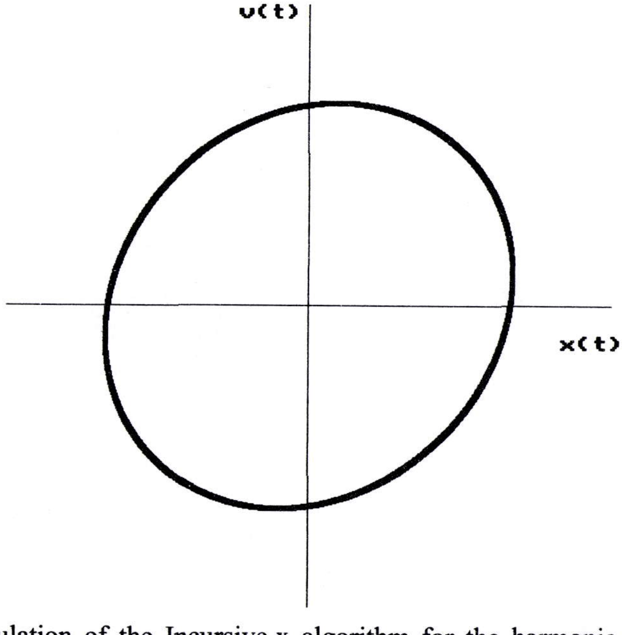

Figure 4b: Simulation of the Incursive-x algorithm for the harmonic oscillator, with 20000 time steps.

5 Conclusion

The paper presented precision and stability considerations for a few discrete models of the harmonic oscillator - classical and incursive.

From this analysis, it results that incursive systems insure better precision than the ones from which they were derived. The latter were unstable, thus unsuitable for simulation, while their incursive versions turned to the orbital stability limit.

Even if the analysis was applied only to the simple case of the harmonic oscillator and only to a few numerical integration methods, it represents a solid argument in favour of incursive computing systems.

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