Orbital Stability and Chaos with Incursive Algorithms for the Nonlinear Pendulum

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Abstract
This paper deals with the Euler and Incursive algorithms of the nonlinear pendulum. The Euler algorithm is unstable. The incursive algorithms show a stable solution as an orbital stability for small values of the time step. For larger values of the time step, the incursive algorithms show an orbital stability for small values of the initial conditions and a chaotic sea for larger initial conditions.

Keywords: incursive algorithm, nonlinear pendulum, orbital stability, chaos, simulation.

1 Introduction

The classical algorithms, like the most used Euler and Runge-Kutta algorithms, show unstable solutions for systems with orbital stability.

The well-known incursive algorithms are able to give the good orbital stability property of such systems. This was already shown for the harmonic oscillator.

The Incursive algorithms were already applied to the nonlinear Lotka-Volterra equation systems. The simulations showed an orbital stability and a chaotic sea depending of the parameters.

This paper wants to show that the incursive algorithms are also able to give an orbital stability, and the emergence of a chaotic sea, for the nonlinear pendulum.

In this paper, the numerical simulations of the nonlinear pendulum give an orbital stability for a wide range of values of the time step in the discrete incursive algorithms.

But due to the nonlinearity of the pendulum, it is shown that chaos, as a chaotic sea, can emerge for large values of the time step or of the initial conditions.

This paper is organized as follows. Firstly, the differential equation of the pendulum is presented. Secondly, the discrete Euler and Incursive algorithms of the pendulum are derived. Thirdly, numerical simulations are performed with these algorithms for different initial conditions and different time steps. The Euler algorithm is unstable and the Incursive algorithms show the good orbital stability or a chaotic sea depending on the values of the initial condition and the time step.

The mathematical study of the stability properties of the Incursive algorithms for nonlinear systems will be presented in a forthcoming paper.

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2 Differential Equation of the Nonlinear Pendulum

The differential equation of the nonlinear pendulum is given by

$$d^2x/dt^2 = -\,(g/l)\,\sin(x) \qquad (1)$$

where g is the gravitational constant, l is the length of the pendulum, t the time and the variable $x(t)$ corresponds to the angle that the pendulum makes with the vertical, measured in radians.

In making the change of variable

$$\omega^2 = g/l \qquad (2)$$

eq. 1 is rewritten as

$$d^2x/dt^2 = -\,\omega^2 \sin(x) \qquad (3)$$

The analytical solution of eq. 3 is well-known and is given by oscillations.

For small amplitudes oscillations, $x \ll 2\pi$, the sin function can be developed in Taylor's series as

$$\sin(x) = x - x^3/3! + x^5/5! - \ldots \qquad (4)$$

Including the first term of relation 4 in eq. 3, the linear harmonic oscillator is obtained

$$d^2x/dt^2 = -\,\omega^2\, x \qquad (5)$$

the solution of which being given by

$$x = x_0 \sin\left(\omega t + \varphi\right) \qquad (6)$$

which is an oscillation with a constant amplitude.

The second order differential eq. 3 can be transformed to two first order differential equations as

$$dx/dt = v \qquad (7a)$$
$$dv/dt = -\,\omega^2 \sin(x) \qquad (7b)$$

where $v(t)$ is the angular velocity.

Let us now derived the incursive algorithms from the nonlinear differential equations 7ab of the pendulum.

3 The Euler and Incursive Algorithms of the Pendulum

The well-known Euler algorithm of these eqs. 7ab is given by

$$x(t+\Delta t) = x(t) + \Delta t\, v(t) \tag{8a}$$
$$v(t+\Delta t) = v(t) - \Delta t\, \omega^2 \sin(x(t)) \tag{8b}$$

This Euler algorithm gives unstable numerical solutions.

In view of obtaining a stable algorithm, let us use the incursive algorithms as follows.

The first incursive algorithm is given by

$$x(t+\Delta t) = x(t) + \Delta t\, v(t) \tag{9a}$$
$$v(t+\Delta t) = v(t) - \Delta t\, \omega^2 \mathbf{\sin(x(t+\Delta t))} \tag{9b}$$

and the second incursive algorithm is the following one

$$v(t+\Delta t) = v(t) - \Delta t\, \omega^2 \sin(x(t)) \tag{10a}$$
$$x(t+\Delta t) = x(t) + \Delta t\, \mathbf{v(t+\Delta t)} \tag{10b}$$

In the first incursive algorithm 9ab, the value of the up-dated angle, $x(t+\Delta t)$, of eq. 9a is propagated to the second eq. 9b.

In the second incursive algorithm, the value of the velocity is firstly calculated in eq. 10a, and this up-dated velocity, $v(t+\Delta t)$, is propagated to the second equation 10b.

This means that the simulations depend on the order in which the calculations of the equations are performed. Each up-dated variable is propagated to the following ordered equation. Let us recall that in the Euler algorithm 8ab, the order in which the equations are calculated is without importance.

It is possible to include explicitly the up-dated variables as follows

$$x(t+\Delta t) = x(t) + \Delta t\, v(t) \tag{11a}$$
$$v(t+\Delta t) = v(t) - \Delta t\, \omega^2 \mathbf{\sin(x(t) + \Delta t\, v(t))} \tag{11b}$$

for the eqs. 9ab, and, for eqs. 10ab,

$$v(t+\Delta t) = v(t) - \Delta t\, \omega^2 \sin(x(t)) \tag{12a}$$
$$x(t+\Delta t) = x(t) + \Delta t\, \mathbf{[v(t) - \Delta t\, \omega^2 \sin(x(t))]} \tag{12b}$$

The difference with the Euler algorithm, in comparing eq. 8a with 12b, deals with a supplementary term in Δt^2 in the incursive algorithm. This term leads to the stability of the incursive algorithm.

Let us now show the results of the numerical simulations of these algorithms for the nonlinear pendulum.

4 Numerical Simulations of the Nonlinear Pendulum with the Euler and the two Incursive Algorithms

When the angular amplitude is small, the $\sin(x(t))$ can be replaced by $x(t)$, and it was already shown that the Euler algorithm is unstable, and the two incursive algorithms give the good orbital stability of the harmonic oscillator.

This section deals with the numerical simulations of the algorithms given in the preceding section for the nonlinear pendulum.

Without lack of generality, for the simulations, the value of the frequency will be taken equal to one, as

$$\omega = 1 \qquad (13)$$

Indeed, it is sufficient to make a re-scaling of the time in eq. 3, as $d\tau = \omega dt$, that gives the following adimensional differential equation of the pendulum

$$d^2x/d\tau^2 = -\sin(x) \qquad (3')$$

Figures 1ab, 2ab, and 3ab, give the simulations of the Euler and the two Incursive algorithms of the nonlinear pendulum, with $\Delta t = 0.025$, and with the following initial conditions: $x(t) = 2\pi/3$ and $v(0) = 0$.

Figures 1ab give the simulation of the Euler algorithm 8ab, respectively, for $x(t)$ and $v(t)$ versus time step t, and in the phase space $(x(t), v(t))$.

The simulation given in the phase space in Fig. 1b shows clearly the unstable solution instead of the orbital stability.

Figures 2ab give the simulation of the first incursive algorithm 9ab, respectively, for $x(t)$ and $v(t)$ versus time step t, and in the phase space $(x(t), v(t))$.
The simulation given in the phase space in Fig. 2b shows clearly the stable solution given by an orbital stability.

Figures 3ab give the simulation of the second incursive algorithm 10ab, respectively, for $x(t)$ and $v(t)$ versus time step t, and in the phase space $(x(t), v(t))$.
The simulation given in the phase space in Fig. 3b shows clearly the stable solution given by an orbital stability.

Figures 3cd give the simulation of the second incursive algorithm 10ab, respectively, for $x(t)$ and $v(t)$ versus time step t, and in the phase space $(x(t), v(t))$; with $\Delta t = 0.025$, and with the following initial conditions: $x(t) = 9\pi/10$ and $v(0) = 0$. With this initial condition, the pendulum is practically at the vertical.
The simulations show the strong effect of the nonlinearity of the pendulum and the orbital stability.

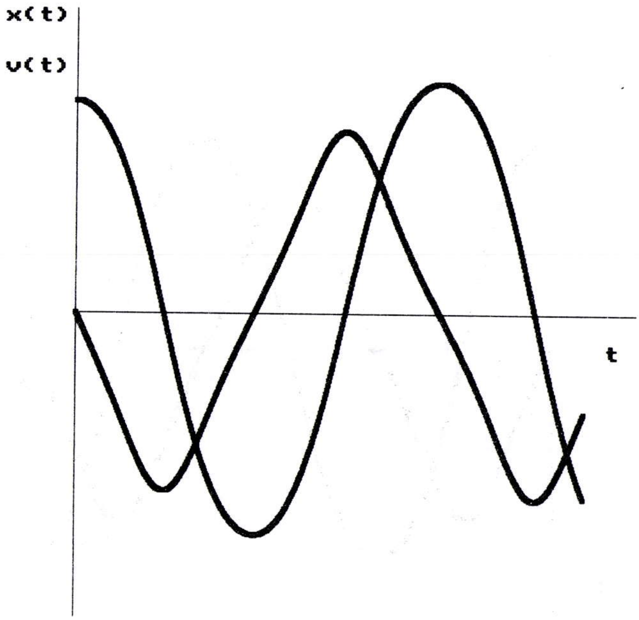

Figure 1a: Simulation of pendulum with the Euler algorithm.

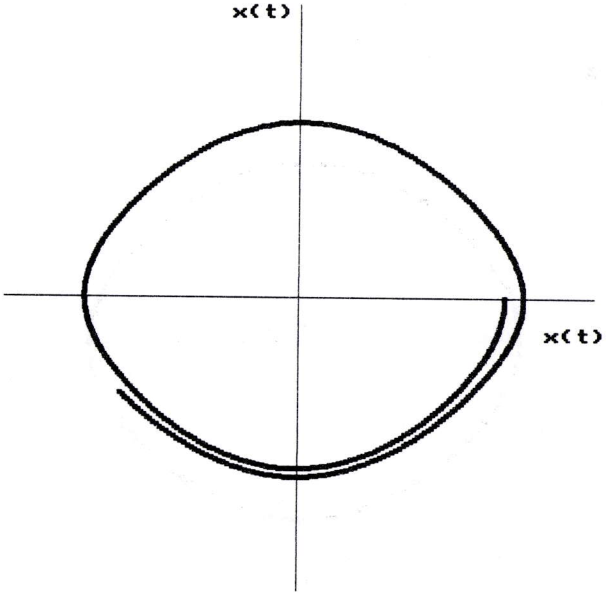

Figure 1b: The Euler algorithm for the pendulum is unstable.

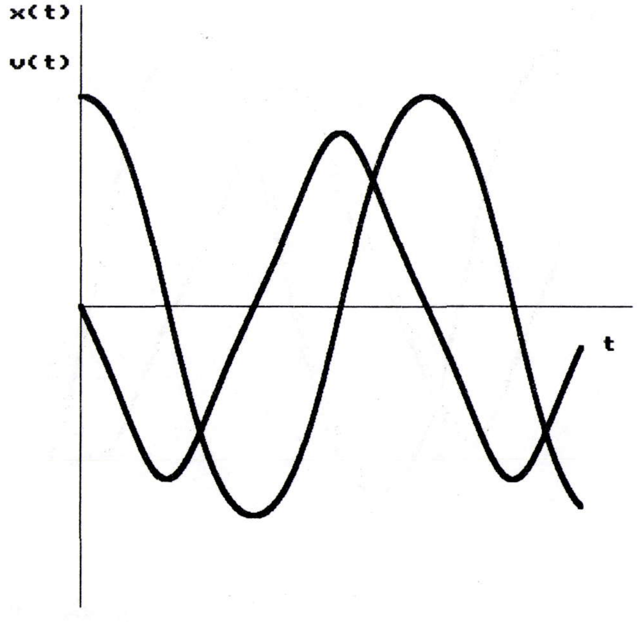

Figure 2a: Simulation of the pendulum with the first incursive algorithm.

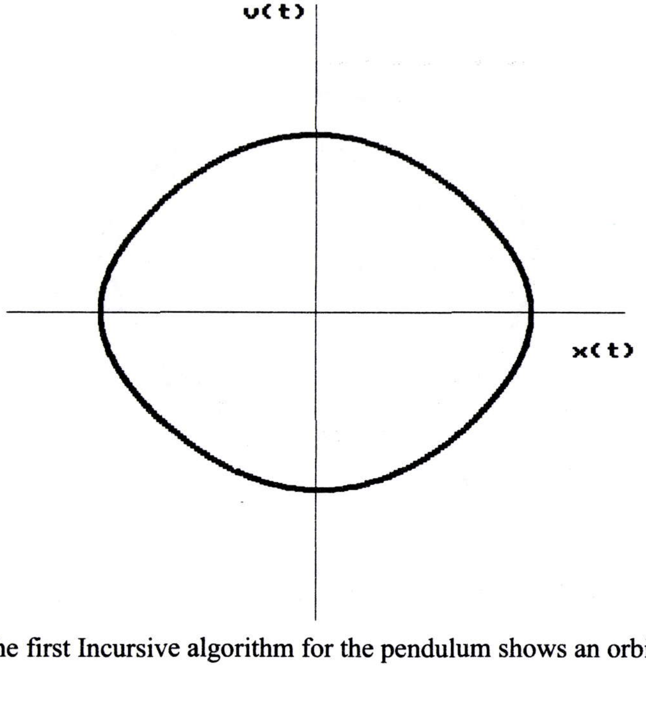

Figure 2b: The first Incursive algorithm for the pendulum shows an orbital stability.

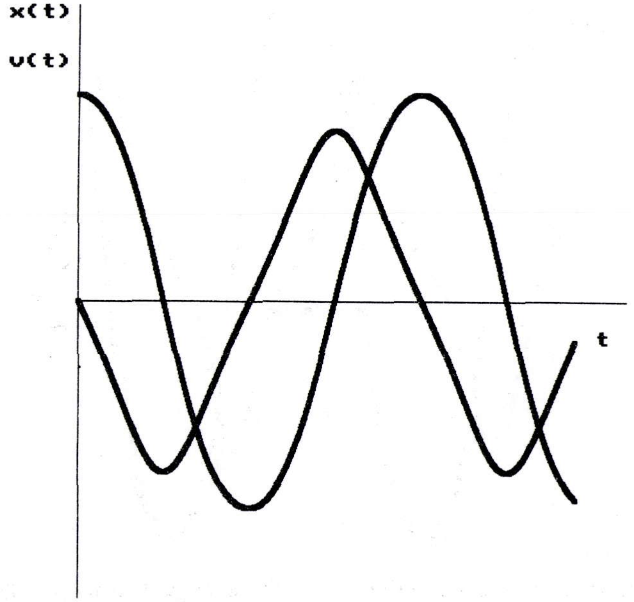

Figure 3a: Simulation of the pendulum with the second incursive algorithm.

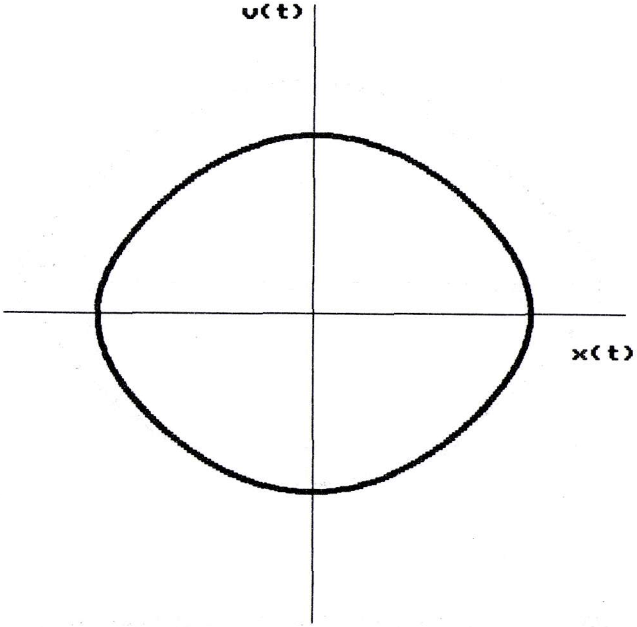

Figure 3b: The second Incursive algorithm for the pendulum shows an orbital stability.

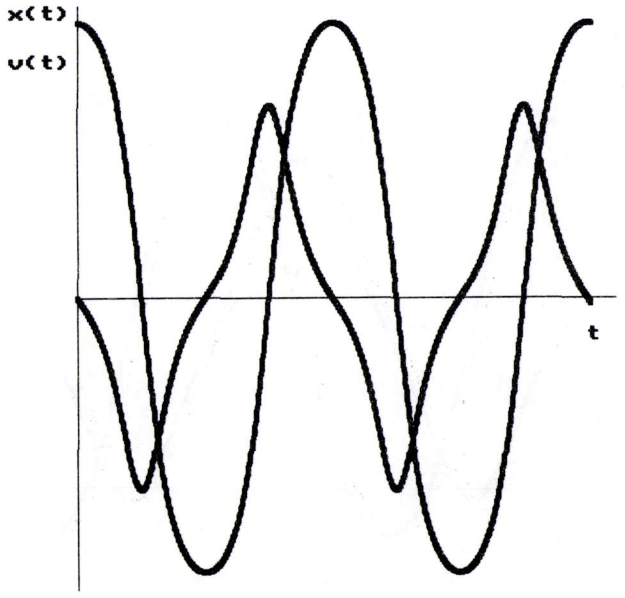

Figure 3c: Simulation of the pendulum with the second incursive algorithm, with an initial condition practically at the vertical. The strong nonlinearity effect is well-seen.

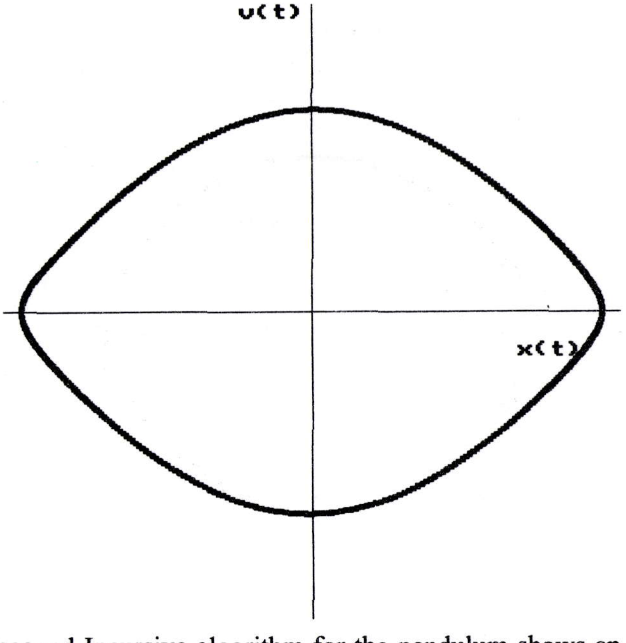

Figure 3d: The second Incursive algorithm for the pendulum shows an orbital stability with an initial condition practically at the vertical. The orbital stability is well-seen.

Let us show that the incursive algorithms give the orbital stability, in the phase space, for a set of initial conditions.

Figure 4a gives the simulations of the pendulum, in the phase space, of the first incursive algorithm 9ab,
for a set of initial conditions: $x(0) = (r/50)$ radians, $r = 0, 1, 2, \ldots, 50$; $v(0) = 0$; and,
for the following time step: $\Delta t = 0.025$.
This simulation shows the orbital stability for all the set of initial conditions.

Figure 4b gives the simulations of the pendulum, in the phase space, of the second incursive algorithm 10ab,
for a set of initial conditions: $x(0) = (r/50)$ radians, $r = 0, 1, 2, \ldots, 50$; $v(0) = 0$; and,
for the following time step: $\Delta t = 0.025$.
This simulation shows the orbital stability for all the set of initial conditions.

Let us give now the simulation of the second Incursive algorithm 10ab in the phase space for the same set of initial conditions, but with larger values of the time step.
Let us point out that the simulations with the first Incursive algorithm 9ab give similar results.

Figures 5abcde give the simulations of the pendulum, in the phase space, of the second incursive algorithm 10ab,
for a set of initial conditions: $x(0) = (r/50)$ radians, $r = 0, 1, 2, \ldots, 50$; $v(0) = 0$; and,
for the following set of time steps: $\Delta t = 0.2, 0.8, 1.1, 1.5$

Figure 5a shows the result of the simulation for the time step $\Delta t = 0.2$. The orbital stability is well-seen.

Figure 5b shows the result of the simulation for the time step $\Delta t = 0.8$. The orbital stability is well-seen, and chaos appears for the large values of the initial conditions.

Figure 5c shows the result of the simulation for the time step $\Delta t = 1.1$. There is an orbital stability zone for the smaller values of the initial conditions. With larger values of the initial conditions, chaos emerges as a chaotic sea.

Figure 5d shows the result of the simulation for the time step $\Delta t = 1.5$. The orbital stability zone decreases and the pattern shows a sophisticated pattern with four isles.

Figure 5e shows the result of the simulation for the time step $\Delta t = 1.55$. The pattern shows chaos between the orbital curves and the four isles.

As the interesting patterns diminish with the increase of the time step, the following figures will be enlarged by a factor 2.

Figures 6a and 6b are enlargements of Fig. 5d and 5e, respectively.

Figure 6b (continuation), give an enlargement of an isle pattern.

Figure 6c shows the result of the enlarged simulation for the time step $\Delta t = 1.6$.
The pattern shows chaos between the orbital curves and the four isles which go away.

Figure 6c (continuation), give an enlargement of an isle pattern.

Figure 6d gives the result of the enlarged simulation for the time step $\Delta t = 1.9$.
The pattern shows a few orbital curves and the four isles which have disappeared.

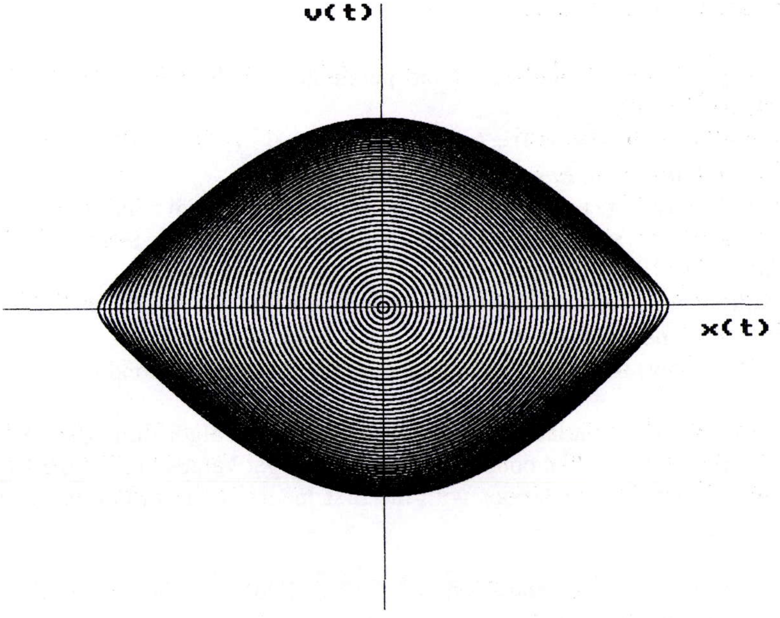

Figure 4a: Simulation of the orbital stability of the first Incursive algorithm of the pendulum for a set of initial conditions.

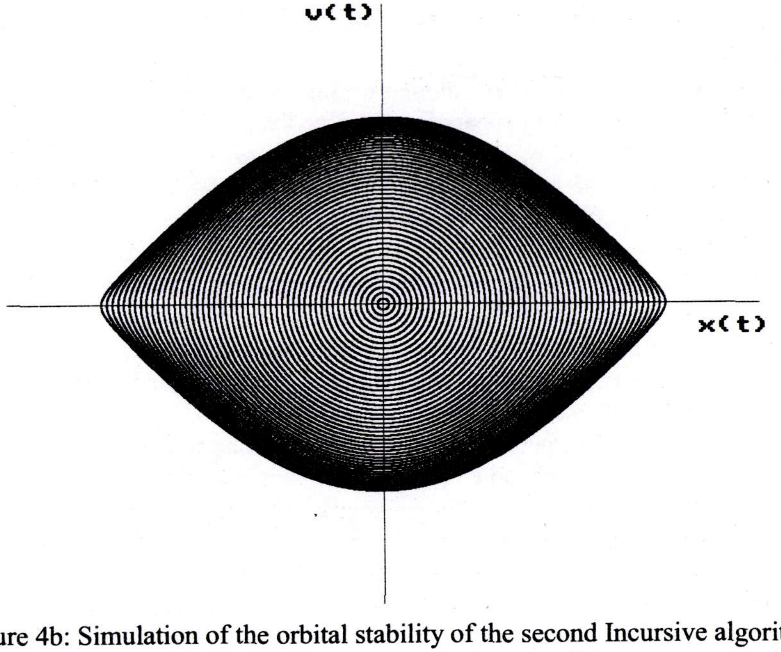

Figure 4b: Simulation of the orbital stability of the second Incursive algorithm of the pendulum for a set of initial conditions.

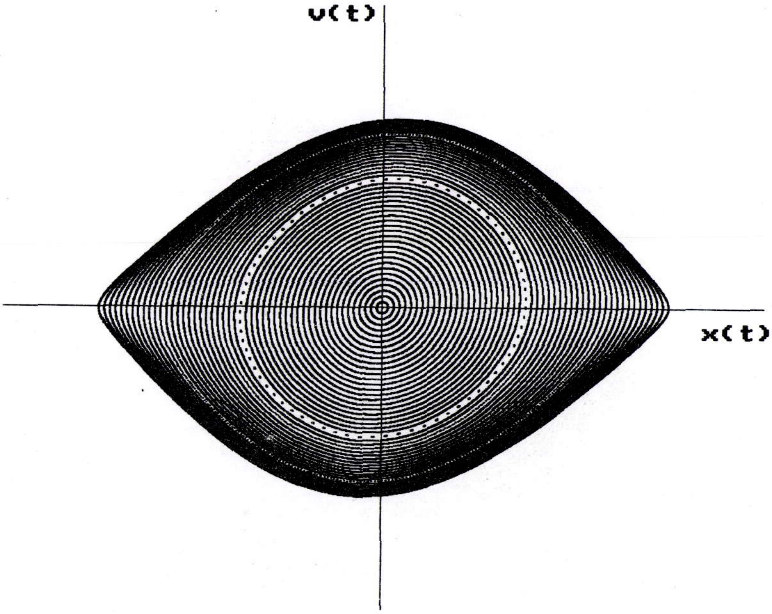

Figure 5a: Simulation of the pendulum with the second Incursive algorithm for a set of initial conditions and the time step $\Delta t = 0.2$. The orbital stability is well-seen.

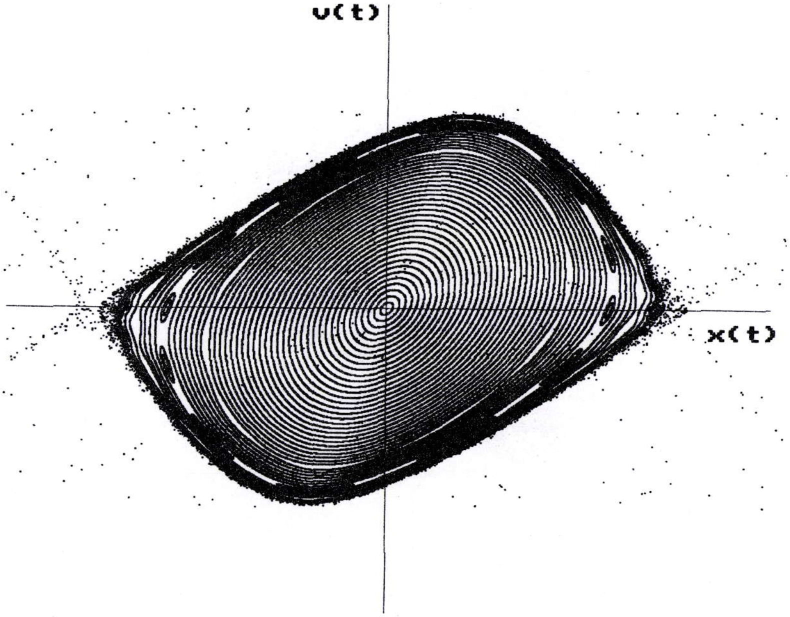

Figure 5b: Simulation of the pendulum with the second Incursive algorithm for a set of initial conditions and the time step $\Delta t = 0.8$.

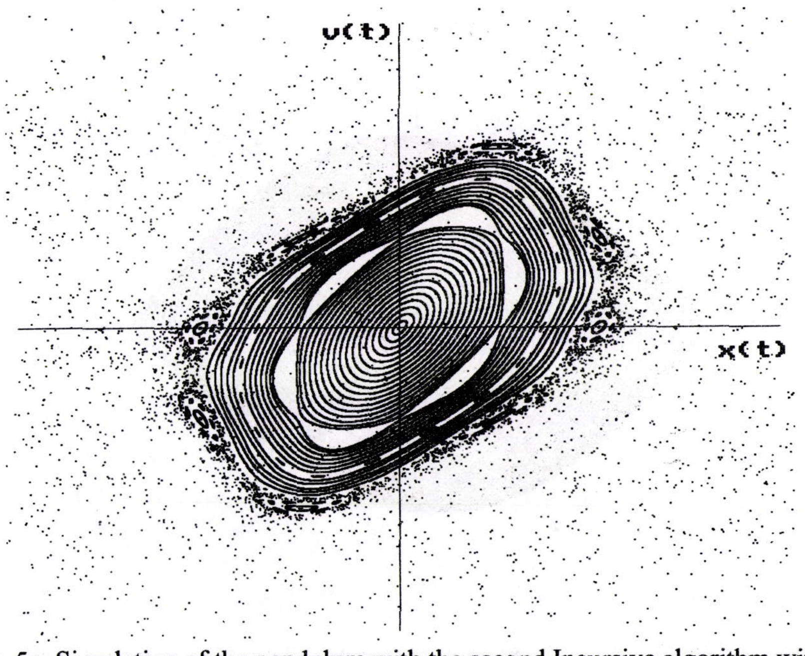

Figure 5c: Simulation of the pendulum with the second Incursive algorithm with the time step $\Delta t = 1.1$. An orbital stability and a chaotic sea are observed.

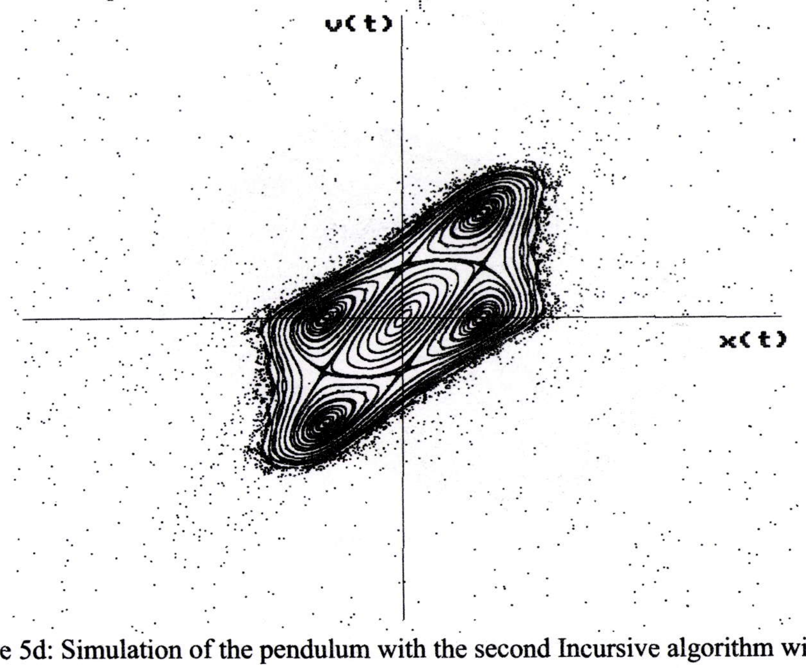

Figure 5d: Simulation of the pendulum with the second Incursive algorithm with the time step $\Delta t = 1.5$. Four isles and chaos are seen around the orbital stability zone.

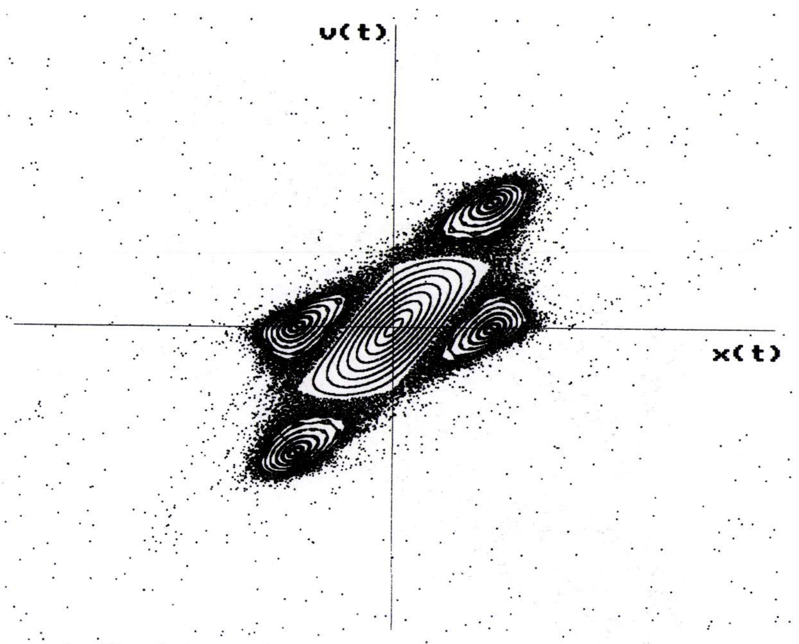

Figure 5e: Simulation of the pendulum with the second Incursive algorithm with the time step $\Delta t = 1.55$.

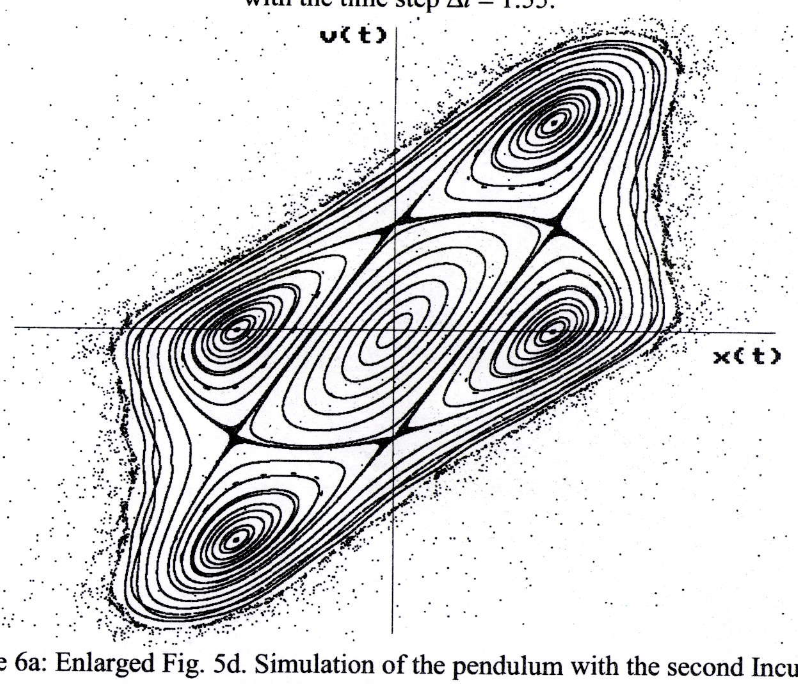

Figure 6a: Enlarged Fig. 5d. Simulation of the pendulum with the second Incursive algorithm for a set of initial conditions and the time step $\Delta t = 1.5$.

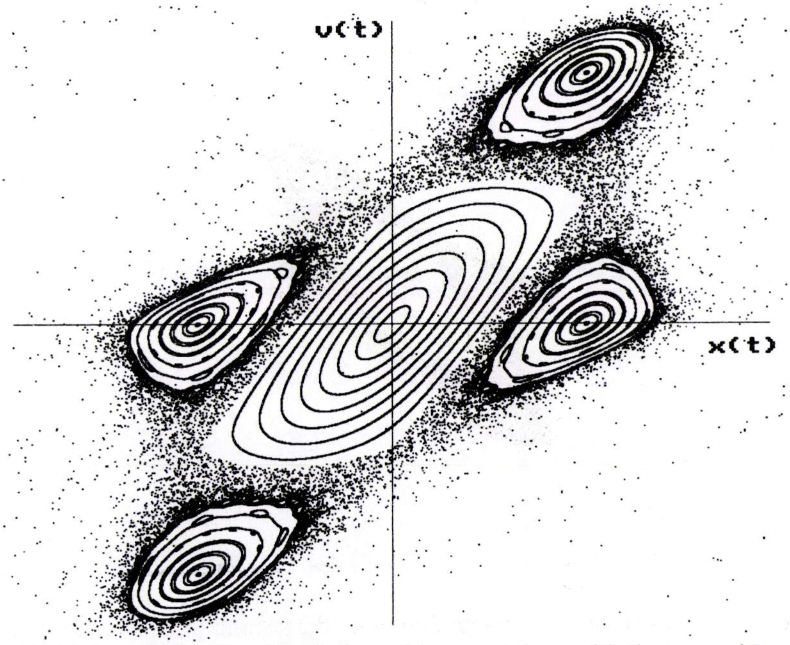

Figure 6b: Enlarged Fig. 5e. Simulation of the pendulum with the second Incursive algorithm for a set of initial conditions and the time step $\Delta t = 1.55$.

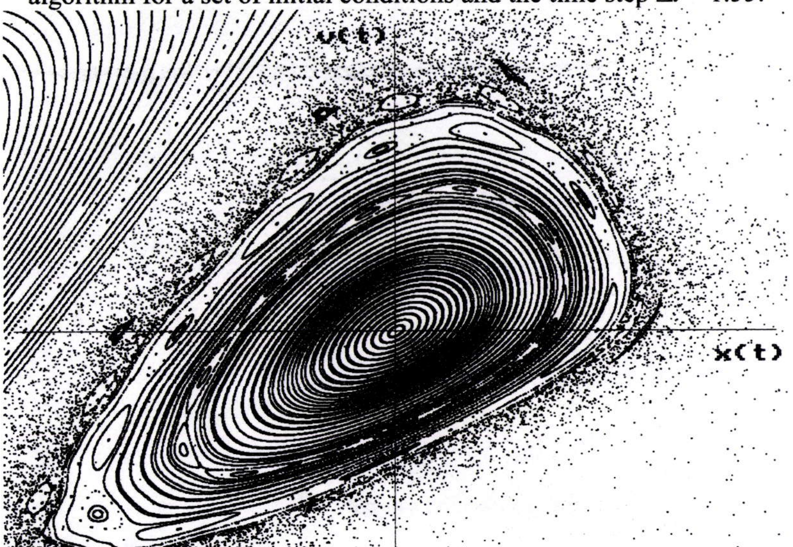

Figure 6b (continuation): Enlargement of the isle at the right.

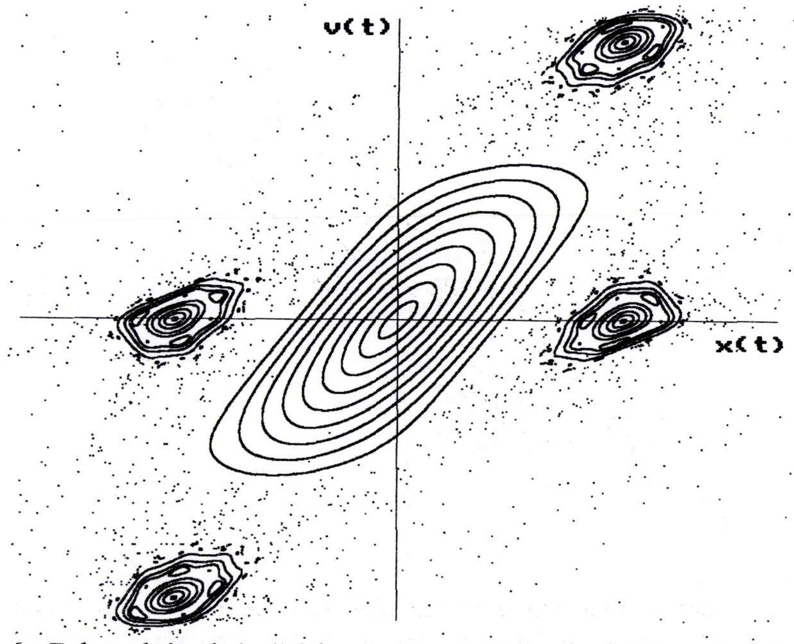

Figure 6c: Enlarged simulation of the pendulum with the second Incursive algorithm for a set of initial conditions and the time step $\Delta t = 1.6$.

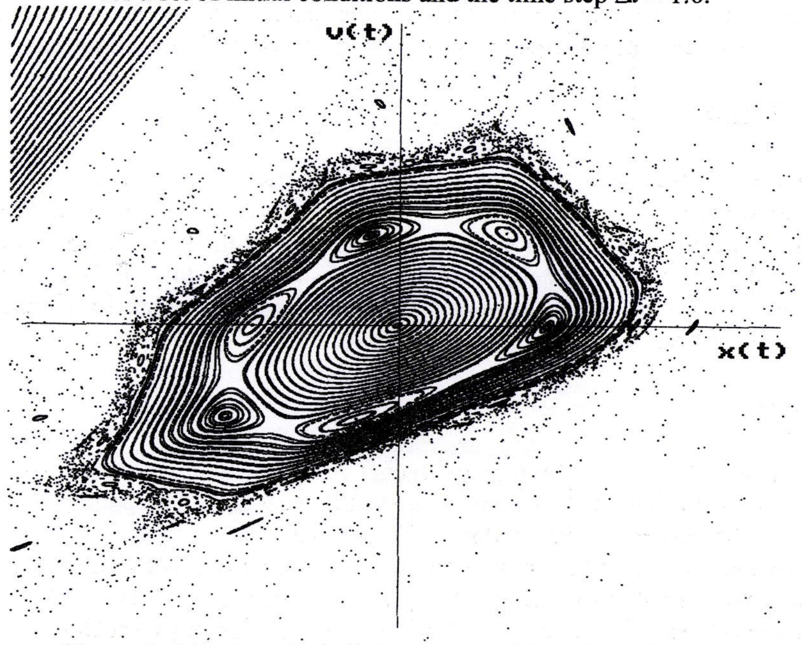

Figure 6c (continuation): Enlargement of the isle at the right.

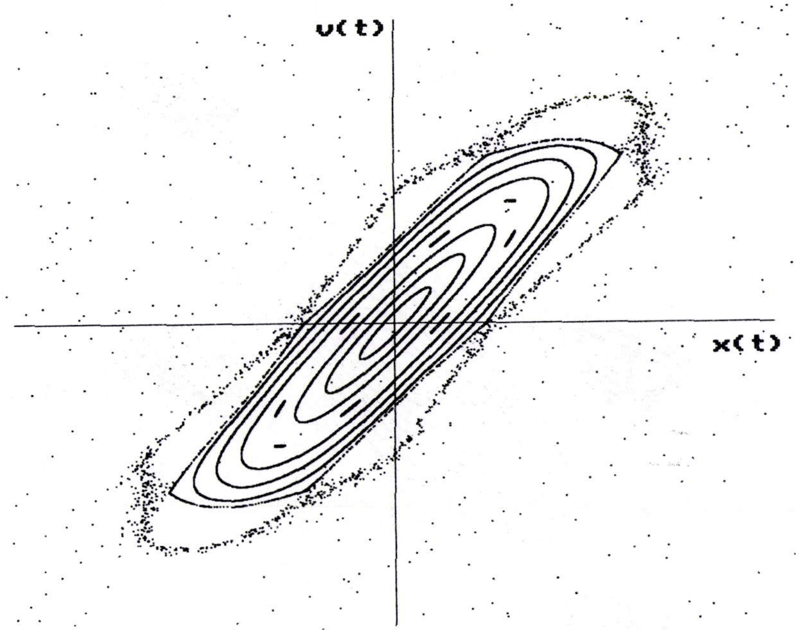

Figure 6d: Enlarged simulation of the pendulum with the second Incursive algorithm for a set of initial conditions and the time step $\Delta t = 1.9$.

All these simulations have shown the complex behavior of the nonlinear pendulum with the incursive algorithms depending on the time steps and initial conditions.

5 Conclusion

This paper showed that the classical Euler algorithm is unstable and that the incursive algorithms are stable for $\omega dt < 2$. For small values of the initial conditions, the nonlinear pendulum is similar to the orbital stability of the harmonic oscillator. The simulations showed an orbital stability and a chaotic sea depending on the time step and on the initial conditions. These incursive algorithms are appropriate for simulating all types of linear and nonlinear systems.

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