Utility and Helpfulness of Probability of the Fuzzy Events in Some Economic Problems

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Abstract. In the paper we present some conceptions of probability of fuzzy events, especially of intuitionistic fuzzy events and discuss them in one perspective and show the utility and helpfulness of using the probability calculus to a valuation of some economic situations.
Section 1. Introduction. Probability of fuzzy events according to the idea of L. Zadeh.
Section 2. Intuitionistic fuzzy sets of K. Atanassov.
Section 3. Intuitionistic fuzzy event (IFE) and its probability according to the results of T. Gerstenkorn and J. Mańko.
Section 4. Probability of IFE by using the theorems of decomposition and extension principle of D. Stoyanova.
Section 5. Probability of IFE according to the ideas of E. Szmidt and J. Kacprzyk.
Section 6. A large example showing utility and helpfulness of using a probability calculus to evaluation of some economic problems. A comparison of different results by using different methods of probability proposals.
Section 7. Final remarks.
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1 Introduction

In 1965 a fundamental paper of L. Zadeh was published initiating a large study of the so-called fuzzy sets. It is very difficult to imagine the origin of the idea and theory of the fuzzy set without numerous papers, preceding this theory, with considerations of mathematicians and logicians creating the bases of multi-valued logic and widening the notion of the set of the Cantor type. Among these scholars one can always find such Polish names as e.g. J.Łukasiewicz (1920,1970), S.Leśniewski (1992), A.Tarski (1956, 1972-1974) and now their successors as e.g. T. Kubiński (1960; with his analysis of vague notion) and G. Malinowski of Łódź University (1993; with a known monograph on multi-valued logic). Nowadays, the using of multi-valued logic is quite common and

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normal and the development of this science is stormy. However the suppression of mental barriers was not easy and was going a long time.

Over many years, Zadeh's theory was putting of some generalizations. One of that theories gaining every now and again a large interest is a theory of Krassimir Atanassov (1983, 1985, 1986,1999) with his conception of the so-called intuitionistic fuzzy set (in other words: bifuzzy set). Our probabilistic problem will be considered in connection with that idea.

2 Intuitionistic fuzzy sets

Let $X \neq \emptyset$ be an arbitrary set in common sense, treated as a space of consideration. By an intuitionistic fuzzy set A in X is meant an object (Atanassov, 1986) of the form

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} , \tag{1}$$

where μ_A , ν_A , $X \to [0,1]$, μ_A - function of membership (as in the theory of L. Zadeh), ν_A – function of non-membership of an element x to the set A, while the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1 . \tag{2}$$

is fulfilled.

The difference

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{3}$$

is called an *intuitionistic index* and the number $\pi_A(x) \in [0,1]$ is treated as a measure of a hesitancy (*hesitancy margin)* bounded with the appreciation of the degree of the membership or non-membership of an element x to the set. The family of all intuitionistic fuzzy sets in the space X will be denoted by IFS(X).

Example 1 (Atanassov (1999))

Let X be a set of all states where governments are elected by voting. Let us assume that we know the rate of electors voting *for* a government in each state. Let us denote this rate by $\mu(x)$. Let $\nu(x) = 1 - \mu(x)$. This number applies to the rate of electors voting *against* the government. The Zadeh's theory does not give any additional information at this moment. But in praxis, there always is a group of people not voting or giving invalid vote. There is $1 - \mu(x) - \nu(x) = \pi(x)$ of that people. In this manner we have construct a set $\{(x, \mu(x), \nu(x)) : x \in X\}$, where condition (2) is fulfilled.

Example 2. Let us assume that we are interested in classification of a businessman from a group X of n men to the category of *clever businessmen. Let* $\mu(x_i)$ (i = 1,2,...,n) denote a degree of belonging of that businessman to the clever ones (more exactly: our opinion of that situation), $\nu(x_i)$ – a degree of non-belonging, $\pi(x_i)$ – a degree of our hesitancy or lack of decision. Evidently, $\mu(x_i) + \nu(x_i) + \pi(x_i) = 1$. Let us now assume that $\mu(x_i) = 0.2$, $\nu(x_i) = 0.5$, then $\pi(x_i) = 0.3$. At some favourable circumstances, e.g. at a sudden boom, the maximal degree of classification of that man to the clever businessmen is the number $\mu_{max}(x_i) = \mu(x_i) + \pi(x_i) = 0.5$. But the situation can also be unfavourable and then $\mu(x_i) = 0.2$ $\nu_{maz}(x_i) = \nu(x_i) + \pi(x_i) = 0.8$. At these circumstances that businessman has barely perceptible chances of his classification to the clever men. Let us now assume that $\mu(x_i) = v(x_i) = 0.5$. In this case $\pi(x_i) = 0$; there is a complete lack of our hesitancy in classification. We can understand such situation that this businessman is very common, poor and nothing can change our opinion of him. Let us now assume quite extremely that $\mu(x_i) = v(x_i) = 0$, i.e. $\pi(x_i) = 1$. This situation shows that depending on the inflow of information all it can occur and we are able easily to change our decision about the value of $\mu(x_i)$ and $v(x_i)$. Lastly, let us assume that $\mu(x_i) = 0.5$ and $v(x_i) = 0.2$ (i.e. $\pi(x_i) = 0.3$). In this case $\mu_{max}(x_i)= \mu(x_i) + \pi(x_i) =0.8$ and $v_{maz}(x_i) = v(x_i) + \pi(x_i) = 0.5$. These values indicate that our businessman has the considerable chance to be recognized as the clever one.

The above example shows the significant part and meaning of $\pi(x)$ in interpreting of a fuzzy set (a vague notion) and gives an easiness of manner of changing the values of $\mu(x)$ and $v(x)$ with coming of some information and evolution of knowledge of an investigator in the case of a concrete problem.

3 Probability of a fuzzy event

If in formula (1) functions μ_A and v_A (therefore also π_A) are measurable in a probability space (X,F,P) with a σ-algebra F of subsets of a set X and with a probability function P, then an intuitionistic fuzzy set A is called an *event*. The family of intuitionistic fuzzy events is denoted by $IFM(X)$.

Definition (Gerstenkorn, Mańko-2001)

The number

$$\widetilde{P}(A) = \int_X [\mu_A(x) + 0.5\pi_A(x)]P(dx) \tag{4}$$

is called *probability* of the event $A \in IFM(X)$. The so defined function fulfils the Kolmogorov's axioms and therefore all properties of the classical probability theory. If $\pi_A(x) = 0$, formula (4) reduces to the known formula of probability of the fuzzy event proposed by Zadeh (1968).

Let us now assume that $X = \{x_1, x_2, ..., x_n\}$ is a finite set, $A \in IFM(X)$ is an event and let be in X defined a probability function $P = \{p_1, p_2,..., p_n\}$. Formula (4) takes in this case the form

$$\widetilde{P}(A) = \sum_{i=1}^{n} [\mu_A(x_i) + 0,5\pi_A(x_i)]p_i \,. \tag{5}$$

Let us now consider a case of the classical probability using the notion of the cardinality (power) of the set.

We call the number

$$\text{card}\, A = \sum_{i=1}^{n} [\mu_A(x_i) + 0{,}5\pi_A(x_i)] \qquad (6)$$

the *cardinality* (power) of the set $A \in IFM(X)$.

That formula is a natural generalization of the formula for the power of a fuzzy set given by de Luca and Termini (1972) and modified to the formula given by us in 2000.

Let us now suppose that the probability distribution in the set X is $P = \{1/n, 1/n, ..., 1/n\}$, i.e. each elementary even has the same probability 1/n. Then, following our paper of 2000, we propose for the probability of the event $A \in IFM(X)$ the number defined by

$$\widetilde{P}(A) = \frac{cardA}{cardX} = \sum_{i=1}^{n} [\mu_A(x_i) + 0{,}5\pi_A(x_i)] \frac{1}{n}. \qquad (7)$$

This expression is a special case of formula (5) and presents the classical Laplace's probability transferred on the ground of intuitionistic fuzzy events.

Example 3. Let $X = \{x_1, x_2, ...x_5\}$ be a set of five businessmen with *good head* for business. Let $P(\{x_i\}) = 1/5$ for $i = 1,2,...,5$. Let $A = \{(x, \mu_A(x), \pi_A(x)\}$ be an intuitionistic set of the form $A = \{(x_1; 0.6, 0.1), (x_2; 0.6, 0.3), (x_3; 0.5, 0.2), (x_4; 0.8, 0.1), (x_5; 0.2, 0.3)$. We randomly draw a good businessman. Then from (7) we have $\widetilde{P}(A) = 0.64$.

4 Other conception of probability of the intuitionistic fuzzy event

We precede the considerations of this section by mention some important notions.

$$A \cup B = \{(x; \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)))\}, \qquad (8)$$

$$A \cap B = \{(x; \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)))\}, \qquad (9)$$

$$A' = \{(x : \nu_A(x), \mu_A(x))\} \qquad (10)$$

As it is known, a crisp set $A_\alpha = \{x : \mu_A(x) \geq \alpha\}$, where $\alpha \in [0,1]$ is called α-*level* set of A. This set is determined by the characteristic function

$$\varphi_{A_\alpha} = \begin{cases} 1 \;\; dla \;\; \mu_A(x) \geq \alpha \\ 0 \;\; dla \;\; \mu_A(x) < \alpha. \end{cases} \qquad (11)$$

Using the operation of α-level, we can achieve the decomposition of the function μ_A on rectangular functions $\alpha \wedge \varphi_{A_\alpha}$ (piece by piece constant), where $\wedge$ is the algebraic operation *minimum* for all levels of α and then

$$\mu_A(x) = \sup[\alpha \wedge \varphi_{A_\alpha}(x)] \quad \text{over all values of } \alpha,$$

i.e. we can express the membership function by using the characteristic function of crisp sets.

If we denote by αA_α a fuzzy set with the membership function $\mu(x) = \alpha \wedge \varphi_{A_\alpha}$, then the fuzzy set A can be expressed by a sum of fuzzy sets αA_α over all levels of α that is

$$A = \bigcup_{\alpha\in[0,1]} \alpha A_\alpha . \quad (12)$$

It means that if we consider instead of a set A its α-levels A_α, we treat in this case the decomposition principle, but if we do on the contrary, that is if we construct the set A by rectangular functions αA_α, we refer to the so-called representation (extension) principle.

D.Stoyanova (1990) introduced analogous notions in the class of IFS(X). So, for $\alpha, \beta \in [0,1]$ and $\alpha + \beta \le 1$ and $A \in IFS(X)$, we have

$(\alpha,\beta) * A = \{(x; \alpha\mu_A(x), \beta + (1-\beta)\nu_A(x)\}$ (product of the pair (α,β) and the set A) (13)

$A_{\alpha,\beta} = \{(x \in X : \cdot\, \mu_A(x) \ge \alpha \ \wedge \ \nu_A(x) \le \beta\}$ ((α,β)-level of the set A), (14)

$N_{\alpha,\beta}(A) = \{(x; 1, 0) : x \in A_{\alpha,\beta}\}$. (bifuzzy analogue) (15)

The decomposition theorem has then the form

$$A = \bigcup_{\alpha,\beta} (\alpha,\beta) * N_{\alpha,\beta}(A) \quad (16)$$

and the extension principle of a function f defined in X gives

$$f(A) = \bigcup_{\alpha,\beta} (\alpha,\beta) * f(N_{\alpha,\beta}(A)) . \quad (17)$$

Taking now in (17) f=P for (X,F,P) we obtain the so-called *fuzzy probability* of the intuitionistic fuzzy event A (Gerstenkorn, Mańko-1988a, 1988b) as

$$P_{IFM}(A) = \bigcup_{\alpha,\beta} (\alpha,\beta) * P(N_{\alpha,\beta}(A)) . \quad (18)$$

This formula is a direct generalization of the conception of R. Yager (1979) of the fuzzy probability of the fuzzy event.

Taking in (17) $X = \{x_1, x_2, ..., x_n\}$ and f=*card*, we obtain the so-called *fuzzy cardinality* (power) of the set $A \in IFS(X)$ in the form (Gerstenkorn, Mańko-1988a)

$$\text{card}_{IFS}(A) = \bigcup_{\alpha,\beta} (\alpha,\beta) * \text{card}\, N_{\alpha,\beta}(A) . \quad (19)$$

and, in consequence, for $p(x_i) = \frac{1}{n}$, i=1,2,...,n,

$$\widetilde{P}(A) = \frac{\text{card}_{IFS}(A)}{\text{card}_{IFS}(X)} = \bigcup_{\alpha,\beta} (\alpha,\beta) * P(N_{\alpha,\beta}(A)) , \quad (20)$$

which is a special case of formula (18).

5 Probability of intuitionistic fuzzy events according to the ideas of E. Szmidt and J. Kacprzyk

In the paper of E. Szmidt and J. Kacprzyk (1999) we find a proposal of the so-called *interval probability* for the intuitionistic fuzzy event $A \in IFM(X)$, where $X = \{x_1, x_2, ..., x_n\}$ and $p(x_i) = \frac{1}{n}$ for i=1,2,...,n. In this case, the number

$$\tilde{\tilde{P}}(A) \in \langle p_{\min}(A), p_{\max}(A) \rangle, \tag{21}$$

is called probability of the event A, where

$$p_{\min}(A) = \frac{1}{N}\sum_{i=1}^{N} \mu_A(x_i) \tag{22}$$

is the so-called *minimal probability*, whereas

$$p_{\max}(A) = p_{\min}(A) + \frac{1}{N}\sum_{i=1}^{N} \pi_A(x_i) \tag{23}$$

is the so-called *maximal probability*.

The interval $\langle p_{\min}(A), p_{\max}(A) \rangle$ determines then the *lower* and *upper limit* of the probability $\tilde{\tilde{P}}(A)$.

6 Probability in application to economic situation. Example

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five domains of the economy of a country or a state, e.g.: x_1 – industry, x_2 – health care, x_3 – education, x_4 – architecture , x_5 – transportation. Let

$A = \{(x_i, \mu_A(x_i), \nu_A(x_i), \pi_A(x_i)\}$, i= 1,2,3,4,5 be an intuitionistic fuzzy set in X describing an influence of the given domain of economy on satisfaction of the society. We take that

A={(x_1; 0.3, 0.6, 0.1), (x_2; 0.6, 0.2, 0.2), (x_3; 0.2, 0.5, 0.3), (x_4; 0.8, 0.2, 0.0), (x_5; 0.4, 0.4, 0.2)}.

Let us assume that each domain is similarly privileged in an experiment consisting in its choosing for an analysis of the importance of the economy domain for the society expectations. We calculate the probability in this experiment. Then, in accordance with (5) and (7), we have

$$\tilde{P}(A) = \frac{1}{5}[(0.3 + 0.05) + (0.6 + 0.1) + (0.2 + 0.15) + (0.8 + 0.0) + (0.4 + 0.1)] = 0.54.$$

Following the procedure (13)-(18), we obtain

$$A_{0.2,0.6} = \{x_1, x_2, x_3, x_4, x_5\}, \qquad A_{0.3,0.5} = \{x_2, x_3, x_4, x_5\},$$

$$A_{o.4,0.4} = \{x_2, x_4, x_5\}, \qquad A_{0.6,0.2} = \{x_2, x_4\},$$

$$A_{0.8,0.2} = \{x_4\}.$$

Other pairs of (α, β) give no new $A_{\alpha,\beta}$..
Then

$$N_{0.2,0.6} = \{(x_1,1,0),(x_2,1,0),(x_3,1,0),(x_4,1,0),(x_5,1,0)\},$$

$$N_{0.3,0.6} = \{(x_2 1,0),(x_3,1,0),(x_4,1,0),(x_5,1,0)\},$$

$$N_{0.4,0.4} = \{(x_2,1,0),(x_4,1,0),(x_5,1,0)\}$$

$$N_{0.6,0.2} = \{(x_2,1,0),(x_4,1,0)\},$$

$$N_{0.8,0.2} = \{(x_4,1,0)\}$$

and also:

$$P(N_{o.2,0.6}) = 1, P(N_{0.3,0.6}) = \frac{4}{5}, P(N_{0.4,0.4}) = \frac{3}{5}, P(N_{0.6,0.2}) = \frac{2}{5}, P(N_{0.8,0.2}) = \frac{1}{5}.$$

Hence, on the ground of (18), we obtain

$$\tilde{\tilde{P}}_{IFM}(A) = (0.2,\ 0.6) * \{(1,0,0)\} \cup (0.3, 0.6) * \{(\frac{4}{5},0,0)\} \cup (0.4, 0.4) * \{(\frac{3}{5},0,0)\} \cup$$

$$(0.6, 0.2) * \{(\frac{2}{5},1,0)\} \cup (0, 0.2) * \{\frac{1}{5},1,0)\} =$$

$$= \{(1, 0.2, 0.6), (\frac{4}{5}, 0.3, 0.6), (\frac{3}{5}, 0.4,\ 0.4), (\frac{2}{5}, 0.6, 0.2), (\frac{1}{5}, 0.8, 0.2)\}$$

and, in the end, in accordance with (18)-(23)

$$p_{\min}(A) = \frac{1}{5}(0.3 + 0.6 + 0.2 + 0.8 + 0.4) = 0.46,$$

$$p_{\max}(A) = 0.46 + \frac{1}{5}(0.1 + 0.2 + 0.3 + 0 + 0.2) = 0.602,$$

which gives

$$\tilde{\tilde{P}}(A) \in [0.46,\ 0.602].$$

7 Final remarks

In presented paper we have emphasized the meaning of unappreciated element defining the intuitionistic fuzzy set which is the hesitancy margin. This parameter contributes a subtle flexibility of the notion of that set. We have given formulae on probability of the intuitionistic fuzzy event as a generalization of the ones known from the paper of Gerstenkorn and Mańko (1999). We have presented different conceptions for calculation of that probability. Its choice depends on the situation and some opportunities of an investigated problem. Therefore it is very difficult to declare which method is a better one.

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