The Magic Square as a Benchmark

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Abstract
We found the magic square a simple problem with a very rich combinatorics: for a magic square of order *n,* there are $n^2!$ manners to fill the *n*x*n* matrix with integers between 1 and n^2, without repetitions, but only very few of them are magic squares [1]. For order 4 there are exactly 7040 magic squares. So we use the magic square as a benchmark to compare mathematical programming, namely mixed integer programming (MIP), with Genetic Algorithms (GAs), that we will show are much more powerful to solve these kind of discrete combinatorial explosive problems. Finally we developed an *artificial intelligence randomized minimax* algorithm that imitates a human solving the magic square and we showed that in most cases its performance, in terms of number of changes of pairs of numbers, is better than the performance of the GA algorithm.
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1 Introduction

In the literature 'magic square' has various meanings. Here we consider the ancient *addition* magic square which is a square matrix *n*x*n* with integer elements between 1 and n^2, without repetitions, and where the sums of the elements of all lines, columns and the two main diagonals are equal to the magic sum which is given by [2]

$$\text{MagicSum}=\Sigma a_{ij}/n= n\,(n^2+1)/2 \qquad (1)$$

Since we found a very difficult task to reach a magic square by simple changes of pairs of elements, even for n=3, for game development we did only consider the *Relaxed Magic Square* where we did not impose that the sum of the elements of the two main diagonals being equal to the magic sum.

By the same argument in the magic rectangles all lines must have the sum given by [2]

$$LineMagicSum=\Sigma a_{ij}\,/n= n(m\,n+1)\,/\,2 \qquad (2)$$

and all columns must have the sum [2]

$$ColumnMagicSum=\Sigma a_{ij} / m = m(m\,n+1)/2 \quad (3)$$

Sun in [6] showed that the magic rectangle has solution only when m and n are both odd or both even, but never $n=m=2$, that is a magic square 2x2 that is simple to show that it has no solution [1].

In the literature exists also another proposal of magic square, the addition-multiplication magic square [3-4] that has also the restriction that the products of all elements of all lines, columns and the two main diagonals being equal to the magic product, but without the restriction of the maximum element being n^2. This is a much more difficult and challenging problem than the traditional magic square and in the near future we will consider the *Relaxed Addition-Multiplication Magic Squares* where it will not be imposed the constraints of the sums and products of the two principal diagonals being equal to the magic sum and magic product, respectively.

2 The Magic Square as a Benchmark

Our initial motivation to study the magic square was to compare MIP with GAs in the solution of explosive combinatorial optimisation problems to make a decision on which method to use to solve a even much more explosive combinatorial problem. Rechenberg in [5] reported that using a 1998 Pentium PC, may be a Pentium II with a 200 MHz clock, obtained a 100x100 magic square in about 2h, which would mean about 20 minutes in a 1GHz modern PC...and our Pentium III with a 1GHz clock, using the Cplex algorithm to implement MIP, took about *4.3 days* to obtain a 7x7 relaxed magic square!...which is a much more simple problem than obtaining a 7x7 magic square! In Table 1 you can find the solution obtained with our MIP model and you can verify that as a matter of fact the sums of the main diagonals are not equal to the magic sum 175.

Table 1: The *relaxed* 7x7 magic square obtained by the Cplex algorithm after *4.3 days* of computation on a Pentium III @ 1GHz. You can verify that the sums of the main diagonals are not equal to the magic sum of 175 and that this solution is completely different from the two solutions presented in Tables 2 and 3.

26	2	43	17	38	10	39
23	19	12	42	13	45	21
31	48	3	14	7	44	28
1	49	40	29	25	4	27
18	36	16	33	11	41	20
30	15	37	32	34	22	5
46	6	24	8	47	9	35

3 Comparing GAs to Humans

Using a game we developped based on the *relaxed* magic square we obtained a 7x7 *relaxed magic square* in 83 moves departing from a sequential square (see table 2) and in 57 moves departing from a random square (see table 3), which is surely a better performance than the MIP's one but we don't believe that we would be capable to build a 100x100 magic square!

Table 2: The 7x7 *relaxed* magic square obtained by the author in 83 moves from a sequential initialisation.

Move 83, Objective=175

49	29	1	4	21	35	36
16	**9**	10	46	40	13	48
8	**2**	17	45	47	31	18
25	23	24	28	26	22	27
14	34	43	7	30	32	15
19	41	38	12	6	39	20
44	37	42	33	5	3	11

Error=98

[old value,new value]=[9 2]

Move 84

49	29	1	4	21	35	36
16	**2**	10	46	40	13	48
8	**9**	17	45	47	31	18
25	23	24	28	26	22	27
14	34	43	7	30	32	15
19	41	38	12	6	39	20
44	37	42	33	5	3	11

Error=0

Table 3: The 7x7 *relaxed* magic square obtained by the author in *57 moves* from a random initialisation.

Move 57, Objective=175

2	35	24	23	39	16	36
17	5	31	29	42	45	6
41	20	46	25	4	18	21
10	12	14	49	44	38	8
43	30	19	1	7	27	48
15	40	32	37	**13**	3	22
47	33	9	11	**26**	28	34

Error=338

[old value,new value]=[13 26]

Move 58

2	35	24	23	39	16	36
17	5	31	29	42	45	6
41	20	46	25	4	18	21
10	12	14	49	44	38	8
43	30	19	1	7	27	48
15	40	32	37	**26**	3	22
47	33	9	11	**13**	28	34

Error=0

4 Comparing GAs to AI Minimax Randomized Algorithm

Before describing in detail our *artificial intelligence randomized minimax* algorithm let's see, in table 4, how it did obtain a magic square of order four from random initialization and in appendix A we did obtain a 4x4 magic square from a random initial filling…in 118 moves!

Table 4: Example of a run of AI minimax algorithm finding a magic square of order four from random initial filling.

Move 1, Objective=34

1	9	3	13
8	6	14	12
11	5	**2**	**15**
7	10	4	16

Error=886

15 <--> 2
Move 2, Objective=34

1	**9**	3	13
8	6	14	**12**
11	5	15	2
7	10	4	16

Error=301

12 <--> 9
Move 3, Objective=34

1	12	3	13
8	6	14	9
11	5	15	2
7	10	4	**16**

Error=175

11 <--> 16
Move 4, Objective=34

1	**12**	3	13
8	6	**14**	9
16	5	15	2
7	10	4	11

Error=90

14 <--> 12
Move 5, Objective=34

1	14	**3**	13
8	6	12	9
16	**5**	15	2
7	10	4	11

Error=46

5 <--> 3
Move 6, Objective=34

1	14	5	13
8	6	12	9
16	3	15	**2**
7	10	4	11

Error=22

2 <--> 1
Move 7, Objective=34

2	14	5	13
8	6	**12**	9
16	3	15	1
7	**10**	4	11

Error=13

10 <--> 12
Move 8, Objective=34

2	**14**	5	**13**
8	6	10	9
16	3	15	1
7	12	4	11

Error=5

13 <--> 14
Move 9, Objective=34

2	13	5	14
8	6	10	**9**
16	3	15	1
7	12	4	11

Error=4

8 <--> 9
Move 10, Objective=34

2	**13**	5	14
9	6	10	8
16	3	15	1
7	**12**	4	11

Error=2

12 <--> 13
Move 11, Objective=34

2	12	5	14
9	6	**10**	8
16	3	15	1
7	13	4	11

Error=4

2 <--> 10
Move 12, Objective=34

10	12	5	14
9	6	2	8
16	3	**15**	1
7	13	**4**	11

Error=388

4 <--> 15

Move 13, Objective=34

10	12	5	14
9	6	2	8
16	**3**	4	1
7	**13**	15	11

Error=575

3 <--> 13

Move 14, Objective=34

10	12	5	14
9	6	**2**	8
16	13	4	1
7	3	15	11

Error=275

7 <--> 2

Move 15, Objective=34

10	**12**	5	14
9	**6**	7	8
16	13	4	1
2	3	15	11

Error=105

12 <--> 6

Move 16, Objective=34

10	6	**5**	14
9	12	7	8
16	13	4	1
2	3	15	11

Error=45

5 <--> 10

Move 17, Objective=34

5	6	10	14
9	12	**7**	8
16	13	4	1
2	3	15	11

Error=30

5 <--> 7

Move 18, Objective=34

7	6	10	14
9	12	5	8
16	13	4	1
2	3	15	11

Error=18

3 <--> 6

Move 19, Objective=34

7	3	10	14
9	12	5	8
16	13	4	1
2	6	15	11

Error=0

Each change corresponds to the permutation that minimizes the error *over a set of random number of cycles of random chosen pairs of numbers*. When is detected a situation where the chosen pair of numbers to be permuted is equal to the previous, then next change corresponds to the change that, now, *maximizes* the error *over a set of random number of cycles of random chosen pairs of numbers*. This prevents the oscillation and *stagnation in local minimum.*

5 Computational Results

In table 5 we compare the AI minimax algorithm to an improved evolutionary algorithm for n=3..20, in terms of number permutations. In most cases our algorithm is much more efficient. Note that the number of permutations is obtained, in both cases, in only one run, and not averaged over a set of successive runs.

Table 5: Computational results of one run of each algorithm in terms of number of permutations.

n	AI Minimax Alg	Improved GA
3	7	121
4	19	57
5	185	129
6	126	48
7	108	3645
8	48	1824
9	84	456
10	95	2985
11	2023	2766
12	320	823
13	3330	562
14	1017	3510
15	1111	893
16	415	7762
17	191	753
18	420	1922
19	625	4507
20	613	6215

6 Conclusions and Future Work

We showed that although very simple, our AI minimax algorithm is very powerful. Nevertheless their runtime are not so small as the permutations number, since each permutation results from a minimization/maximization over a relatively great number of cycles, and the calculation of the new error associated to a given permutation is time consuming. We also showed that the magic square is a good benchmark to compare optimization algorithms since it has an explosive combinatorics that increases with $n^2!$, n being the order of the magic square.

In the near future we are planning to test our algorithm to solve a much more complex problem: the *relaxed addition-multiplication magic square* where it is *relaxed* the constraint of the maximum element being n^2 which turns its combinatorics much more explosive [3-4].

References

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