An Intuitively Simple Property of Limit Figures of Quadratic Transformations

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Abstract In this paper, we deal with intuitively simple properties of the limit figures of two-dimensional inhomogeneous quadratic transformations. The divergence-convergence boundary of homogeneous quadratic transformations was investigated in detail in Da-te (1978). In an inhomogeneous case, there exist, possibly, the region of initial points converging to a fixed point other than the origin due to the linear terms. Then, there appears a boundary with a finite area as a limit figure. Next, in certain cases, the convergence regions or the divergence regions consist of infinite number of separated regions. We show the examples of the properties, and investigate them.

Keywords: quadratic transformation, inhomogeneous quadratic transformation, limit figure, divergence-convergence boundary, intuitively simple property

1 Introduction

The behavior of a point in iteration processes of a quadratic transformation depends not only on the coefficients of the transformation but also essentially on the initial point. It is the difference in kind from a linear transformation and brings about the complexity of its behavior. It was investigated in detail in Da-te (1978), especially in a two-dimensional real homogeneous case, and the properties of the limit figures were exhaustively cleared. In the one-dimensional complex inhomogeneous quadratic transformations, illustration of nice fractals can be found in Mandelbrot (1982), etc.

In this paper, we deal with intuitively simple properties of a divergence-convergence boundary (DCB) in the two-dimensional real inhomogeneous quadratic transformations. And we illustrate some DCBs, and investigate their characteristic properties. When a transformation has a stable fixed point other than the origin, there exists a set of initial points converging to the fixed point, which forms a region. Then, there appears a region, which is neither a convergence region nor a divergence region. Or in certain cases, the convergence region separates the divergence region into infinite number of regions, and vice versa. These properties come from the influence of linear terms, and they are not observed in the case of DCBs of homogeneous quadratic transformations.

2 Quadratic Transformation and its DCB

In this section, we introduce a quadratic transformation, its divergence-convergence boundary (DCB), and an algorithm to illustrate DCB.

2.1 Divergence-Convergence Boundary of Homogeneous Case

An n-dimensional homogeneous quadratic transformation is written in the form:

$$x'^k = f^k(x) = P^k_{rs} x^r x^s$$

$$(k = 1, 2, \ldots, n;\ r, s = 1, 2, \ldots, n;\ \boldsymbol{x} \in \boldsymbol{R}^n;\ P^k_{rs} \in \boldsymbol{R}), \quad (1)$$

where f can be considered a mapping from $\boldsymbol{R}^n$ into itself.

Many properties and canonical forms of homogeneous quadratic transformations were investigated in Date and Iri (1976).

A divergence-convergence boundary (DCB) is defined in Date (1978), and the shapes of DCBs in two dimensions are investigated and classified in detail.

For a homogeneous quadratic transformation, we introduce a convergence region C as

$$C = \{\boldsymbol{x}^{(0)} \mid \lim \|\boldsymbol{f}^m(\boldsymbol{x}^{(0)})\| = 0 \text{ for } m \to \infty\}, \quad (2)$$

where $\boldsymbol{x}^{(0)}$ is the coordinates of initial point. Then, we can define a divergence-convergence boundary B as

$$B = \{r(\Theta)\,\Theta \mid r(\Theta) < \infty\}, \quad (3)$$

where $r(\Theta) = \sup\{\alpha \mid \alpha\Theta \in C\}$ for $\Theta \in S$ is a mapping from the unit sphere $S = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| = 1\}$ into $\boldsymbol{R} \cup \{\infty\}$. A divergence region D is defined as $D = \boldsymbol{R}^n - B - C$.

The DCB is a limit figure, which is obtained by infinite number of iterations of a transformation.

There are many algorithms to illustrate images of DCBs, and an algorithm that gives the images more quickly by using the properties of the DCB in a homogenous case was introduced in Da-te (1978). The algorithm used in this paper is shown in the next section.

We show two examples of the approximate image of convergence regions in homogeneous quadratic transformations in Fig. 1. In these figures, the black region represents a convergence region and the white region does a divergence region. The boundary of these two regions corresponds to a DCB. In Fig. 1(a), the upper and the lower curves of the DCB are smooth. On the other hand, the left and the right curves are complicated and considered to have a self-similar structure. In Fig. 1(b), the convergence region has many spikes, and they are considered to extend to infinity.

2.2 Divergence-Convergence Boundary of Inhomogeneous Case

An n-dimensional quadratic transformation is written in the form:

$$x'^k = f^k(x) = P^k_{rs} x^r x^s + P^k_t x^t$$

$$(k = 1, 2, \ldots, n;\ r, s, t = 1, 2, \ldots, n;\ \boldsymbol{x} \in \boldsymbol{R}^n;\ P^k_{rs}, P^k_t \in \boldsymbol{R}), \quad (4)$$

where f can be considered a mapping from $\boldsymbol{R}^n$ into itself.

In an inhomogeneous case, we modify the definition of a DCB of homogenous quadratic transformations. The DCB is, intuitively, a set of initial points that neither converge to the origin nor diverge to infinity in transformation process.

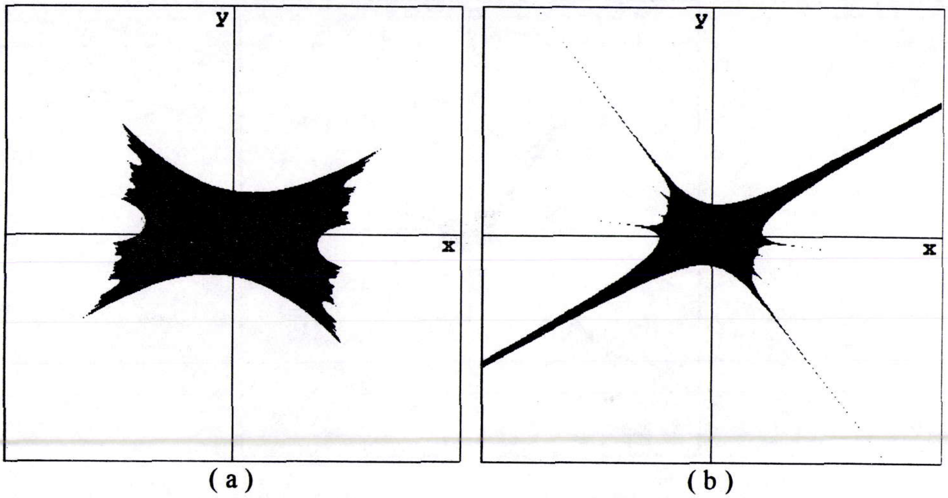


Fig. 1: Examples of DCB in homogeneous quadratic transformations

In this paper, we illustrate the images of DCBs by the following process.

1. choose a pixel for initial point $\boldsymbol{x}^{(0)}$
2. for $m = 1, 2, \ldots, M$,

    if $| \boldsymbol{f}^m (\boldsymbol{x}^{(0)}) | >$ INF, then $\boldsymbol{x}^{(0)} \in D$,

    if $| \boldsymbol{f}^m (\boldsymbol{x}^{(0)}) | <$ EPS, then $\boldsymbol{x}^{(0)} \in C$,

    where INF ($\in \boldsymbol{R}$) is a sufficiently large fixed number, EPS ($\in \boldsymbol{R}$) is a sufficiently small fixed number, and M ($\in N$) is fixed.

We apply the same procedure for all pixels.

The values of INF, EPS, M are given in consideration of the programming language system or the coefficients of transformation, etc.

Fig. 2 is an example of the images of DCBs in inhomogeneous quadratic transformations. As shown this figure, in inhomogeneous cases, there exist, generally, many points of DCB on a certain straight line through the origin. The convergence region in Fig. 2 has a self-similar structure, i.e. the shape of its portion at the end of each branch is similar to the whole.

For certain inhomogeneous quadratic transformations, a set of initial points converging to a fixed point other than the origin forms a region. We call the set a boundary region in this paper. When there is a boundary region, there is a region that is neither a convergence region nor a divergence region as a limit figure.

3 Simple Properties of DCB

For a homogeneous quadratic transformation, there exists only one point of DCB on

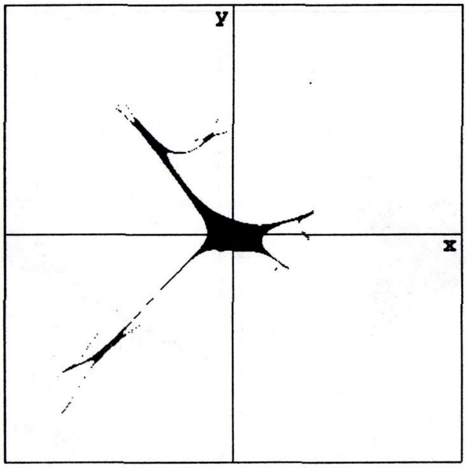


Fig. 2: An Example of DCB in inhomogeneous quadratic transformations

any straight line through the origin. On the other hand, in an inhomogeneous case, there are generally many points of DCB on certain directions (Fig. 2).

Moreover, $r(\Theta)$ defined in 2.1 was proved to be lower semicontinuous with respect to Θ in Da-te (1978), and every DCB of homogeneous transformations doesn't form a region. Then, there doesn't exist a boundary region in homogeneous cases.

In this section, we investigate these characteristic properties of the DCB in an inhomogeneous case.

3.1 DCB of Quadratic Transformation with Stable Fixed Point Other than the Origin

Certain inhomogeneous transformations have a stable fixed point other than the origin. In this case, there exists a set of initial points converging to the fixed point, i.e. a boundary region.

For example, the transformation

$$x' = x^2 + 0.7x - 1.2y,$$
$$y' = x^2 + xy - 0.008333x + 0.7y \qquad (5)$$

has three fixed points, (x, y) = (0, 0), (–0.4, 0.23333), (–0.2, 0.08333), and two fixed points, (0, 0), (–0.4, 0.2333) are stable. We show the DCB of the transformation in Fig. 3. In this figure, the black region represents a convergence region, the white region does a divergence region, and the gray region is a set of initial points converging to the fixed point (–0.4, 0.23333), i.e. a boundary region.

The boundary region in Fig. 3 has many spikes, and they are bounded. At the end of each spike, there appears the convergence region. In the divergence region, there are many doted curves, but in reality they form stratified curves.

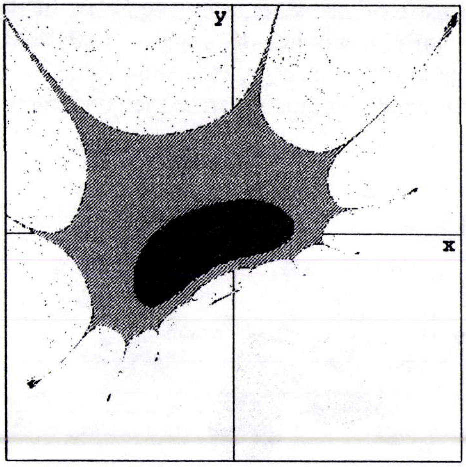


Fig. 3: A DCB with boundary region

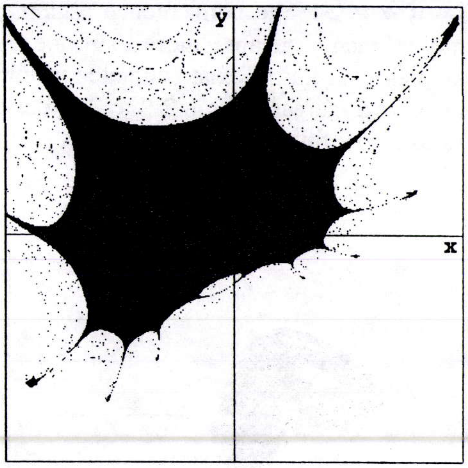


Fig. 4: A DCB without boundary region

The number of fixed points and their stability depend on the parameters of a transformation. It seems to change largely the shape of DCB at the bifurcation parameters.

For example, the transformation

$$x' = x^2 + 0.7\,x - 1.2\,y,$$
$$y' = x^2 + x\,y + 0.01\,x + 0.7\,y \qquad (6)$$

has no fixed point other than the origin. We show the DCB of this transformation in Fig. 4.

As shown in this figure, there doesn't exist the gray region, and the convergence regions have highly similar structure to the union of the convergence regions and the boundary regions in Fig. 3.

As mentioned above, if there exists a fixed point (x^*, y^*), satisfying $x^{*\prime} = x^*$ and $y^{*\prime} = y^*$, other than the origin, and this fixed point is stable, then there exists a boundary region. Then, there appears a boundary region as a limit figure.

3.2 DCB Consisting of Many Separated Regions

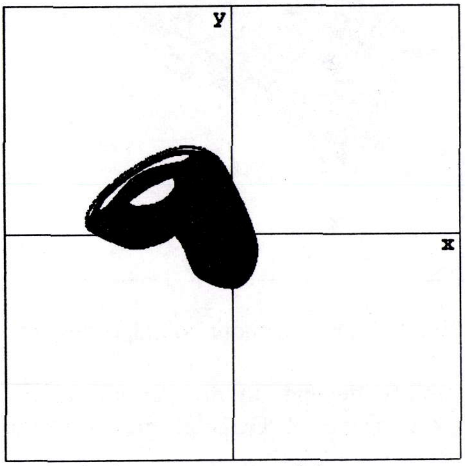
In Fig 5, we show the images of DCBs of the transformations

$$x' = x\,y,$$
$$y' = x^2 + y^2 + a\,x \qquad (7)$$

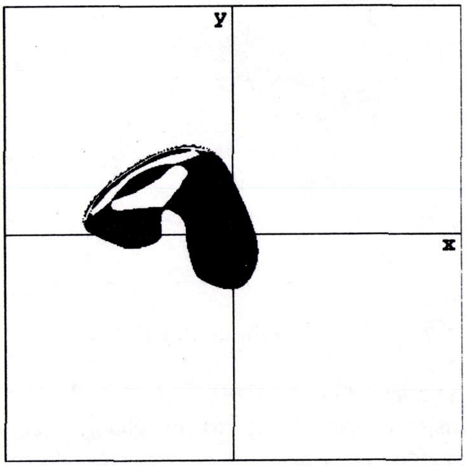
with $a = 2.0$, 2.05, 2.1. All of these transformations don't have any stable fixed point other than the origin.

In all of these figures, many convergence regions and divergence regions appear in turn on the direction from the origin to $(x, y) = (-1, 1)$. In homogeneous cases, there exists only one point of DCB on any direction, and there doesn't exist the above property. This property is one of the characteristic properties in inhomogeneous cases.

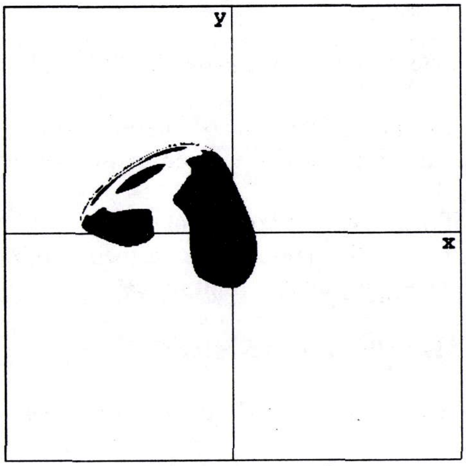
In case of $a = 2.0$, the divergence regions consist of many regions (Fig.5(a)), and in case of a = 2.1, the convergence regions consist of many regions (Fig.5(c)). In this context, “many” means “more than one” or “infinite number of ”, and in both above examples, it seems to exist countable infinite number of the regions. As the value of the parameter a increases from 2.0 to 2.1, the divergence regions separate the convergence regions into infinite number of regions.



(a) parameter a = 2.0



(b) parameter a = 2.05



(c) parameter a = 2.1

Fig.5: DCBs consisting of many separated regions

We have not yet obtained the conditions for the existence of these properties in the shape of DCBs. In the DCB with these properties, it seems to exist always countable infinite number of convergence regions or divergence regions experimentally, but we have not yet investigated it theoretically.

4 Conclusion

We have dealt with the divergence-convergence boundary of two-dimensional inhomogeneous quadratic transformations, and extracted some characteristic properties. In general the shape of DCBs of inhomogeneous transformations has more complicated structure than that of homogeneous cases. Certain inhomogeneous transformations have a stable fixed point other than the origin, and a set of initial points converging to the fixed point forms a region. Then, there appears a boundary with a finite area as a limit figure. Certain transformations have convergence regions or divergence regions consisting of infinite number of separated regions. These properties come from the influence of linear terms, and they are not observed in the DCBs of homogeneous cases.

When a transformation has a stable fixed point other than the origin, there exists a boundary region in the DCB of the transformation. When a transformation has a stable periodic points or pseudo-periodic points, there exists a boundary region for the transformation.

For certain transformations, there exist many points of DCBs on a certain direction. In the special case of them, convergence regions or divergence regions consist of infinite number of separated regions. The conditions for the existence of these properties have not been cleared.

And we have not yet sufficiently investigated the validity of approximate images of DCBs.

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