Evaluation of Bounds on Service Cycle Times in Acyclic Fork-Join Queueing Networks

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Abstract

We present a new approach to get bounds on the service cycle time in acyclic fork-join queueing networks. The approach is based on (max, +)-algebra representation of network dynamics and involves analysis of limiting behaviour of a product of random matrices. As a result, a new upper bound on the cycle time is established which takes into consideration the network topology.

Keywords: Fork-join queueing networks, Cycle time, (Max,+)-algebra, Stochastic difference equation, Product of random matrices

1 Introduction

Fork-join networks, as introduced in (Baccelli, 1989; Baccelli, 1992), present a class of queueing systems which allow customers (jobs, tasks) to be split into several parts, and to be merged into one when they circulate through the system. The fork-join formalism proves to be useful in the description of dynamical processes in a variety of actual complex systems, including production processes in manufacturing, transmission of messages in communication networks, and parallel data processing in multi-processor computer systems. As an illustration of the fork and join operations, one can consider respectively splitting a message into packets in a communication network, each intended for transmitting via separate ways, and merging packets at a destination node of the network to restore the message (Baccelli, 1989).

One of the problems of interest in the analysis of stochastic queueing networks is to evaluate the service cycle time of a network. Both the cycle time and its inverse

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which can be regarded as a throughput present performance measures commonly used to describe efficiency of the network operation.

A natural way to represent the dynamics of fork-join queueing networks relies on the implementation of recursive state equations of the Lindley type (Baccelli, 1989). Since the recursive equations associated with the fork-join networks can be expressed only in terms of the operations of maximum and addition, there is a possibility to represent the dynamics of the networks in terms of the $(\max, +)$-algebra which is actually an algebraic system just supplied with the same two operations (Cuninghame-Green, 1979; Baccelli, 1992; Maslov, 1994).

In this paper, a new approach to get bounds on the service cycle time for acyclic fork-join queueing networks (AFJQN's) is developed. We exploit the $(\max, +)$-algebra representation of network dynamics derived in (Krivulin, 1996), which allows one to describe the evolution of a network by a stochastic vector difference equation. We consider a $(\max, +)$-algebra product of random matrices involved in the equation, and give algebraic bounds on the product. Furthermore, the limiting behaviour of the product is examined, and appropriate bounds on its associated limit matrix are obtained. Finally, we apply the above results to get bounds on the service cycle time, including a new upper bound which takes into account the network topology.

The rest of the paper is organized as follows. Section 2 serves as an introduction to the problem under consideration, including a brief description of AFJQN's and their related performance measures. Section 3 starts with an overview of basic facts about $(\max, +)$-algebra. Furthermore, we investigate alternating $(\max, +)$-algebra products of matrices of particular types, and give some useful inequalities.

In Section 4, we present a dynamic equation which represents the network dynamics, and give an example. We also show that the service cycle time of a network is determined by the limiting behaviour of a product of random matrices. We examine properties of the matrix product and offer algebraic bounds in Sections 5 and 6.

In Section 7, the above results are applied to establish existence conditions for a limiting matrix and obtain appropriate bounds on the matrix. Finally, in Section 8, we present bounds on the service cycle time, and give related examples.

2 Acyclic Fork-Join Queueing Networks

Consider a network with n single-server nodes and customers of a single class. The topology of the network is described by an oriented acyclic graph $\mathcal{G} = (\mathbf{N}, \mathbf{A})$, where the set $\mathbf{N} = \{1, \ldots, n\}$ represents the nodes, and $\mathbf{A} = \{(i,j)\} \subset \mathbf{N} \times \mathbf{N}$ does the arcs determining the transition routes of customers.

For every node $i \in \mathbf{N}$, we denote the sets of its immediate predecessors and successors respectively as $\mathbf{P}(i) = \{j | (j,i) \in \mathbf{A}\}$ and $\mathbf{S}(i) = \{j | (i,j) \in \mathbf{A}\}$. In specific cases, there may be one of the conditions $\mathbf{P}(i) = \emptyset$ and $\mathbf{S}(i) = \emptyset$ encountered. Each node i with $\mathbf{P}(i) = \emptyset$ is assumed to represent an infinite external arrival stream of customers; provided that $\mathbf{S}(i) = \emptyset$, it is considered as an output node intended to release customers from the network.

Each node $i \in \mathbf{N}$ includes a server and its buffer with infinite capacity, which together present a single-server queue operating under the first-come, first-served discipline. At the initial time, the servers are assumed to be free of customers; the buffers in all nodes i with $\mathbf{P}(i) \neq \emptyset$ are empty, whereas the buffer at each node with no predecessors is assumed to contain an infinite number of customers.

Furthermore, we suppose that, in addition to the usual service procedure, special join and fork operations are performed in its nodes, respectively before and after service. The join operation is actually thought to cause each customer which comes into node i, not to enter the buffer at the server but to wait until at least one customer from every node $j \in \mathbf{P}(i)$ arrives. As soon as these customers arrive, they, taken one from each preceding node, are united into one customer which then enters the buffer to become a new member of the queue.

The fork operation at node i is initiated every time the service of a customer is completed; it consists in giving rise to several new customers instead of the original one. As many new customers appear in node i as there are succeeding nodes included in the set $\mathbf{S}(i)$. These customers simultaneously depart the node, each being passed to separate node $j \in \mathbf{S}(i)$. We assume that the execution of fork-join operations when appropriate customers are available, as well as the transition of customers within and between nodes require no time.

For the queue at node i, we denote the kth arrival and departure epochs respectively as $u_i(k)$ and $x_i(k)$. Furthermore, the service time of the kth customer at server i is indicated by τ_{ik}. We assume that τ_{ik} are given nonnegative random variables (r.v's). With the condition that the network starts operating at time zero, it is convenient to set $x_i(0) = 0$, and $x_i(k) = -\infty$ for all $k < 0$, $i = 1, \ldots, n$.

It is easy to set up equations which relates $x_i(k)$ and $u_i(k)$. Specifically, the dynamics of the queue at any node i is described as

$$x_i(k) = \max(\tau_{ik} + u_i(k), \tau_{ik} + x_i(k-1)). \tag{1}$$

As it immediately follows from the above description of the fork-join operations, the kth arrival epoch into the queue at node i is represented as

$$u_i(k) = \begin{cases} \max\limits_{j \in \mathbf{P}(i)} x_j(k), & \text{if } \mathbf{P}(i) \neq \emptyset, \\ -\infty, & \text{if } \mathbf{P}(i) = \emptyset. \end{cases} \tag{2}$$

We consider the evolution of the network as sequences of service cycles performed in the network nodes. In each node, the 1st cycle starts at the initial time, and it is terminated as soon as the server in the node completes its 1st service, the 2nd cycle is terminated as soon as the server completes its 2nd service, and so on. Clearly, the completion time of the kth cycle in node i can be represented as $x_i(k)$.

In many applications, one is interested in evaluating the limits

$$\gamma_i = \lim_{k\to\infty} \frac{1}{k} x_i(k), \qquad \gamma = \lim_{k\to\infty} \frac{1}{k} \max_i x_i(k)$$

for all $i = 1, \ldots, n$, provided that they exist. The limit γ is normally referred to as the service cycle time of the network. The system throughput presents another performance measure of interest, which is calculated as the inverse of γ.

It has been shown in (Krivulin, 1998) that if for each $i = 1, \ldots, n$, the service times $\tau_{i1}, \tau_{i2}, \ldots$, present independent and identically distributed (i.i.d.) r.v.'s with finite mean and variance, then it holds

$$\max_i \mathbb{E}[\tau_{i1}] \leq \gamma \leq \mathbb{E}[\max_i \tau_{i1}].$$

Note that both the lower and upper bound do not depend on the topology of the underlying network. Below a new upper bound will be given based on the $(\max, +)$-algebra approach. The bound allows one to take into account the network topology.

3 The (Max,+)-Algebra

The $(\max, +)$-algebra presents an idempotent commutative semiring (idempotent semifield) which is defined as the triple $\langle \mathbb{R}_\varepsilon, \oplus, \otimes \rangle$ with $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, $\varepsilon = -\infty$, and binary operations $\oplus$ and $\otimes$ defined as

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y,$$

for all $x, y \in \mathbb{R}_\varepsilon$.

As it is easy to see, the operations $\oplus$ and $\otimes$ retain most of the properties of the ordinary addition and multiplication, including associativity, commutativity, and distributivity of $\otimes$ over $\oplus$. This allows algebraic manipulations to be performed under the usual conventions regarding brackets and precedence of $\otimes$ over $\oplus$. Note that the operation $\oplus$ is idempotent. In other words, for any $x \in \mathbb{R}_\varepsilon$, one has $x \oplus x = x$.

There are the null and identity elements in the algebra, namely ε and 0, to satisfy the conditions $x \oplus \varepsilon = \varepsilon \oplus x = x$, and $x \otimes 0 = 0 \otimes x = x$, for any $x \in \mathbb{R}_\varepsilon$. The null element ε and the operation $\otimes$ are related by the usual absorption rule involving $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

3.1 Matrix Algebra

The $(\max,+)$-algebra of matrices is readily introduced in the regular way. Specifically, for any $(n \times n)$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, we have

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \quad \text{and} \quad \{A \otimes B\}_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}.$$

As in the conventional algebra, both the matrix operations $\oplus$ and $\otimes$ are associative, whereas only the operation $\oplus$ is commutative. The distributivity property of $\otimes$ over $\oplus$ is also valid in the matrix algebra.

The matrices

$$\mathcal{E} = \begin{pmatrix} \varepsilon & \dots & \varepsilon \\ \vdots & \ddots & \vdots \\ \varepsilon & \dots & \varepsilon \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & & \varepsilon \\ & \ddots & \\ \varepsilon & & 0 \end{pmatrix}$$

present the null and identity elements, respectively.

The matrix operations $\oplus$ and $\otimes$ possess monotonicity properties; that is, the component-wise matrix inequalities $A \leq C$ and $B \leq D$ result in

$$A \oplus B \leq C \oplus D, \qquad A \otimes B \leq C \otimes D$$

for any matrices of an appropriate size.

Let $A \neq \mathcal{E}$ be a square matrix. In the same way as in the conventional algebra, one can define $A^0 = E$, and $A^m = A \otimes A^{m-1} = A^{m-1} \otimes A$ for any integer $m \geq 1$.

Note that idempotency of $\oplus$ leads, in particular, to the identity

$$(A \oplus B)^m = \bigoplus_{i=0}^{m} A^i \otimes B^{m-i}, \tag{3}$$

for any square matrices A and B of the same size.

Consider an $(n \times n)$-matrix A. It can be treated as an adjacency matrix of an oriented graph with n nodes, provided each entry $a_{ij} \neq \varepsilon$ implies the existence of the arc (i,j) in the graph, while $a_{ij} = \varepsilon$ does the lack of the arc.

It is easy to verify that for any integer $m \geq 1$, the matrix A^m has its the entry $a_{ij}^{(m)} \neq \varepsilon$ if and only if there exists a path from node i to node j in the graph, which consists of m arcs. Furthermore, if the graph associated with the matrix A is acyclic, we have $A^m = \mathcal{E}$ for all $m > p$, where p is the length of the longest path in the graph. Otherwise, provided that the graph is not acyclic, one can construct a path of any length, and then it holds that $A^m \neq \mathcal{E}$ for all $m \geq 0$.

Let $A = (a_{ij})$ be an arbitrary matrix. The matrix G obtained from A by replacing each entry $a_{ij} > \varepsilon$ by 0 is referred to as the support matrix associated with A.

For any matrix A, we denote its maximal element as

$$\|A\| = \bigoplus_{i,j} a_{ij}.$$

Suppose that G is the support matrix of A. Then we can write the obvious inequality

$$A \leq \|A\| \otimes G. \tag{4}$$

Finally, we introduce the ordinary matrix addition $+$ as an external operation. We consider both the operations $\otimes$ and $\oplus$ as taking precedence over $+$ in any algebraic expressions below. Clearly, the operation $+$ is distributive over $\oplus$.

3.2 Distributivity Properties

Let A_{ij} be $(n \times n)$-matrices for all $i = 1, \dots, k$ and $j = 1, \dots, m$. Distributivity of the operation $\otimes$ over $\oplus$ immediately gives the equality

$$\bigotimes_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} = \bigoplus_{1 \leq j_1, \dots, j_k \leq m} A_{1j_1} \otimes \cdots \otimes A_{kj_k}, \tag{5}$$

which leads, in particular, to the inequality

$$\bigotimes_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} \geq \bigoplus_{j=1}^{m} \bigotimes_{i=1}^{k} A_{ij}. \tag{6}$$

In a similar way, one can easily get the equality

$$\sum_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} = \bigoplus_{1 \leq j_1, \dots, j_k \leq m} (A_{1j_1} + \cdots + A_{kj_k}), \tag{7}$$

and then the inequality

$$\sum_{i=1}^{k} \bigoplus_{j=1}^{m} A_{ij} \geq \bigoplus_{j=1}^{m} \sum_{i=1}^{k} A_{ij}. \tag{8}$$

Consider the matrix operations $\otimes$ and $+$. Although there is no way to formulate any general distributivity property associated with these operations, in particular cases involving support and diagonal matrices, some useful inequalities can be established.

Specifically, it is not difficult to verify that for any matrices A and B, and support matrices G_1 and G_2, it holds

$$G_1 \otimes (A + B) \otimes G_2 \leq G_1 \otimes A \otimes G_2 + G_1 \otimes B \otimes G_2. \tag{9}$$

Consider a $(\max, +)$-algebra diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ with all off-diagonal elements equal to ε. As it is easy to see both matrix operations $\otimes$ and $+$ being applied to diagonal matrices produce the same result. In other words, we have $D_1 \otimes D_2 = D_1 + D_2$.

Let D_1 and D_2 be diagonal matrices. Then for any matrices A and B, it holds

$$\begin{aligned} & D_1 \otimes (A + B) \otimes D_2 \\ &= D_1 \otimes A \otimes D_2 + B = D_1 \otimes A + B \otimes D_2 = A + D_1 \otimes B \otimes D_2. \end{aligned} \tag{10}$$

3.3 Products of Diagonal and Support Matrices

In this section, we consider $\otimes$-products of alternating diagonal and support matrices, which take the form

$$D_0 \otimes (G \otimes D_1) \otimes \cdots \otimes (G \otimes D_m) = D_0 \otimes \bigotimes_{j=1}^{m} (G \otimes D_j), \tag{11}$$

where $D_0, D_1, \ldots, D_m$ are diagonal matrices, G is a support matrix. Some useful inequalities will be given which offer bounds on the product in terms of both the ordinary matrix addition $+$ and $\otimes$-multiplication.

First suppose that the diagonal matrices in eq. 11 can have both positive and negative elements on the main diagonal.

Lemma 1 *Let G be a support matrix, and D_j, $j = 0, 1, \ldots, m$, be diagonal matrices. Then it holds*

$$D_0 \otimes \bigotimes_{j=1}^{m} (G \otimes D_j) \leq \sum_{j=0}^{m} G^j \otimes D_j \otimes G^{m-j}. \tag{12}$$

The above inequality can easily be proved with eq. 9 by using induction on m. Note that for $m = 1$, we have from the obvious identity $D_0 \otimes G = D_0 \otimes G + G$, and eq. 10

$$D_0 \otimes G \otimes D_1 = (D_0 \otimes G + G) \otimes D_1 = D_0 \otimes G + G \otimes D_1.$$

Lemma 2 *Let G be a support matrix, $D_j^{(i)}$ be diagonal matrices for all $i = 1, \ldots, k$ and $j = 0, 1, \ldots, m_i$, and $m = m_1 + \cdots + m_k$. Then it holds*

$$\bigotimes_{i=1}^{k} D_0^{(i)} \otimes \bigotimes_{j=1}^{m_i} (G \otimes D_j^{(i)}) \leq \sum_{i=1}^{k} \sum_{j=M_{i-1}}^{M_i} G^j \otimes D_{j-M_{i-1}}^{(i)} \otimes G^{m-j}, \tag{13}$$

where $M_0 = 0$, and $M_i = m_1 + \cdots + m_i$ for all $i = 1, \ldots, k$.

The proof of the lemma can be given using eqs. 12 and 9.

Let us now suppose that the diagonal matrices in the products under consideration have only nonnegative elements on the main diagonal.

Lemma 3 *Let G be a support matrix, $D_j^{(1)}, D_j^{(2)}$ be diagonal matrices with non-negative elements on the main diagonal for all $j = 0, 1, \ldots, m$. Then for any s, $1 \leq s \leq m$, it holds*

$$\begin{aligned} & D_0^{(1)} \otimes \bigotimes_{j=1}^{m} (G \otimes D_j^{(1)}) + D_0^{(2)} \otimes \bigotimes_{j=1}^{m} (G \otimes D_j^{(2)}) \\ \geq \; & D_0^{(1)} \otimes \bigotimes_{j=1}^{s} (G \otimes D_j^{(1)}) \otimes \bigotimes_{j=s+1}^{m} (G \otimes D_j^{(2)}). \end{aligned} \tag{14}$$

Proof: Let us first introduce the matrices

$$D_j = D_j^{(1)} \otimes D_j^{(2)} = D_j^{(1)} + D_j^{(2)}$$

for all $j = 0, 1, \ldots, m$.

By applying induction on m with eqs. 10,9, it is easy to verify the inequality

$$D_0^{(1)} \otimes \bigotimes_{j=1}^{m} (G \otimes D_j^{(1)}) + D_0^{(2)} \otimes \bigotimes_{j=1}^{m} (G \otimes D_j^{(2)}) \geq D_0 \otimes \bigotimes_{j=1}^{m} (G \otimes D_j).$$

Since the diagonal elements of the matrices $D_j^{(1)}$ and $D_j^{(2)}$ are nonnegative, it holds that $D_j \geq D_j^{(1)}$ and $D_j \geq D_j^{(2)}$ for all $j = 0, 1, \ldots, m$. Therefore, for any s, $1 \leq s \leq m$, we get

$$D_0 \otimes \bigotimes_{j=1}^{m} (G \otimes D_j) \geq D_0^{(1)} \otimes \bigotimes_{j=1}^{s} (G \otimes D_j^{(1)}) \otimes \bigotimes_{j=s+1}^{m} (G \otimes D_j^{(2)}).$$

4 Algebraic Representation of Network Dynamics

In this section, we briefly show how the dynamics of AFJQN's can be described based on the (max, +)-algebra approach. Further details can be found in (Krivulin, 1996; Krivulin, 1998).

Let us consider eq. 1 and eq. 2. Clearly, with the (max, +)-algebra operations, they can be rewritten in their equivalent forms as

$$x_i(k) = \tau_{ik} \otimes u_i(k) \oplus \tau_{ik} \otimes x_i(k-1), \tag{15}$$

$$u_i(k) = \begin{cases} \bigoplus\limits_{j \in \mathbf{P}(i)} x_j(k), & \text{if } \mathbf{P}(i) \neq \emptyset, \\ \varepsilon, & \text{if } \mathbf{P}(i) = \emptyset. \end{cases} \tag{16}$$

In order to get eqs. 15 and 16 in a matrix-vector form let us introduce

$$\boldsymbol{u}(k) = \begin{pmatrix} u_1(k) \\ \vdots \\ u_n(k) \end{pmatrix}, \quad \boldsymbol{x}(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix}, \quad \mathcal{T}_k = \begin{pmatrix} \tau_{1k} & & \varepsilon \\ & \ddots & \\ \varepsilon & & \tau_{nk} \end{pmatrix}.$$

As it easy to see, eq. 15 leads us to the equation

$$\boldsymbol{x}(k) = \mathcal{T}_k \otimes \boldsymbol{u}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{x}(k-1). \tag{17}$$

Furthermore, eq. 16 can be rewritten in a vector form as

$$\boldsymbol{u}(k) = G^T \otimes \boldsymbol{x}(k), \tag{18}$$

where G^T denotes the transpose of the support matrix G with elements

$$g_{ij} = \begin{cases} 0, & \text{if } i \in \mathbf{P}(j), \\ \varepsilon, & \text{otherwise.} \end{cases}$$

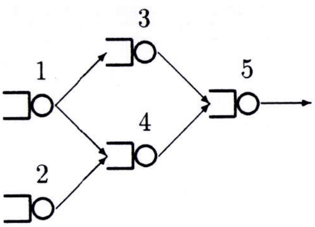
Note that G can be considered as an adjacency matrix of the network graph.

By combining eqs. 17 and 18, we arrive at the equation

$$\boldsymbol{x}(k) = \mathcal{T}_k \otimes G^T \otimes \boldsymbol{x}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{x}(k-1).$$

By iterating the above implicit equation, with the condition that $G^q = \mathcal{E}$ for all $q > p$, we immediately obtain the explicit dynamic equation

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1), \tag{19}$$



$$G = \begin{pmatrix} \varepsilon & \varepsilon & 0 & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

Fig. 1: An acyclic fork-join network.

where

$$A(k) = \bigoplus_{j=0}^{p} (\mathcal{T}_k \otimes G^T)^j \otimes \mathcal{T}_k = \bigoplus_{j=0}^{p} \mathcal{T}_k \otimes (G^T \otimes \mathcal{T}_k)^j. \tag{20}$$

An example of AFJQN having $n = 5$ nodes together with its associated support matrix G are shown in Fig. 1.

Taking into account that for the graph $\mathcal{G}$, the length of its longest path $p = 2$, we arrive at eq. 19 with the state transition matrix calculated from eq. 20 as

$$A(k) = (E \oplus \mathcal{T}_k \otimes G^T \oplus (\mathcal{T}_k \otimes G^T)^2) \otimes \mathcal{T}_k$$
$$= \begin{pmatrix} \tau_{1k} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_{2k} & \varepsilon & \varepsilon & \varepsilon \\ \tau_{1k} \otimes \tau_{3k} & \varepsilon & \tau_{3k} & \varepsilon & \varepsilon \\ \tau_{1k} \otimes \tau_{4k} & \tau_{2k} \otimes \tau_{4k} & \varepsilon & \tau_{4k} & \varepsilon \\ \tau_{1k} \otimes (\tau_{3k} \oplus \tau_{4k}) \otimes \tau_{5k} & \tau_{2k} \otimes \tau_{4k} \otimes \tau_{5k} & \tau_{3k} \otimes \tau_{5k} & \tau_{4k} \otimes \tau_{5k} & \tau_{5k} \end{pmatrix}.$$

Consider the service cycle time γ of the network. It is clear that now we can represent it as

$$\gamma = \lim_{k \to \infty} \frac{1}{k} \|\boldsymbol{x}(k)\|,$$

where $\|\boldsymbol{x}(k)\|$ denotes the maximal element of $\boldsymbol{x}(k)$.

Let us represent the vector $\boldsymbol{x}(k)$ in the form

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1) = \cdots = A(k) \otimes \cdots A(1) \otimes \boldsymbol{x}(0),$$

and denote

$$A_k^T = A^T(1) \otimes \cdots \otimes A^T(k) = \bigotimes_{i=1}^{k} \bigoplus_{j=0}^{p} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^j. \tag{21}$$

Clearly, in order to get information about the growth rate of $\boldsymbol{x}(k)$, one can examine the limiting behaviour of the matrix A_k^T. Below we investigate the limit $A^T = \lim_{k \to \infty} A_k^T / k$, and give related existence conditions.

5 Subadditivity Property

Let us consider the product of matrices A_k defined by eq. 21, and introduce the family of matrices $\{A_{lk}^T | \, l, k = 0, 1, \ldots; \, l < k\}$ with

$$A_{lk}^T = A^T(l+1) \otimes \cdots \otimes A^T(k).$$

Note that $A_k^T = A_{0k}^T$.

The next lemma states that the family $\{A_{lk}^T | \, l < k\}$ possesses subadditivity property.

Lemma 4 *For all* $l < r < k$*, it holds*

$$A_{lk}^T \leq A_{lr}^T + A_{rk}^T.$$

Proof: With eqs. 5 and 7, we can write

$$\begin{aligned}
A_{lr}^T + A_{rk}^T &= \bigotimes_{i=l+1}^{r} \bigoplus_{j=0}^{p} T_i \otimes (G \otimes T_i)^j + \bigotimes_{i=r+1}^{k} \bigoplus_{j=0}^{p} T_i \otimes (G \otimes T_i)^j \\
&= \bigoplus_{0 \leq m_{l+1}, \ldots, m_r \leq p} \ \bigotimes_{i=l+1}^{r} T_i \otimes (G \otimes T_i)^{m_i} + \bigoplus_{0 \leq m_{r+1}, \ldots, m_k \leq p} \ \bigotimes_{i=r+1}^{k} T_i \otimes (G \otimes T_i)^{m_i} \\
&= \bigoplus_{\substack{0 \leq m_{l+1}, \ldots, m_r \leq p \\ 0 \leq m_{r+1}, \ldots, m_k \leq p}} \left(\bigotimes_{i=l+1}^{r} T_i \otimes (G \otimes T_i)^{m_i} + \bigotimes_{i=r+1}^{k} T_i \otimes (G \otimes T_i)^{m_i} \right).
\end{aligned}$$

By imposing more restrictive conditions on the indices $m_{l+1}, m_{l+2}, \ldots, m_k$ in the last term, we get

$$A_{lr}^T + A_{rk}^T \geq \bigoplus_{m=0}^{p} \bigoplus_{\substack{m_{l+1} + \cdots + m_r = m \\ m_{r+1} + \cdots + m_k = m \\ m - m_r \leq m_{r+1}}} \left(\bigotimes_{i=l+1}^{r} T_i \otimes (G \otimes T_i)^{m_i} + \bigotimes_{i=r+1}^{k} T_i \otimes (G \otimes T_i)^{m_i} \right).$$

Consider the sum in parenthesis. Lemma 3 allows us to take any integer s such that $m - m_r \leq s \leq m_{r+1}$, so as to write

$$\begin{aligned}
& \bigotimes_{i=l+1}^{r} T_i \otimes (G \otimes T_i)^{m_i} + \bigotimes_{i=r+1}^{k} T_i \otimes (G \otimes T_i)^{m_i} \\
& \geq \bigotimes_{i=l+1}^{r-1} T_i \otimes (G \otimes T_i)^{m_i} \otimes T_r \otimes (G \otimes T_r)^{s-m+m_r} \\
& \quad \otimes T_{r+1} \otimes (G \otimes T_{r+1})^{m_{r+1}-s} \otimes \bigotimes_{i=r+2}^{k} T_i \otimes (G \otimes T_i)^{m_i} \\
& = \bigotimes_{i=l+1}^{k} T_i \otimes (G \otimes T_i)^{s_i},
\end{aligned}$$

where

$$s_i = \begin{cases} m_i, & \text{if } l \le i < r, \\ s - m + m_r, & \text{if } i = r, \\ m_{r+1} - s, & \text{if } i = r+1, \\ m_i, & \text{if } r+1 < i \le k \end{cases}$$

with $s_{l+1} + \cdots + s_k = m$.

Finally, with the condition that $G^q = \mathcal{E}$ for all $q > p$, we have

$$\begin{aligned} A_{lr}^T + A_{rk}^T &\ge \bigoplus_{m=0}^{p} \bigoplus_{s_{l+1}+\cdots+s_k=m} \bigotimes_{i=l+1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^{s_i} \\ &= \bigoplus_{0 \le s_{l+1},\ldots,s_k \le p} \bigotimes_{i=l+1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^{s_i} = \bigotimes_{i=l+1}^{k} \bigoplus_{j=0}^{p} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^j \\ &= A_{lk}^T. \end{aligned}$$

6 Algebraic Bounds on A_k

The next lemma offers algebraic bounds on the matrix A_k^T.

Lemma 5 *It holds that*

$$\bigoplus_{r=0}^{\lfloor p/k \rfloor} \bigotimes_{i=1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^r \le A_k^T \le \left\| \bigoplus_{i=1}^{k} \mathcal{T}_i \right\| \otimes \bigoplus_{r=1}^{p} G^r + \sum_{i=1}^{k} \bigoplus_{0 \le r+s \le p} G^s \otimes \mathcal{T}_i \otimes G^r, \quad (22)$$

where $\lfloor r \rfloor$ *is the greatest integer equal to or less than* r.

Proof: With eq. 6, and considering that $G^q = \mathcal{E}$ if $q = kr > p$, we immediately obtain the lower bound

$$A_k^T \ge \bigoplus_{r=0}^{p} \bigotimes_{i=1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^r = \bigoplus_{r=0}^{\lfloor p/k \rfloor} \bigotimes_{i=1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^r.$$

Note that as k becomes greater than p, the lower bound degenerates into

$$A_k^T \ge \bigotimes_{i=1}^{k} \mathcal{T}_i = \sum_{i=1}^{k} \mathcal{T}_i.$$

In order to derive the upper bound, we first apply eq. 5 to represent the matrix A_k^T in the form

$$A_k^T = \bigotimes_{i=1}^{k} \bigoplus_{j=0}^{p} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^j = \bigoplus_{0 \le m_1,\ldots,m_k \le p} \bigotimes_{i=1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^{m_i}.$$

Application of eq. 13 to the inner $\otimes$-product, with $M_0 = 0$, $M_i = m_1 + \cdots + m_i$ for all $i = 1, \ldots, k$, and $m = m_1 + \cdots + m_k$, gives

$$\begin{aligned}\bigotimes_{i=1}^{k} \mathcal{T}_i \otimes (G \otimes \mathcal{T}_i)^{m_i} &\leq \sum_{i=1}^{k} \sum_{j=M_{i-1}}^{M_i} G^j \otimes \mathcal{T}_i \otimes G^{m-j} \\ &= \sum_{i=1}^{k} \sum_{j=M_{i-1}+1}^{M_i} G^j \otimes \mathcal{T}_i \otimes G^{m-j} + \sum_{i=1}^{k} G^{M_{i-1}} \otimes \mathcal{T}_i \otimes G^{m-M_{i-1}}.\end{aligned}$$

With eq. 8, we further obtain

$$\begin{aligned}A_k^T &\leq \bigoplus_{0 \leq m_1, \ldots, m_k \leq p} \sum_{i=1}^{k} \sum_{j=M_{i-1}+1}^{M_i} G^j \otimes \mathcal{T}_i \otimes G^{m-j} \\ &+ \bigoplus_{0 \leq m_1, \ldots, m_k \leq p} \sum_{i=1}^{k} G^{M_{i-1}} \otimes \mathcal{T}_i \otimes G^{m-M_{i-1}} = S_1 + S_2.\end{aligned}$$

Let us now define $D_k = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_k$. Since $D_k \geq \mathcal{T}_i$ for each $i = 1, \ldots, k$, we get with eq. 4

$$\begin{aligned}S_1 &\leq \bigoplus_{0 \leq m_1, \ldots, m_k \leq p} \sum_{i=1}^{k} \sum_{j=M_{i-1}+1}^{M_i} G^j \otimes D_k \otimes G^{m-j} \\ &= \bigoplus_{0 \leq m_1 + \cdots + m_k \leq p} \sum_{j=1}^{m_1 + \cdots + m_k} G^j \otimes D_k \otimes G^{m-j} \\ &= \bigoplus_{r=1}^{p} \sum_{s=1}^{r} G^s \otimes \left(\bigoplus_{i=1}^{k} \mathcal{T}_i \right) \otimes G^{r-s} \leq \left\| \bigoplus_{i=1}^{k} \mathcal{T}_i \right\| \otimes \bigoplus_{r=1}^{p} G^r.\end{aligned}$$

Let us represent S_2 in its equivalent form as

$$S_2 = \bigoplus_{m=0}^{p} \bigoplus_{0 \leq m_1, \ldots, m_k \leq m} \sum_{i=1}^{k} G^{M_{i-1}} \otimes \mathcal{T}_i \otimes G^{m-M_{i-1}}.$$

By applying eq. 7 and then eq. 8, we finally have

$$\begin{aligned}S_2 &= \bigoplus_{m=0}^{p} \sum_{i=1}^{k} \bigoplus_{j=0}^{m} G^j \otimes \mathcal{T}_i \otimes G^{m-j} \\ &\leq \sum_{i=1}^{k} \bigoplus_{m=0}^{p} \bigoplus_{j=0}^{m} G^j \otimes \mathcal{T}_i \otimes G^{m-j} = \sum_{i=1}^{k} \bigoplus_{0 \leq r+s \leq p} G^s \otimes \mathcal{T}_i \otimes G^r.\end{aligned}$$

7 Limiting Behaviour of A_k

Now we give simple existence conditions for $\lim_{k\to\infty} A_k/k$ to exist, and present bounds on the limiting matrix A. We start with the following theorem which can be proved based on the Subadditive Ergodic Theorem (Kingman, 1973) as well as on the result of Lemma 4.

Theorem 6 *If for each* $i = 1,\ldots,n$, *the service times* $\tau_{i1}, \tau_{i2}, \ldots$, *present i.i.d. r.v.'s with* $\mathbb{E}[\tau_{i1}] < \infty$, *then there exists a fixed matrix* A *such that*

1. $\lim_{k\to\infty} A_k^T/k = A^T$ *with probability 1,*
2. $\lim_{k\to\infty} \mathbb{E}[A_k^T]/k = A^T$.

Theorem 7 *If in addition to the condition of Theorem 6,* $\mathbb{D}[\tau_{i1}] < \infty$ *for each* $i = 1,\ldots,n$, *then it holds*

$$\mathbb{E}[\mathcal{T}_1] \leq A^T \leq \mathbb{E}\Big[\bigoplus_{0\leq r+s\leq p} G^s \otimes \mathcal{T}_i \otimes G^r\Big]. \tag{23}$$

Proof: First note that Theorem 6 allows us to conclude that the limiting matrix A exists with probability one. Moreover, we can write

$$A^T = \lim_{k\to\infty} \mathbb{E}[A_k^T]/k.$$

Clearly, eq. 23 can be obtained from eq. 22 after taking expectation and dividing by k. To prove the right inequality, one has to show that

$$\frac{1}{k}\mathbb{E}\Big\|\bigoplus_{i=1}^{k} \mathcal{T}_i\Big\| \longrightarrow 0 \quad \text{as } k \to \infty.$$

The last assertion follows from the bounds for the mean value of maximum of i.i.d. r.v.'s and related asymptotic results established in (Gumbel, 1954; Hartly, 1954). In fact, these results allow us to write the relation

$$\mathbb{E}\Big\|\bigoplus_{i=1}^{k} \mathcal{T}_i\Big\| = O(\sqrt{k})$$

as k tends to ∞ (see also (Krivulin, 1998)).

8 Evaluation of Bounds on Service Cycle Time

In this section we show how the above bounds on the limiting matrix A can be applied to calculating bounds on the service cycle time γ. The next result is a consequence of Theorem 7.

Lemma 8 *Under the conditions of Theorem 7, for any finite fixed vector* $\boldsymbol{x}(0)$, *it holds*

$$\|\mathbb{E}[\mathcal{T}_1]\| \leq \gamma \leq \left\| \mathbb{E}\Big[\bigoplus_{0\leq r+s\leq p} G^s \otimes \mathcal{T}_1 \otimes G^r \Big] \right\|. \qquad (24)$$

As it is easy to see, eq. 23 can also be exploited to derive bounds on the service cycle times of particular nodes in a network. Let us introduce the matrix

$$B(k) = \bigoplus_{0\leq r+s\leq p} (G^s \otimes \mathcal{T}_k \otimes G^r)^T,$$

and denote its (i,j)-entry by $b_{ij}(k)$. It is not difficult to verify that the service cycle time of node i, $i = 1,\ldots,n$, satisfies the double inequality

$$\mathbb{E}[\tau_{i1}] \leq \gamma_i \leq \bigoplus_{j=1}^{n} \mathbb{E}[b_{ij}(1)].$$

Consider the network depicted in Fig. 1. To get bounds on γ, let us first calculate the matrix

$$B(1) = (\mathcal{T}_1 \oplus G \otimes \mathcal{T}_1 \oplus \mathcal{T}_1 \otimes G \oplus G^2 \otimes \mathcal{T}_1 \oplus G \otimes \mathcal{T}_1 \otimes G \oplus \mathcal{T}_1 \otimes G^2)^T.$$

Furthermore, application of eq. 24 gives

$$\mathbb{E}[\tau_{11}] \oplus \cdots \oplus \mathbb{E}[\tau_{51}] \leq \gamma \leq \mathbb{E}(\tau_{11} \oplus \tau_{31} \oplus \tau_{41} \oplus \tau_{51}) \oplus \mathbb{E}(\tau_{21} \oplus \tau_{41} \oplus \tau_{51}).$$

Now suppose that for all $i = 1,\ldots,5$, and $k = 1,2,\ldots$, the r.v.'s τ_{ik} are independent and they have exponential probability distribution of mean 1. Then we have

$$\mathbb{E}[B(1)] = \begin{pmatrix} 1.0000 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1.0000 & \varepsilon & \varepsilon & \varepsilon \\ 1.5000 & \varepsilon & 1.0000 & \varepsilon & \varepsilon \\ 1.5000 & 1.5000 & \varepsilon & 1.0000 & \varepsilon \\ 2.0833 & 1.8333 & 1.5000 & 1.5000 & 1.0000 \end{pmatrix}.$$

Evaluation of bounds on the cycle time of the network leads us to the inequality

$$1.0000 \leq \gamma \leq 2.0833.$$

Finally, one can easily obtain bounds on the cycle times of the network nodes γ_i, $i = 1, \ldots, 5$:

$$\gamma_1 = \gamma_2 = 1.0000,$$
$$1.0000 \leq \gamma_3, \gamma_4 \leq 1.5000,$$
$$1.0000 \leq \gamma_5 \leq 2.0833.$$

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