Uncertain Logics, Variables and Systems

Z. Bubnicki
Institute of Control and Systems Engineering
Wroclaw University of Technology
Wyb. Wyspianskiego 27, 50-370 Wroclaw, POLAND
E-mail: bubnicki@ists.pwr.wroc.pl

Abstract
The definitions of two versions of uncertain logics and variables are given. The review of main concepts and results concerning the application of the uncertain variables to the analysis and decision making in a class of systems with unknown parameters in their mathematical models is presented. The special and related problems concerning the application to pattern recognition and operation systems, the learning processes and the distributed knowledge representation are described.
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1 Introduction

Uncertainty is one of the main features of anticipatory systems. There exists a great variety of definitions and formal models of uncertainties and uncertain systems [e.g. 25, 26, 27]. The most popular approaches are based on probabilistic model, fuzzy sets theory and related formalisms such as evidence and possibility theory. In this paper the uncertainty is understood in a narrow sense of the word and concerns an incomplete or imperfect knowledge of something which is necessary to solve the problem. In our considerations it is the knowledge of the parameters in the mathematical description of a system. The purpose of this paper is to present a short description of so called uncertain variables and a review of main concepts and results concerning the application of the uncertain variables to the analysis and decision making in a class of systems with unknown parameters in their mathematical models [2, 4, 5, 8, 9, 11, 12, 17-24]. The unknown parameters will be assumed to be uncertain variables and the systems with uncertain parameters will be called uncertain systems.

The uncertain variables, related to random variables and fuzzy numbers are described by their certainty distributions which correspond to probability distributions for the random variables and to membership functions for the fuzzy numbers. The certainty distribution is given by an expert and evaluates his opinion on approximate values of the uncertain variable. The definitions of the uncertain variables are based on definitions of uncertain logics. Two versions of the uncertain logic and the corresponding versions of the uncertain variable are introduced in Sec. 2 and 3. The definitions of the uncertain variables contain not only the formal description but also their interpretation, which is

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of much importance. It is worth to note that from the formal point of view (without taking into account the interpretation) the probabilistic measure is a special case of the fuzzy measure and the probability distribution is a special case of the membership function in the formal definition of the fuzzy number when the meaning of the membership function is not described. The uncertain variable in the first version may be formally considered as a very special case of the fuzzy number (exactly speaking – the possibilistic number) with a specific interpretation of the membership function. Nevertheless for the sake of simplicity and unification it is better to introduce it independently (as has been done in the paper) and not as a special case of much more complicated formalism with different semantics.

In Sec. 4 the applications of the uncertain variables to basic analysis and decision making (control) problems are presented for the system described by a function (functional system) and described by a relation (relational system). In the second case the analysis consists in finding the output property (i.e. the property concerning the output vector or the set to which the output vector belongs) for the given input property. The decision making is an inverse problem: for the given output property one should find the input property which implies the required output property [1, 3]. For the system with uncertain parameters the modified versions of these problems adequate to the description of uncertainty are presented. In Sec. 6 the applications of the uncertain variables for a closed-loop control system with a dynamical plant are indicated and in Sec. 7 special problems are described. The details and examples may be found in the papers listed in References.

2 Uncertain Logics

We shall present two versions of an uncertain logic: logic L and logic C. Consider a universal set Ω, $\omega \in \Omega$, a set $\overline{X} \subseteq R^k$, a function $g : \Omega \to \overline{X}$, a crisp property (predicate) $P(\bar{x})$ and the crisp property $\Psi(\omega, P)$ generated by P and g: "For $\bar{x} = g(\omega) \stackrel{\Delta}{=} \bar{x}(\omega)$ assigned to ω the property P is satisfied". Let us introduce now the property $G_\omega(x) =$ "$\bar{x}(\omega) \cong x$" for $x \in X \subseteq \overline{X}$, which means: "$\bar{x}$ is approximately equal to x" or "x is the approximate value of $\bar{x}$". The properties P and G_ω generate the soft property $\overline{\Psi}(\omega, P)$ in Ω: "the approximate value of $\bar{x}(\omega)$ satisfies P", i.e.

$$\overline{\Psi}(\omega, P) = \text{"}\bar{x}(\omega) \mathrel{\tilde{\in}} D_x\text{"}, \qquad D_x = \{\bar{x} \in \overline{X} : P(\bar{x})\}, \tag{1}$$

which means: "$\bar{x}$ approximately belongs to D_x". Denote by $h_\omega(x)$ the logic value of G:

$$w\,[\bar{x}(\omega) \cong x] \stackrel{\Delta}{=} h_\omega(x), \quad \bigwedge_{x \in X} h_\omega(x) \geq 0, \quad \max_X h_\omega(x) = 1. \tag{2}$$

Definition 1. The *uncertain logic L* is defined by Ω, $\bar{X}$, X, crisp predicates $P(\bar{x})$, the properties $G_\omega(x)$ and corresponding functions $h_\omega(x)$ for $\omega \in \Omega$. In this logic we consider soft properties (1) generated by P and G_ω. The logic value of $\bar{\Psi}$ is

$$w\,[\bar{\Psi}(\omega, P)] \stackrel{\Delta}{=} v\,[\bar{\Psi}(\omega, P)] = \begin{cases} \max\limits_{x \in D_x} h_\omega(x) & \text{for } D_x \neq \varnothing \\ 0 & \text{for } D_x = \varnothing \end{cases}$$

and is called a *certainty index*. The operations are defined as follows:

$$v\,[\neg \bar{\Psi}(\omega, P)] = 1 - v\,[\bar{\Psi}(\omega, P)], \tag{3}$$

$$v\,[\Psi_1(\omega, P_1) \vee \Psi_2(\omega, P_2)] = \max\{v[\Psi_1(\omega, P_1), v\,[\Psi_2(\omega, P_2)]\}, \tag{4}$$

$$v\,[\Psi_1(\omega, P_1) \wedge \Psi_2(\omega, P_2)] = \begin{cases} 0 & \text{if for each } x \ w\,(P_1 \wedge P_2) = 0 \\ \min\{v[\Psi_1(\omega, P_1), v\,[\Psi_2(\omega, P_2)]\} & \text{otherwise} \end{cases} \tag{5}$$

where Ψ_1 is $\bar{\Psi}$ or $\neg\bar{\Psi}$, and Ψ_2 is $\bar{\Psi}$ or $\neg\bar{\Psi}$ □

It is easy to note that G_ω is a special case of $\bar{\ }$ for $D_x = \{x\}$ (a singleton) and

$$v\,[\bar{x}(\omega) \cong x] = h_\omega(x), \qquad v\,[\bar{x}(\omega) \not\cong x] = 1 - h_\omega(x).$$

For the logic L one can prove the following statements [11, 20]:

$$v\,[\bar{\Psi}(\omega, P_1 \vee P_2)] = v\,[\bar{\Psi}(\omega, P_1) \vee \bar{\Psi}(\omega, P_2)], \tag{6}$$

$$v\,[\bar{\Psi}(\omega, P_1 \wedge P_2)] \leq \min\{v\,[\bar{\Psi}(\omega, P_1)], v\,[\bar{\Psi}(\omega, P_2)]\}, \tag{7}$$

$$v\,[\bar{\Psi}(\omega, \neg P)] \geq v\,[\neg \bar{\Psi}(\omega, P)]. \tag{8}$$

The interpretation (semantics) of the uncertain logic L is the following: The uncertain logic operates with crisp predicates P, but for the given ω it is not possible to state if P is true or false because the function g and consequently the value $\bar{x}$ is unknown. The function $h_\omega(x)$ is given by an expert, who "looking at" ω obtains some information concerning $\bar{x}$ and uses it to evaluate his opinion that $\bar{x} \cong x$. For the same (Ω, X) we may have the different logics (the different h_ω) determined by different experts.

Definition 2 (the *uncertain logic* C). The first part is the same as in Def.1. The certainty index of $\bar{\Psi}$ and the operations are defined as follows:

$$v_C[\bar{\Psi}(\omega,P) = \frac{1}{2}\{v[\bar{\Psi}(\omega,P)]+1-v[\bar{\Psi}(\omega,\neg P)]\} = \frac{1}{2}[\max_{x\in D_x} h_\omega(x)+1-\max_{x\in \bar{D}_x} h_\omega(x)],$$

$$\neg\bar{\Psi}(\omega,P) = \bar{\Psi}(\omega,\neg P), \tag{9}$$

$$\bar{\Psi}(\omega,P_1)\vee\bar{\Psi}(\omega,P_2) = \bar{\Psi}(\omega,P_1\vee P_2), \tag{10}$$

$$\bar{\Psi}(\omega,P_1)\wedge\bar{\Psi}(\omega,P_2) = \bar{\Psi}(\omega,P_1\wedge P_2) \tag{11}$$

□

One can note that G_ω is a special case of $\bar{\Psi}$ and

$$v_C[\bar{x}(\omega)\cong x] = \frac{1}{2}[h_\omega(x)+1-\max_{\tilde{x}\in X-\{x\}} h_\omega(\tilde{x})]. \tag{12}$$

For the logic C one can prove the following statements [11, 20]:

$$v_C[\bar{\Psi}(\omega,P_1\vee P_2)] \ge \max\{v_C[\bar{\Psi}(\omega,P_1)], v_C[\bar{\Psi}(\omega,P_2)]\}, \tag{13}$$

$$v_C[\bar{\Psi}(\omega,P_1\wedge P_2)] \le \min\{v_C[\bar{\Psi}(\omega,P_1)], v_C[\bar{\Psi}(\omega,P_2)]\}, \tag{14}$$

$$v_C[\neg\bar{\Psi}(\omega,P)] = 1-v_C[\bar{\Psi}(\omega,P)]. \tag{15}$$

The definition of implication is not introduced here because it is not used in the further considerations concerning uncertain variables and the decision making problem.

3 Uncertain Variables

The variable $\bar{x}$ for a fixed ω will be called an uncertain variable. Two versions of uncertain variables will be defined by: $h(x)$ given by an expert and the definitions of certainty indexes $w(\bar{x} \mathrel{\tilde{\in}} D_x)$, $w(\bar{x} \mathrel{\tilde{\notin}} D_x)$, $w(\bar{x} \mathrel{\tilde{\in}} D_1 \vee \bar{x} \mathrel{\tilde{\in}} D_2)$, $w(\bar{x} \mathrel{\tilde{\in}} D_1 \wedge \bar{x} \mathrel{\tilde{\in}} D_2)$.

Definition 3. *L-uncertain variable* $\bar{x}$ is defined by *X*, the function $h(x) = v(\bar{x} \cong x)$ given by an expert and the following definitions:

$$v(\bar{x} \mathrel{\tilde{\in}} D_x) = \max_{x \in D_x} h(x) \text{ for } D_x \neq \varnothing \quad \text{and} \quad 0 \text{ for } D_x = \varnothing, \tag{16}$$

$$v(\bar{x} \mathrel{\tilde{\notin}} D_x) = 1 - v(\bar{x} \mathrel{\tilde{\in}} D_x), \tag{17}$$

$$v(\bar{x} \mathrel{\tilde{\in}} D_1 \vee \bar{x} \mathrel{\tilde{\in}} D_2) = \max\{v(\bar{x} \mathrel{\tilde{\in}} D_1), v(\bar{x} \mathrel{\tilde{\in}} D_2)\}, \tag{18}$$

$$v(\bar{x} \mathrel{\tilde{\in}} D_1 \wedge \bar{x} \mathrel{\tilde{\in}} D_2) = \begin{cases} \min\{v(\bar{x} \mathrel{\tilde{\in}} D_1), v(\bar{x} \mathrel{\tilde{\in}} D_2)\} & \text{for } D_1 \cap D_2 \neq \varnothing \\ 0 & \text{for } D_1 \cap D_2 = \varnothing. \end{cases} \tag{19}$$

The function $h(x)$ will be called *L-certainty distribution* □

The definition of *L*-uncertain variable is based on logic *L*. Then for (1) the properties (6), (7), (8) are satisfied. In particular, (8) becomes

$$v(\bar{x} \mathrel{\tilde{\in}} \overline{D}_x) \geq v(\bar{x} \mathrel{\tilde{\notin}} D_x) = 1 - v(\bar{x} \mathrel{\tilde{\in}} D_x).$$

Definition 4. *C-uncertain variable* is defined by *X*, $h(x) = v(\bar{x} \cong x)$ given by an expert and the following definitions:

$$v_C(\bar{x} \mathrel{\tilde{\in}} D_x) = \frac{1}{2}[\max_{x \in D_x} h(x) + 1 - \max_{x \in \overline{D}_x} h(x)], \tag{20}$$

$$v_C(\bar{x} \mathrel{\tilde{\notin}} D_x) = 1 - v_C(\bar{x} \mathrel{\tilde{\in}} D_x), \tag{21}$$

$$v_C(\bar{x} \mathrel{\tilde{\in}} D_1 \vee \bar{x} \mathrel{\tilde{\in}} D_2) = v_C(\bar{x} \mathrel{\tilde{\in}} D_1 \cup D_2), \tag{22}$$

$$v_C(\bar{x} \,\tilde{\in}\, D_1 \wedge \bar{x} \,\tilde{\in}\, D_2) = v_C(\bar{x} \,\tilde{\in}\, D_1 \cap D_2) \tag{23}$$

□

The definition of C-uncertain variable is based on logic C. Then for (1) the properties (13), (14), (15) are satisfied. According to (9) and (15) $\bar{x}$. The function $v_C(\bar{x} \cong x) \triangleq h_C(x)$ may be called a *C-certainty distribution*. To determine h_C and $v_C(\bar{x} \,\tilde{\in}\, D_x)$ it is necessary to know $h(x)$ and to use (12) and (20), respectively. The uncertain logics L and C are chosen as the bases for the uncertain variables because of the advantages of these approaches. In both cases $w(\bar{x} \,\tilde{\notin}\, D_x) = 1 - w(\bar{x} \,\tilde{\in}\, D_x)$, in the first case it is easy to determine the certainty indexes for $\vee$ and $\wedge$, in the second case in the definition of $M(\bar{x}) = \sum_{i=1}^{m} x_i h(x_i)(\sum_{i=1}^{m} h(x_i))^{-1}$ the values of $h(x)$ for $\overline{D}_x$ are also taken into account. In the discrete case, i.e. for $X = \{x_1, x_2, ..., x_m\}$

$$M(\bar{x}) = \sum_{i=1}^{m} x_i h(x_i)(\sum_{i=1}^{m} h(x_i))^{-1} \tag{24}$$

is called a mean value of L-uncertain variable $\bar{x}$. In the continuous case (h is a continuous function) the sums in (24) should be replaced by integrals. The mean value of C-uncertain variable is defined in the same way, with h_C in the place of h. To compare the uncertain variables with probabilistic and fuzzy approaches, let us take into account the following definitions for the discrete case ($x_i \in R^1$), using Ω, ω and g introduced in Sec.2.

The *random variable* $\tilde{x}$ is defined by X and probability distribution $P(\tilde{x} = x) \triangleq p(x)$. The function $p(x)$ does not depend on the subjective opinion of an expert, may be determined in empirical way and describes the whole set Ω ($h_\omega(x)$ describes the fixed, particular ω).

In the fuzzy approach there exist three basic definitions of the fuzzy set based on the number set X: (a) The *fuzzy number* $\hat{x}(d)$ for the given fixed value $d \in X$ is defined by X and the membership function $\mu(x, d)$ which may be considered as a logic value of the soft property "if $\hat{x} = x$ then $\hat{x} \cong d$". (b) The linguistic fuzzy variable $\hat{x}$ is defined by X and a set of membership functions $\mu_i(x)$ corresponding to different description of the size of $\hat{x}$ (small, medium, large etc.). E.g. $\mu_1(x)$ may be considered as a logic value of the soft property "if $\hat{x} = x$ then $\hat{x}$ is small". (c) The fuzzy number $\hat{x}(\omega)$ is defined by X and $\mu_\omega(x)$ which is a logic value of the soft property "it is possible that $P(\omega, x)$" for the given $P(\omega, x)$. In the cases (a) and (b) μ does not depend on ω and the difference between $\hat{x}(d)$ or $\hat{x}$ and the uncertain variable $\bar{x}(\omega)$ is evident. In the case (c) the function g may not exist, $\bar{x}(\omega)$ may be considered as a special case of $\hat{x}(\omega)$ (where relation $P(\omega, x)$ is reduced to the function g), with a special interpretation of $\mu_\omega(x) = h_\omega(x) =$ certainty index that $\bar{x}(\omega) \cong x$. The further difference is connected with the definitions of the certainty indices for $\bar{x} \mathrel{\tilde{\in}} D_x$, $\bar{x} \mathrel{\tilde{\notin}} D_x$, $\bar{x} \mathrel{\tilde{\in}} D_1 \vee \bar{x} \mathrel{\tilde{\in}} D_2$ and $\bar{x} \mathrel{\tilde{\in}} D_1 \wedge \bar{x} \mathrel{\tilde{\in}} D_2$. The function $w(\bar{x} \mathrel{\tilde{\in}} D_x) \triangleq m(D_x)$ may be considered as a measure defined for the family of sets D_x. Two measures have been defined here: $v(\bar{x} \mathrel{\tilde{\in}} D_x) \triangleq m_L(D_x)$ and $v_C(\bar{x} \mathrel{\tilde{\in}} D_x) \triangleq m_C(D_x)$. Taking into account the measures known in our area (e.g. [25]) it is easy to show, that m_C is neither belief nor plausibility measure and m_L is a possibility measure with very specific semantics: $m_C(D_x)$ is a certainty index that $\bar{x} \mathrel{\tilde{\in}} D_x$.

4 Uncertain Systems

Systems whose description contains uncertain parameters may be called uncertain systems. For such systems one may formulate analysis and decision making problems analogous to those for deterministic functional and relational systems [1, 3]. Consider a static system described by a function $y = \Phi(u, x)$ where $u \in U$, $y \in Y$, $x \in X$ are input, output and unknown parameter vector, respectively (U, Y, X are number vector spaces). The parameter x is assumed to be a value of an uncertain variable $\bar{x}$ with $h_x(x)$ given by an expert.

Analysis problem for a functional system: For the given Φ, $h_x(x)$ and u determine $h_y(y; u)$ and $M_y(\bar{y})$. Using (16) one obtains

$$h_y(y; u) = v(\bar{y} \cong y) = \max_{x \in D_x(y;u)} h_x(x) \tag{25}$$

where $D_x(y; u) = \{x \in X : \Phi(u, x) = y\}$. Having $h_y(y; u)$ one can determine $M_y(\bar{y})$ according to (24). The analysis problem may be extended for a system described by a relation $R(u, y, x) \subset U \times Y \times X$ in the following way:

Analysis problem for a relational system: For the given R, $h_x(x)$, u and D_y determine $v[D_y \mathrel{\tilde{\subseteq}} D_y(u; x)]$ where

$$D_y(u; x) = \{y \in Y : (u, y, x) \in R(u, y, x)\}$$

is the set of all possible outputs for the fixed u. To solve the problem one should determine $D_x(D_y, u) = \{x \in X : D_y \subseteq D_y(u; x)\}$. Then

$$v[D_y \tilde{\subseteq} D_y(u; \bar{x})] = v[\bar{x} \tilde{\in} D_x(D_y, u)] = \max_{x \in D_x(D_y, u)} h_x(x). \tag{26}$$

The value (26) denotes the certainty index of the property: the set of all possible outputs approximately contains the set D_y given by a user. In the above formulations $\bar{x}$ has been considered as L-uncertain variable.

The analogous problems may be formulated for C-uncertain variables: for the same data as in the above formulations one should determine C-certainty distribution $h_{C,y}(y; u)$ and $M_{C,y}(\bar{y})$ in the case of the functional system and $v_C[D_y \tilde{\subseteq} D_y(u; \bar{x})]$ for the second case. For the solution one should use (25) and (26), find

$$v[\bar{x} \tilde{\in} \overline{D}_x(D_y, u)] = \max_{x \in \overline{D}_x(D_y, u)} h_x(x)$$

where $\overline{D}_x = X - D_x$, and determine $h_{C,y}(y; u)$, $v_C[D_y \tilde{\subseteq} D_y(u; \bar{x})]$ using the relationships (12) and (20), i.e.

$$v_C[D_y \tilde{\subseteq} D_y(u, \bar{x})] = v_C[\bar{x} \tilde{\in} D_x(D_y, u)] = \frac{1}{2}\{v[\bar{x} \tilde{\in} D_x(D_y, u)] + 1 - v[\bar{x} \tilde{\in} \overline{D}_x(D_y, u)] .$$

Decision making problem for a functional system may be formulated as follows:
Version I: to find the decision u^* maximizing $v(\bar{y} \cong y^*)$ where y^* is a desirable output variable.
Version II: to find u^* such that $M_y(\bar{y}; u) = y^*$.

In both versions $h_y(y; u)$ should be determined according to (25). Then, in version I u^* is the value of u maximizing $h_y(y^*; u)$ and in version II u^* is obtained from the equation $M_y(\bar{y}; u) = y^*$. If $\bar{x}$ is considered as C-uncertain variable then $h_{C,y}$, $M_{C,y}$ and $v_C(\bar{y} \cong y^*)$ should be determined.

Decision making problem for a relational system: For the given R, $h_x(x)$ and D_y find the decision u^* maximizing the certainty index of the property: the set of all possible outputs approximately belongs to D_y given by a user. Then

$$u^* = \arg\max_u v[D_y(u;\bar{x}) \tilde{\subseteq} D_y] = \arg\max_u \max_{x \in \hat{D}_x(D_y,u)} h_x(x) \quad (27)$$

where

$$\hat{D}_x(D_y, u) = \{x \in X : D_y(u; x) \subseteq D_y\}$$

or

$$\hat{D}_x(D_y, u) = \{x \in X : u \in D_u(x)\}$$

where $D_u(x) \subset U$ is the largest set such that the implication $u \in D_u(x) \to y \in D_y$ is satisfied, i.e. $D_u(x) = \{u \in U : D_y(u, x) \subseteq D_y\}$. Let $\hat{x} = \arg\max h_x(x)$, i.e. $h(\hat{x}) = 1$. Then the set of all decisions u^* is $D_u = \{u \in U : \hat{x} \in D_x(u)\}$ and $v[D_y(u;\bar{x}) \tilde{\subseteq} D_y] = 1$. The considerations using v_C for C-uncertain variables are analogous to those for the analysis problem.

The formulations of the analysis and decision problems may be extended for the system (the plant) described by a function $y = \Phi(u, z, x)$ or by a relation $R(u, y, z, x)$ where $z \in Z$ is a vector of external disturbances [19, 23]. In particular, the decision problem for the functional plant in version I may be formulated as follows: For the given z and y^* find u^* maximizing $v(\bar{y} \cong y^*)$. Then

$$u^* = \arg\max_{u \in U} \overline{\Phi}(u, z) \stackrel{\Delta}{=} \Psi(z)$$

where $\overline{\Phi}(u, z) = h_y(y^*; u, z)$ and h_y is determined according to (25) with $\Phi(u, z, x)$ in the place of $\Phi(u, x)$. If u^* is a unique value maximizing $\overline{\Phi}$ for the given z then we obtain the function $u^* = \Psi(z)$ i.e. the decision algorithm in an open-loop decision system. Assume that the equation $\Phi(u, z, x) = y$ has a unique solution with respect to u: $u_d \stackrel{\Delta}{=} \Phi_d(z, x)$. This relationship together with $h_x(x)$ may be considered as a knowledge of the decision making and may be called an *uncertain decision algorithm*. Using it one can obtain the deterministic algorithm

$$u_d^* = \arg\max_{u \in U} h_u(u; z) \stackrel{\Delta}{=} \Psi_d(z)$$

where

$$h_u(u; z) = \max_{x \in D_x(u;z)} h_x(x)$$

and $D_x(u; z) = \{x \in X : u = \Phi_d(z, x)\}$. The algorithms Ψ and Ψ_d are based on the knowledge of the plant $KP = <\Phi, h_x>$ and on the knowledge of the decision making $KD = <\Phi_d, h_x>$, respectively. The results of these two approaches may be different, i.e. in general $\Psi_d(z) \neq \Psi(z)$.

Example 1: Consider a plant without the disturbances z, with $u, y, x \in R^1$. The relation R is given by inequality $xu \leq y \leq 2xu$, $D_y = [y_1, y_2]$, $y_1 > 0$, $y_2 > 2y_1$. Then $D_u(x) = [\frac{y_1}{x}, \frac{y_2}{2x}]$, $D_x(u) = [\frac{y_1}{u}, \frac{y_2}{2u}]$. Assume that x is a value of an uncertain variable $\bar{x}$ with triangular distribution $h_x(x)$: $x \in [0,1]$, $\hat{x} = \frac{1}{2}$. It is easy to note that u^* minimizing v in (27) is any value from $[2y_1, y_2]$ and $v(u^*) = 1$. For C-uncertain variable we obtain

$$v_C(u) = \begin{cases} y_2(2u)^{-1} & \text{when} \quad u \leq y_1 + 0.5y_2 \\ 1 - y_1 u^{-1} & \text{when} \quad y_1 \leq u \leq y_1 + 0.5y_2 \\ 0 & \text{when} \quad u \leq y_1 . \end{cases}$$

It is easy to see that $u_C^* = y_1 + 0.5y_2$ where u_C^* is the value maximizing $v_C(u)$, and $v_C(u_C^*) = y_2(2y_1 + y_2)^{-1}$. E.g. for $y_1 = 2$, $y_2 = 12$ the results are the following: $u^* \in [4, 12]$ and $v = 1$; $u_C^* = 8$ and $v_C = 0.75$.

5 Closed-Loop Control System. Uncertain Controller

The approach based on uncertain variables may be applied to closed-loop control systems containing continuous dynamic plant with unknown parameters which are assumed to be values of uncertain variables. The plant may be described by a classical model or by a relational knowledge representation. Now let us consider two control algorithms for the classical model of the plant, analogous to the algorithms Ψ and Ψ_d presented in Sec. 4: the control algorithm based on KP and the control algorithm based on KD which may be obtained from KP or may be given directly by an expert. The plant is described by the equations

$$\dot{s}(t) = f[s(t), u(t); x],$$
$$y(t) = \eta[s(t)]$$

where s is a state vector, or by the transfer function $K_P(p; x)$ in the linear case. The controller with the input y (or the control error ε) is described by the analogous model with a vector of parameters b which is to be determined. Consequently, the performance index

$$Q = \int_0^T \varphi(y, x)\, dt \stackrel{\Delta}{=} \Phi(b, x)$$

for the given T and φ is a function of b and x. In particular, for onedimensional plant

$$Q = \int_0^{\infty} \varepsilon^2(t)dt = \Phi(b, x).$$

The closed-loop control system is then considered as a static plant with the input b, the output Q and the unknown parameter x, for which we can formulate and solve the decision problem described in Sec. 4. The control problem consisting in the determination of b in the known form of the control algorithm may be formulated as follows.

Control problem: For the given models of the plant and the controller find the value $\hat{b}$ minimizing $M(\overline{Q})$, i.e. the mean value of the performance index.

The procedure of the problem solving is then the following:

1. To determine the function $Q = \Phi(b, x)$.
2. To determine the certainty distribution $h_q(q; b)$ for $\overline{Q}$ using the function Φ and the distribution $h_x(x)$ in the same way as in the formula (25) for $\overline{y}$.
3. To determine the mean value $M(\overline{Q}; b)$.
4. To find $\hat{b}$ minimizing $M(\overline{Q}; b)$.

In the second approach corresponding to the determination of Ψ_d for the static plant, it is necessary to find the value $b(x)$ minimizing $Q = \Phi(b, x)$ for the fixed x. The control algorithm with the uncertain parameter $b(x)$ may be considered as a knowledge of the control in our case, and the controller with this parameter may be called an *uncertain controller*. The deterministic control algorithm may be obtained in two ways, giving the different results. The first way consists in substituting $M(\overline{b})$ in the place of $b(x)$ in the deterministic control algorithm, where $M(\overline{b})$ should be determined using the function $b(x)$ and the certainty distribution $h_x(x)$. The second way consists in determination of the relationship between $u_d = M(\overline{u})$ and the input of the controller, using the form of the uncertain control algorithm and the certainty distribution $h_x(x)$. It may be very difficult for the dynamic controller.

The problem may be easier if the state of the plant $s(t)$ is put at the input of the controller. Then the uncertain controller has the form

$$u = \Psi(s, x)$$

which may be obtained as a result of nonparametric optimization, i.e. Ψ is the optimal control algorithm for the given model of the plant with the fixed x and for the given form of a performance index. Then

$$u_d = M(\overline{u}; s) \stackrel{\Delta}{=} \Psi_d(s)$$

where $M(\overline{u}; s)$ is determined using the distribution

$$h_u(u;s) = v[\bar{x} \mathrel{\tilde{\in}} D_x(u;s)] = \max_{x \in D_x(u;s)} h_x(x)$$

and

$$D_x(u;s) = \{x \in X : u = \Psi(s,x)\}.$$

Example 2: Let us consider the time-optimal control of the plant with $K_P(p;x) = xp^{-2}$, subject to constraint $|u(t)| \le M$. It is well known that the optimal control algorithm is the following

$$u(t) = M \operatorname{sgn}(\varepsilon + |\dot{\varepsilon}|\dot{\varepsilon}(2xM)^{-1})$$

where $\varepsilon = -y$. For the given $h_x(x)$ we can determine $h_u(u;\varepsilon,\dot{\varepsilon})$ which is reduced to three values $v_1 = v(\bar{u} \cong M)$, $v_2 = v(\bar{u} \cong -M)$, $v_3 = v(\bar{u} \cong 0)$. Then

$$u_d(t) = M(\bar{u}) = M(v_1 - v_2)(v_1 + v_2 + v_3)^{-1}.$$

It is easy to see that

$$v_1 = \max_{x \in D_{x1}} h_x(x), \qquad v_2 = \max_{x \in D_{x2}} h_x(x)$$

where

$$D_{x1} = \{x : x \operatorname{sgn} \varepsilon > -|\dot{\varepsilon}|\dot{\varepsilon}(2M|\varepsilon|)^{-1}\},$$

$$D_{x2} = \{x : x \operatorname{sgn} \varepsilon < -|\dot{\varepsilon}|\dot{\varepsilon}(2M|\varepsilon|)^{-1}\}$$

and $v_3 = h_x(-|\dot{\varepsilon}|\dot{\varepsilon}(2M\varepsilon)^{-1})$.

The certainty distribution of $\bar{x}$ has triangular form: $h_x = d^{-1}(x - a + d)$ for $a - d \le x \le a$, $h_x = -d^{-1}(x - a - d)$ for $a \le x \le a + d$, $h_x = 0$ otherwise, $0 \le d \le a$. For $\varepsilon > 0$, $\dot{\varepsilon} < 0$ and $x_g < a$ it is easy to obtain the following control algorithm

$$u_d = M(\bar{u}) = \begin{cases} M & \text{for } d \le a - x_g \\ M \dfrac{a - x_g}{3d - 2(a - x_g)} & \text{for } d \ge a - x_g \end{cases}$$

where $x_g = -|\dot{\varepsilon}|\dot{\varepsilon}(2M\varepsilon)^{-1}$. E.g. for $M = 0.5$, $\dot{\varepsilon} = -3$, $\varepsilon = 1$, $a = 16$ and $d = 10$ we obtain $u_d = 0.2$.

6 Special and Related Problems

A. Logical knowledge representation

The relation $R(u, y, x)$ considered in Sec. 4 may be described by a set of facts, i.e. logical formulas concerning u, y and x, which describe a *logical knowledge representation*. If the properties $u \in D_u$ and $y \in D_y$ are described by logical formulas using the simple formulas from the knowledge representation then the analysis and decision problems analogous to those in Sec. 4 may be solved by applying so called *logic-algebraic method*. The main idea of this method consists in replacing individual reasoning concepts based on inference rules by unified algebraic procedures based on the rules in two-value logic algebra [1, 3, 4].

B. Pattern recognition

The uncertain variables may be applied to a knowledge-based pattern recognition problem. Let an object described by a vector of features u belong to a class $j \in J = \{1, 2, ..., M\}$. The knowledge representation in the form of a relation $R(u, j, x)$ is reduced to the sets $D_u(j) \subset U$ for $j = 1, 2, ..., M$. Then

$$D_j(u, x) = \{j \in J : u \in D_u(j)\}$$

is the set of all possible j for the given value u. The recognition problem may consists in finding the certainty index

$$v_j = v[j \,\tilde{\in}\, D_j(u, \bar{x})] = \max_{x \in D_x(j)} h_x(x)$$

where $D_x(j) = \{x \in X : j \in D_j(u, x)\}$, or the certainty index $v(\Delta_j)$ for the given $\Delta_j \subset J$

$$v(\Delta_j) = v[D_j(u, \bar{x}) \,\tilde{\subseteq}\, \Delta_j] = \max_{x \in D_x(\Delta_j)} h_x(x)$$

where $D_x(\Delta_j) = \{x \in X : D_j(u, x) \subseteq \Delta_j\}$. In the first case v_j is the certainty index that j for the object to be classified belongs to the set of all possible j, and in the second case $v(\Delta_j)$ is the certainty index that the set of all possible j belongs to Δ_j [9].

C. Task distribution in the complex of operations

Let us consider a complex of parallel operations executed by a group of executors (operators in production system or computers for computational operations). The operations are described by relations $T_i \le x_i u_i$ where $T_i = y_i$ is the execution time and u_i is the size of a task (e.g. the amount of a raw material) in i-th operation, $i \in \overline{1,k}$. The requirement is $y \le \alpha$ where $y = T = \max T_i$ is the execution time of the whole complex. The **decision problem** is as follows: for the given $h_{x1}(x_1)$, ..., $h_{xk}(x_k)$ and α, find the task distribution $u^* = (u_1^*, ..., u_k^*)$ maximizing $v(\bar{y} \,\tilde{\in}\, [0, \alpha]) = v(\bar{y} \,\tilde{\le}\, \alpha)$, subject to constraints $u_1 + ... + u_k = U$ where U is the size of the task to be distributed. It is easy to see that

$$v(\bar{y} \mathrel{\tilde{\leq}} \alpha) = v(\bar{y}_i \mathrel{\tilde{\leq}} \alpha) \wedge \ldots \wedge v(\bar{y}_k \mathrel{\tilde{\leq}} \alpha) = \min v_i(u_i)$$

where

$$v_i(u_i) = v(\bar{y}_i \mathrel{\tilde{\leq}} \alpha) = \max_{x_i \in D_{xi}(u_i)} h_{xi}(x_i)$$

and $D_{xi}(x_i)$ is described by $x_i \leq \alpha u_i^{-1}$ [15, 16, 21, 22].

D. Descriptive and prescriptive concepts. Generalization

The knowledge of the decision making described in Sec. 4 may be obtained from the knowledge of the plant or may be given directly by an expert. It is possible to compare these two approaches by comparing the deterministic decision algorithm $\Psi_d(z)$ obtained by the determinization of the uncertain decision algorithm based on the knowledge of the plant or the *uncertain decision algorithm* arbitrary formulated by an expert. Under some assumptions so called *principle of equivalency* may be given. The same problem concerns the descriptive and prescriptive approach for the fuzzy description and the *fuzzy decision algorithm* (or fuzzy controller). The considerations using uncertain variables, random variables and fuzzy numbers are similar from the formal point of view and may be generalised by introducing *soft variables* and *evaluating functions*. Then uncertain variables, random variables and fuzzy descriptions are considered as special cases of the description based on the soft variables [23, 18].

E. Learning processes

For the knowledge-based system with unknown parameters x in the knowledge representation, the learning process consisting in *step by step* knowledge validation and updating has been described. The validation and updating may concern the knowledge of the plant, i.e. $R(u, y, x)$ or directly the knowledge of the decision making, i.e. $D_u(x)$ where $D_u(x) \subset U$ is the largest set of the decisions such that the implication $u \in D_u(x) \rightarrow y \in D_y$ is satisfied. In the process of the current estimation of x it is possible to use *a priori* knowledge in the form of the certainty distribution $h_x(x)$ [6, 7, 8, 9, 10, 13, 15, 16]. The analysis of the convergence of the learning process may be based on the stability conditions for uncertain discrete systems [14].

7 Conclusion

The uncertain variables are proved to be a convenient tool solving the decision making problems in a class of uncertain systems, including specific problems for uncertain anticipatory systems. The analysis and decision problems based on the uncertain variables may be extended for some classes of complex uncertain systems with the distributed knowledge representation. In some cases it is possible to apply a decomposition, and to compare the results with a direct approach to the system as a whole. In particular, such an approach is possible for a complex production system with a cascade or a multilevel structure [12, 22]. The problems concerned with the applications of the uncertain variables and certainty distributions in learning knowledge-based systems and in systems with the distributed knowledge representation (including anticipatory systems, decision making in complex manufacturing systems and in computer operating systems) form the main new directions in the considered area.

References

[1] BUBNICKI Z., *Logic-algebraic method for a class of knowledge based systems*, In F. Pichler and R. Moreno-Diaz (eds) Computer Aided Systems Theory. Lecture Notes in Computer Science, Berlin: Springer Verlag, Vol. 1333 (1997), pp. 420-428.

[2] BUBNICKI Z., *Uncertain variables and their applications in uncertain control systems*. In M. H. Hamza (ed) Modelling, Identification and Control, Zurich: Acta Press, (1998), pp. 305-308.

[3] BUBNICKI Z., *Logic-algebraic method for knowledge-based relation systems*. Systems Analysis Modelling Simulation, Vol. 33 (1998), pp. 21-35.

[4] BUBNICKI Z., *Uncertain variables and logic-algebraic method in knowledge based systems*. In M. H. Hamza (ed) Proc. of IASTED Int. Conf. on Intelligent Systems and Control, Halifax, Canada, Zurich: Acta Press, 1998, pp. 135-139.

[5] BUBNICKI Z., *Uncertain logics, variables and systems*, In L. Guangquan (ed) Proc. of the 3rd Workshop of Int. Inst. for General Systems Studies, Beidahe, Qinhuangdao, P. R. China, Tianjin People's Pub. House, 1998, pp. 7-14.

[6] BUBNICKI Z., *Learning processes and logic-algebraic method in knowledge-based control systems*, In S. G. Tzafestas and G. Schmidt (eds) Progress in System and Robot Analysis and Control Design. Lecture Notes in Control and Information Sciences, London: Springer Verlag, Vol. 243 (1999,), pp. 183-194.

[7] BUBNICKI Z., *Learning control systems with relational plants*. Proceedings of the European Control Conference, Karlsruhe, Germany, 1999.

[8] BUBNICKI Z., *Uncertain variables and learning algorithms in knowledge-based control systems*, Artificial Life and Robotics, Vol.3 (1999), No. 3.

[9] BUBNICKI Z., *The application of learning algorithms and uncertain variables in the knowledge-based pattern recognition*. Proceedings of the 5th International Symposium on Artificial Life and Robotics, Oita, Japan, Vol.1, 2000, pp. I-1–I-5.

[10] BUBNICKI Z., *Learning algorithms in a class of knowledge-based systems*. Artificial Life and Robotics, Vol. 4 (2000), No. 1, pp. 22-26.

[11] BUBNICKI Z., *Uncertain variables in the computer aided analysis of uncertain systems*; In: Computer Aided Systems Theory. Lecture Notes in Comp. Sci., Springer Verlag, Vol. 1798 (2000), pp. 528-542.

[12] BUBNICKI Z., *Knowledge validation and updating in a class of uncertain distributed knowledge systems*; Proc. 16th IFIP World Computer Congress. Intelligent Information Processing, Beijing, China, 2000, pp. 516-523.

[13] BUBNICKI Z., *Learning processes in a class of knowledge-based systems*, Kybernetes, Vol. 29 (2000), No. 7/8, pp. 1016-1028.

[14] BUBNICKI Z., *General approach to stability and stabilization for a class of uncertain discrete non-linear systems*, International Journal of Control, Vol. 73 (2000), No. 14, pp.1298-1306.

[15] BUBNICKI Z., *Learning control system for a class of production operations with parametric uncertainties*. In Peter G. Groumpos, Antonios P. Tzes (eds.) Preprints of IFAC Symposium on Manufacturing, Modeling, Management and Control, Patras, Rio, Greece, 2000, pp. 228-233.

[16] BUBNICKI Z., *Learning process in an expert system for job distribution in a set of parallel computers*. Proc. of the 14th Int. Conf. on Systems Engineering, Coventry University, United Kingdom , Vol. 1, 2000, pp. 78-83.

[17] BUBNICKI Z., *Uncertain variables and their applications for a class of uncertain systems*, International Journal of Systems Science, Vol. 32 (2001), No. 5, pp. 651-659.

[18] BUBNICKI Z., *A unified approach to decision making and control in knowledge-based uncertain systems*, In Daniel M. Dubois (ed) Computing Anticipatory Systems: CASYS'00 – Fourth International Conference, N. York: American Institute of Physics, 2001, pp. 545-557.

[19] BUBNICKI Z., *Uncertain variables – a new tool for analysis and design of knowledge-based control systems*, In: Modelling, Identification and Control, Acta Press, Zurich, 2001, pp. 928-930.

[20] BUBNICKI Z., *Uncertain variables and their application to decision making*, IEEE Trans. on SMC, Part A: Systems and Humans, Vol. 31 (2001), No. 6.

[21] BUBNICKI Z., *Application of uncertain variables to control for a class of production operations with parametric uncertainties*, Proc. IFAC Symposium on Manufacturing, Modeling, Management and Control, Prague, 2001, pp. 29-34.

[22] BUBNICKI Z., *Application of uncertain variables and logics to complex intelligent systems*. Proceedings of the 6th International Symposium on Artificial Life and Robotics, Tokyo, Japan, Vol.1, 2001, pp. 220-223.

[23] BUBNICKI Z., *A unified approach to descriptive and prescriptive concepts in uncertain decision systems*, Proceedings of the European Control Conference, Porto, Portugal, 2001, pp. 2458-2463.

[24] BUBNICKI Z., *Uncertain Logics, Variables and Systems*, London, Berlin: Springer-Verlag, (in press).

[25] KLIR G. J., and FOLGER, T. A., *Fuzzy Sets, Uncertainty, and Information*, Englewood Cliffs, NJ: Prentice-Hall, 1988.

[26] Ranze K. C., Stuckenschmidt H., *Modelling uncertainty in expertise*. In: Cuena J (ed) Proc. of the XV. IFIP World Computer Congress, Information Technologies and Knowledge Systems. Österreichische Computer Gesellschaft, Vienna, 1998, pp. 105-116.

[27] Zadeh L., Kacprzyk J. (eds), *Fuzzy Logic for the Management of Uncertainty*. N. York: J. Wiley, 1992.