A Logical System for Reasoning with Inconsistent Deontic Modalities

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Abstract

In this paper we present a class of paraconsistent deontic systems $D^*\tau$ which may constitute, for instance, a framework for the formal study of normative theory in law, in which it is important to manipulate directly the concept of contradiction.

Key Words: Paraconsistent deontic logic, normative theory and inconsistency.

1 Introduction

Deontic logic was introduced by E. Mally in 1926 (*Die Logik des Willens - Grundgesetze des Sollens*, Graz: Leuchner-Lubensky). In his work, Mally has essentially developed a logical system which dealt with expressions such as *I want that...* Later, A. Hofstadter and J. McKinsey attempted to formalize a logic of imperatives, which is closed related to deontic logic. As is well known, in the two systems above the deontic operators were only ornamental; that is, they collapsed.

Only with the publication of [von Wright 54] did deontic logic receive significant development. von Wright also showed how deontic logic and the usual modal logics were related.

When proposing a formalism for normative knowledge, the primary contender is deontic logic. One argument in favor the need for deontic systems claims that such logics are necessary because the language used in laws and norms actually use the concepts expressed by the deontic modalities such as "permitted", "obligatory" and "prohibited".

In applications, for instance in AI and law, some authors have proposed the use of deontic logics, because they are able to represent the relations between "what ought to be the case and what is the case", or "between the ideal and the actual" ([Valente 95]).

However, there are a large number of well known difficulties in reasoning with legal norms:

- In the deontic model, there is no explicit distinction between the statements of fact and norms.
- There is no explicit distinction between facts and norms
- There is no explicit classification of cases.

Another difficult, as becomes evident in some classical examples (Chisholm paradox, Good Samaritan paradox, and others), is that deontic logic normally assumes that it is always desirable to comply with all norms. This leads most deontic systems into paradoxes.

In reasoning with legal norms, the central concern is the application of norms to cases and the results of this application.

Finally, the usual deontic systems, because they are extensions of classical logic, do not permit us to work with conflicting norms.

Roughly speaking, the usual deontic logics are inadequate; there are structural limitations on their use to model legal normative knowledge. Many authors have recognized these aspects, for instance, [Alchurrón and Bulygin 71], [Alchurrón & Makinson 81], and [Da Costa 96]. The literature on deontic logic consists entirely, these days, of work intended to overcome such limitations.

In this paper we present a class of first order paraconsistent deontic systems $D^{*}\tau$ which may constitute, for instance, a framework for the formal study of normative theory in the law, in which it is important to manipulate directly the concept of contradiction.

2 Paraconsistent, paracomplete, and non-alethic logics

Let T be a theory whose underlying logic is L. T is *inconsistent* when it contains theorems of the form A and $\neg A$ (the negation of A). If T is not inconsistent, it is called *consistent*. T is said to be *trivial* if all formulas of T are also theorems of T. Otherwise, T is called *non-trivial*. So, in trivial theories, the extensions of the concepts of formula and theorem coincide. When L is the classical logic (or several other ones, such as intuitionistic logic), a theory is trivial iff it is inconsistent. A *paraconsistent* logic is a logic which can be used as the basis for inconsistent but non-trivial theories. A theory is called *paraconsistent* if its underlying logic is a paraconsistent logic.

Similarly, we can be introduced the concept of *paracomplete* logic. A logic is called *paracomplete* if it can function as the underlying logic of theories in which there are formulas such that these formulas and their negations are both false. A theory is called *paracomplete* if its underlying logic is a paracomplete logic.

As a consequence, paraconsistent theories do not satisfy the principle of non-contradiction which can be stated as follows: from among two contradictory propositions (i.e., one is the negation of the other) one is false. Moreover, paracomplete theories do not satisfy the principle of excluded middle, formulated in the following form: from among two contradictory propositions, one is true.

Finally, logics which are simultaneously paraconsistent and paracomplete are called *non-alethic* logics.

3 Paraconsistent Deontic systems $D^*\tau$

Annotated logics are a kind of paraconsistent and, in general, paracomplete and non-alethic logics. Several interesting applications were found in Artificial Intelligence (e.g. reasoning about inconsistent knowledge bases, declarative semantics for inheritance networks, object-oriented data bases, paraconsistent frames, and others). Because of the importance of such systems in artificial intelligence a number of authors began to study such systems from a foundational point of view: [Da Costa, Subrahmanian, & Vago 91], [Da Costa, Abe, & Subrahmanian 91], [Abe 92, 94a], [Sylvan & Abe 96], among others.

In this section we introduce the annotated deontic systems $D^*\tau$. The symbol $\tau = < |\tau|, \leq >$ indicates some finite lattice called the *lattice of truth values*. We use the symbol $\leq$ to denote the ordering under which τ is a complete lattice, $\perp$ and $\top$ to denote, respectively, the bottom element and the top element of τ. Also, $\wedge$ and $\vee$ denote, respectively, the greatest lower bound and least upper bound operators with respect to subsets of $|\tau|$. We also fix an operator $\sim: |\tau| \rightarrow |\tau|$ which will work as the "meaning" of the negation of the deontic system $D^*\tau$.

The language of $D^*\tau$ has the following primitive symbols:

1. Individual variables: a denumerable infinite set of variable symbols: $x_1, x_2, \ldots$
2. Logical connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and $\rightarrow$ (implication).
3. For each n, zero or more n-ary function symbols (n is a natural number).
4. For each $n \neq 0$, n-ary predicate symbols.
5. The equality symbol: =
6. Annotational constants: each member of τ is called an annotational constant.
7. Deontic operator: O (obligatory).
8. Quantifiers: $\forall$ (for all) and $\exists$ (there exists).
9. Auxiliary symbols: parentheses and comma.

For each n the number of n-ary function symbols may be zero or non-zero, finite or infinite. A 0-ary function symbol is called a *constant*. Also, for each $n \geq 1$, the number of n-ary predicate symbols may be finite or infinite. We suppose that $D^*\tau$ possesses at least one predicate symbol.

We define the notion of *term* as usual. Given a predicate symbol p of arity n and n terms $t_1, \ldots, t_n$, a *basic formula* is an expression of the form $p(t_1, \ldots, t_n)$. An *annotated atomic formula* is an expression of the form $p_\lambda(t_1, \ldots, t_n)$, where λ is an annotational constant. We introduce the general concept of (*annotated*) *formula* in the standard way. Among several intuitive readings, an atomic annotated formula $p_\lambda(t_1, \ldots, t_n)$ can be read: *it is believed that* $p(t_1, \ldots, t_n)$*'s truth value is at least* λ.

Definition 3.1 Let A and B be formulas. We put

1. $A \leftrightarrow B =_{\text{Def.}} (A \rightarrow B) \wedge (B \rightarrow A)$
2. $\urcorner A =_{\text{Def.}} A \rightarrow ((A \rightarrow A) \wedge \neg(A \rightarrow A))$
3. $PA =_{\text{Def.}} \urcorner O \urcorner A$

The symbol '$\leftrightarrow$' is called *biconditional*, '$\neg^{*}$' is called *strong negation*, and 'P' is called the operator *permitted*'.

Let A be a formula. Then: $\neg^0 A$ indicates A, $\neg^1 A$ indicates $\neg A$, and $\neg^k A$ indicates $\neg(\neg^{k-1} A)$, ($k \in$ N, $k > 0$, N is the set of natural numbers).

Also, if $\mu \in \tau$, $\sim^0 \mu$ indicates μ, $\sim^1 \mu$ indicates $\sim\mu$, and $\sim^k \mu$ indicates $\sim(\sim^{k-1}\mu)$, ($k \in$ N, $k >$ 0).

If A is an atomic formula $p_\lambda(t_1, \dots, t_n)$, then a formula of the form $\neg^k p_\lambda(t_1, \dots, t_n)$ ($k \geq 0$) is called a *hyper-literal*. A formula other than hyper-literals is called a *complex* formula.

The postulates (axiom schemata and primitive rules of inference) of $D^*\tau$ are the following: A, B, and C are any formulas whatsoever, F and G are complex formulas, $p(t_1, \dots, t_n)$ is a basic formula, and λ, μ, μ_j are annotational constants.

1) $A \to (B \to A)$
2) $(A \to (B \to C) \to ((A \to B) \to (A \to C))$
3) $((A \to B) \to A) \to A$
4) $\dfrac{A,\ A \to B}{B}$
5) $A \wedge B \to A$
6) $A \wedge B \to B$
7) $A \to (B \to (A \wedge B))$
8) $A \to A \vee B$
9) $B \to A \vee B$
10) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
11) $(F \to G) \to ((F \to \neg G) \to \neg F)$
12) $F \to (\neg F \to A)$
13) $F \vee \neg F$
14) $p_\perp(t_1, \dots, t_n)$.
15) $\neg^k p_\lambda(t_1, \dots, t_n) \to \neg^{k-1} p_{\sim\lambda}(t_1, \dots, t_n)$, $k \geq 1$
16) $p_\lambda(t_1, \dots, t_n) \to p_\mu(t_1, \dots, t_n)$, $\lambda \geq \mu$
17) $p_{\lambda 1}(t_1, \dots, t_n) \wedge p_{\lambda 2}(t_1, \dots, t_n) \wedge \dots \wedge p_{\lambda m}(t_1, \dots, t_n) \to p_\lambda(t_1, \dots, t_n)$, where $\lambda = \bigvee_{i=1}^{m} \lambda_i$
18) $O(A \to B) \to (OA \to OB)$
19) $PA \to OPA$
20) $OA \to PA$
21) $\dfrac{A}{OA}$
22) $A(t) \to \exists x A(x)$
23) $\dfrac{A(x) \to B}{\exists x A(x) \to B}$
24) $\forall x A(x) \to A(t)$
25) $\forall x OA \to O\forall x A$
26) $\dfrac{B \to A(x)}{B \to \forall x A(x)}$
27) $x = x$
28) $x = y \to A[x] \leftrightarrow A[y]$

29) $\urcorner\!\!\urcorner(x = y) \rightarrow O_{\urcorner\!\urcorner}(x = y)$
with the usual restrictions.

Theorem 3.2 In $D^*\tau$, $\urcorner\!\!\urcorner$ has all the properties of the classical negation. For instance, we have:

1. $\vdash A \vee \urcorner\!\!\urcorner A$
2. $\vdash \urcorner\!\!\urcorner(A \wedge \urcorner\!\!\urcorner A)$
3. $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \urcorner\!\!\urcorner B) \rightarrow \urcorner\!\!\urcorner A)$
4. $\vdash A \rightarrow \urcorner\!\!\urcorner\urcorner\!\!\urcorner A$
5. $\vdash \urcorner\!\!\urcorner A \rightarrow (A \rightarrow B)$
6. $\vdash (A \rightarrow \urcorner\!\!\urcorner A) \rightarrow B$

Corollary 3.2.1 In $D^*\tau$, the connectives $\urcorner\!\!\urcorner$, $\wedge$, $\vee$, and $\rightarrow$ together with the quantifiers $\forall$ and $\exists$ have all properties of the classical negation, conjunction, disjunction, conditional and universal and existential quantifiers, respectively. If A, B, and C are any formulas whatsoever, we have, for instance,

1. $(A \wedge B) \leftrightarrow \urcorner\!\!\urcorner(\urcorner\!\!\urcorner A \vee \urcorner\!\!\urcorner B)$
2. $\urcorner\!\!\urcorner \forall A \leftrightarrow \exists x \urcorner\!\!\urcorner A$
3. $\exists xB \vee C \leftrightarrow \exists x(B \vee C)$
4. $B \vee \exists xC \leftrightarrow \exists x(B \vee C)$

Corollary 3.2.2 The "corresponding" classical deontic predicate calculus is contained in $D^*\tau$ though it constitutes a strict subcalculus of the later.

Theorem 3.3 If A is a complex formula, then
$\vdash \neg A \leftrightarrow \urcorner\!\!\urcorner A$

Theorem 3.4: $D^*\tau$ is non-trivial.

4 Semantical analysis: Kripke structures

Definition 4.1: A Kripke model for $D^*\tau$ is a set theoretical structure
$K = [W, R, I]$ where
W is a nonempty set of elements called 'worlds'
R is a binary relation on W such that
1. For each $w \in W$, there exists $w' \in W$ such that $w\,R\,w'$
2. For $w, w', w'' \in W$, and if $w\,R\,w'$ and $w\,R\,w''$, then $w'\,R\,w''$.

I is an interpretation function with the usual properties with the exception that for each n-ary predicate symbol p we associate a function $p_I\colon W^n \rightarrow |\tau|$.

Given a Kripke model K for the language L of $D^*\tau$, the *diagram* language $L(K)$ is obtained as usual. Given a free variable term a of $L(K)$ we define, as usual, the individual $K(a)$ of K. We use i and j as meta-variables for names.

Definition 4.2 If A is a closed formula of $D^*\tau$, and $w \in W$, we define the relation $K,w \Vdash A$ (K,w *force* A) by recursion on A:

1) If A is atomic of the form $p_\lambda(t_1, \ldots, t_n)$, then $K,w \Vdash A$ iff $p_I(K(t_1), \ldots, K(t_n)) \geq \mu$.
2) If A is of the form $\neg^k p_\lambda(t_1, \ldots, t_n)$ $(k \geq 1)$, $K,w \Vdash A$ iff $K,w \Vdash \neg^{k-1} p_{\sim\lambda}(t_1, \ldots, t_n)$.
3) Let A and B formulas. Then, $K,w \Vdash (A \wedge B)$ iff $K,w \Vdash A$ and $K,w \Vdash B$; $K,w \Vdash (A \vee B)$ iff $K,w \Vdash A$ or $K,w \Vdash B$; $K,w \Vdash (A \to B)$ iff it is not the case that $K,w \Vdash A$ or $K,w \Vdash B$;
4) If A is a complex formula, then $K,w \Vdash (\neg A)$ iff it is not the case that $K,w \Vdash A$.
5) If A is of the form $(\exists x)B$, then $K,w \Vdash A$ iff $K,w \Vdash B_x[i]$ for some i in $L(K)$.
6) If A is of the form $(\forall x)B$, then $K,w \Vdash A$ iff $K,w \Vdash B_x[i]$ for all i in $L(K)$.
7) If A is of the form OB then $K,w \Vdash A$ iff $K,w' \Vdash B$ for each $w' \in W$ such that $w R w'$.

Theorem 4.3 Let $K = [W, R, I]$ be a Kripke structure for $D^*\tau$ (or $D^*\tau$-structure) and F a complex formula. Then we have not simultaneously $K,w \Vdash \neg F$ and $K,w \Vdash F$.

Proof. It follows from the condition 4 of the preceding definition. o

Definition 4.4 Let $K = [W, R, I]$ be a $D^*\tau$-structure. The Kripke structure K *forces* a formula A (in symbols, $K \Vdash A$), if $K,w \Vdash A$ for each $w \in W$. A formula A is called $D^*\tau$-valid if for any $D^*\tau$-structure K, $K \Vdash A$. We symbolize this fact by $\Vdash A$.

Theorem 4.5 Let $K = [W, R, I]$ be a $D^*\tau$-structure. For all formulas A, B of $D^*\tau$ we have

1. If A is an instance of a propositional tautology then, $K \Vdash A$
2. If $K \Vdash A$ and $K \Vdash A \to B$, then $K \Vdash B$
3. $K \Vdash O(A \to B) \to (OA \to OB)$
4. $K \Vdash PA \to OPA$
5. $K \Vdash OA \to PA$
6. If $K \Vdash A$ then $K \Vdash OA$

Theorem 4.6 Let $p(t_1, \ldots, t_n)$ be a basic formula and $\lambda, \mu, \rho \in |\tau|$. We have

1. $\Vdash p_\perp(t_1, \ldots, t_n)$
2. $\Vdash p_\lambda(t_1, \ldots, t_n) \to p_\mu(t_1, \ldots, t_n)$, if $\lambda \geq \mu$
3. $\Vdash p_\lambda(t_1, \ldots, t_n) \wedge p_\mu(t_1, \ldots, t_n) \to p_\rho(t_1, \ldots, t_n)$, where $\rho = \lambda \vee \mu$

Proof 1. For any Kripke structure K, we have $p_I(K(t_1), \ldots, K(t_n)) \geq \perp$, for all $w \in K$. So, $K \models p_\perp(t_1, \ldots, t_n)$ for every K, and therefore $\Vdash p_\perp(t_1, \ldots, t_n)$.
2. Let us suppose that there exists a K such that it is not the case that $K \Vdash p_\lambda(t_1, \ldots, t_n) \to p_\mu(t_1, \ldots, t_n)$, that is $K \Vdash p_\lambda(t_1, \ldots, t_n)$ and it is not the case that $K \Vdash p_\mu(t_1, \ldots, t_n)$, for some $w \in K$. So, $p_I(K(t_1), \ldots, K(t_n)) \geq \lambda$ and not $p_I(K(t_1), \ldots, K(t_n)) \geq \mu$, which contradicts the hypothesis. Therefore, we have $\Vdash p_\lambda(t_1, \ldots, t_n) \to p_\mu(t_1, \ldots, t_n)$, if $\lambda \geq \mu$.
3. Similar to the preceding, using conditions 1 and 2 of Definition 4.2.

Theorem 4.7 Let A and B be arbitrary formulas and F a complex formula. Then:

1. $\Vdash ((A \to B) \to ((A \to \urcorner B) \to \urcorner A))$
2. $\Vdash (A \to (\urcorner A \to B))$
3. $\Vdash (A \vee \urcorner A)$

4. $\Vdash (\neg F \leftrightarrow \urcorner F)$
5. $\Vdash A \leftrightarrow \urcorner\urcorner A$
6. $\Vdash \forall x A \leftrightarrow \exists x \urcorner A$
7. $\Vdash (A \wedge B) \leftrightarrow \urcorner(\urcorner A \vee \urcorner B)$
8. $\Vdash \forall A \leftrightarrow \exists x \urcorner A$
9. $\Vdash \forall x B \vee C \leftrightarrow \exists x(B \vee C)$
10. $\Vdash B \vee \exists x C \leftrightarrow \exists x(B \vee C)$

Corollary 4.7.1 In the same conditions of the preceding theorem, we have not simultaneously $K \Vdash \urcorner A$ and $K \Vdash A$.

Theorem 4.8 There are Kripke structures $K = [W, R, I]$ such that for some hyper-literals A and B and some worlds w and $w' \in W$, we have $K,w \Vdash \neg A$ and $K,w \Vdash A$ and it is not the case that $K,w' \Vdash B$.

Proof Let $W = \{\{a\}\}$ and $R = \{(\{a\},\{a\})\}$ (that is $w = \{a\}$) and $p(t_1, \ldots, t_n)$ and $q(t'_1, \ldots, t'_n)$ basic (closed) formulas such that $p_I \equiv \top$ and $q_I \equiv \perp$. As $\top \geq \top$, it follows that $p_\top(t_1, \ldots, t_n) \geq \top$. Also, $\top \geq \sim\top$. So, $p_I \geq \sim\top$. Therefore, $K,w \Vdash p_\top(t_1, \ldots, t_n)$ and $K,w \Vdash p_{\sim\top}(t_1, \ldots, t_n)$. By condition 2 of Definition 4.2, it follows that $K,w \Vdash \neg p_\top(t_1, \ldots, t_n)$. On the other hand, as it is false that $\perp \geq \top$, it follows that it is not the case that $q_I \geq \top$, and so, it is not the case that $K,w \Vdash q_\perp(t'_1, \ldots, t'_n)$. o

Theorem 4.9 For some systems $D^*\tau$ there are Kripke structures $K = [W, R, I]$ such that for some hyper-literal formula A and some world $w \in W$, we don't have $K,w \Vdash A$ nor $K,w \Vdash \neg A$.

Proof Let us define the operator $\sim: |\tau| \rightarrow |\tau|$ by setting $\sim\top = \top$. Then, let I be the interpretation such that $p_I \equiv \perp$. So, it is no the case that $p_I \geq \top$ and also, it is not the case that $p_I \geq \sim\top$ (or, equivalently, not $K,w \Vdash p_\top(t_1, \ldots, t_n)$ and not $K,w \Vdash \neg p_\top(t_1, \ldots, t_n)$). o

Corollary 4.9.1 For some systems $D^*\tau$ there are Kripke structures $K = [W, R, I]$ such that for some hyper-literal formulas A and B, and some worlds $w, w' \in W$, we have $K,w \Vdash \neg A$ and $K,w \Vdash A$ and we don't have $K,w \Vdash B$ nor $K,w \Vdash \neg B$.

Proof Consequence of the theorems 4.8 and 4.9. o

The earlier results show us that there are systems $D^*\tau$ such that we have "inconsistent" worlds, "paracomplete" worlds, or both.

Now we present a strong version these results linking with paraconsistent, paracomplete, and non-alethic logics.

Definition 4.10: A Kripke structure $K = [W, R, I]$ is called *paraconsistent* if there are basic formulas $p(t_1, \ldots, t_n)$, $q(t_1, \ldots, t_n)$, and annotational constants $\lambda, \mu \in |\tau|$ such that $K,w \Vdash p_\lambda(t_1, \ldots, t_n)$, $K,w \Vdash \neg p_\lambda(t_1, \ldots, t_n)$, and it is not the case that $K,w \Vdash q_\mu(t_1, \ldots, t_n)$.

Definition 4.11 A deontic system $D^*\tau$ is called *paraconsistent* if there is a Kripke structure $K = [W, R, I]$ for $D^*\tau$ such that K is paraconsistent.

Theorem 4.12 $D^*\tau$ is a paraconsistent system iff $\#|\tau| \geq 2$.

Proof Define a structure $K = [\{w\}, \{(w, w)\}, I]$ such that $\begin{cases} q_I = \bot \\ p_I = \mathrm{T} \end{cases}$

It is clear that $p_I \geq \top$, and so $K \Vdash p_\top(t_1, \ldots, t_n)$. Also, $p_I \geq \sim\top$, and, so $K \Vdash p_{\sim\top}(t_1, \ldots, t_n)$, or $K \Vdash \neg p_\top(t_1, \ldots, t_n)$. Also, it is not the case that $q_I(t_1, \ldots, t_n) \geq \bot$, so it is not the case that $K,w \Vdash q_\bot(t_1, \ldots, t_n)$. o

Definition 4.13 A Kripke structure $K = [W, R, I]$ is called *paracomplete* if there is a basic formula $p(t_1, \ldots, t_n)$, annotational constant $\lambda \in |\tau|$ such that it is false that $K,w \Vdash p_\lambda(t_1, \ldots, t_n)$ and it is false that $K,w \Vdash \neg p_\lambda(t_1, \ldots, t_n)$. A deontic system $D^*\tau$ is called *paracomplete* if there is a Kripke structures $K = [W, R, I]$ for $D^*\tau$ such that K is paracomplete.

Definition 4.14 A Kripke structure $K = [W, R, I]$ is called *non-alethic* if K are both paraconsistent and paracomplete. A deontic system $D^*\tau$ is called *non-alethic* if there is a Kripke structure $K = [W, R, I]$ for $J\tau$ such that K is non-alethic.

Theorem 4.15 If $\# |\tau| \geq 2$, then there are deontic systems $D^*\tau$ which are paracomplete and systems $D^*\tau$' that are not paracomplete, $\# |\tau'| \geq 2$.
Proof Similar to the preceding theorem. o

Corollary 4.15.1 If $\# |\tau| \geq 2$, then there are systems $D^*\tau$ which are non-alethic and systems $D^*\tau$' that are not non-alethic, $\# |\tau'| \geq 2$.

5 Soundness and Completeness

Theorem 5.1 Let U be a maximal non-trivial maximal (with respect to inclusion of sets) subset of the set of formulas F. Let A and B formulas whatsoever. Then
1. If A is an axiom of $D^*\tau$, then $A \in U$
2. $A \wedge B \in U$ iff $A \in U$ and $B \in U$.
3. $A \vee B \in U$ iff $A \in U$ or $B \in U$.
4. $A \to B \in U$ iff $A \notin U$ or $B \in U$.
5. If $p_{\lambda_1}(t_1, \ldots, t_n)$ and $p_{\lambda_2}(t_1, \ldots, t_n) \in U$, then $p_\lambda(t_1, \ldots, t_n) \in U$, where $\lambda = \lambda_1 \vee \lambda_2$
6. $\neg^k p_\lambda(t_1, \ldots, t_n) \in U$ iff $\neg^{k-1} p_{\sim\lambda}(t_1, \ldots, t_n) \in U$.
7. If A and $A \to B \in U$, then $B \in U$.
8. $A \in U$ iff $\urcorner A \notin U$. Moreover $A \in U$ or $\urcorner A \in U$.
9. If A is a complex formula, $A \in U$ iff $\neg A \notin U$. Moreover $A \in U$ or $\neg A \in U$.
10. If $A \in U$, then $OA \in U$.

Proof. Let us show only 5. In fact, if $p_{\lambda_1}(t_1, \ldots, t_n)$ and $p_{\lambda_2}(t_1, \ldots, t_n) \in U$, then $p_{\lambda_1}(t_1, \ldots, t_n) \wedge p_{\lambda_2}(t_1, \ldots, t_n)$ by 2. But it is an axiom $p_{\lambda_1}(t_1, \ldots, t_n) \wedge p_{\lambda_2}(t_1, \ldots, t_n) \to p_\lambda$, where $\lambda = \lambda_1 \vee \lambda_2$. It follows that $p_{\lambda_1}(t_1, \ldots, t_n) \wedge p_{\lambda_2}(t_1, \ldots, t_n) \to p_\lambda(t_1, \ldots, t_n) \in U$, and so $p_\lambda(t_1, \ldots, t_n) \in U$, by 7. o

We give a Henkin-type proof of the completeness theorem for the logics $D^*\tau$.

For this we define a relation R on the set of all free-variable terms of $D^*\tau$ as usual and we indicate by $\overset{\circ}{t}$ the equivalence class determined by t. Also, we will consider the quotient set F/R, where F indicate the set of all formulas.

Given a set U of formulas, define $U/O = \{A \mid OA \in U\}$. Let us consider the canonical structure $K = [W, R, I]$ where $W = \{U \mid U$ is a maximal non-trivial set$\}$ and the interpretation function is as usual with the exception that given a n-ary predicate symbol p we associate the function $p_I : W^n \rightarrow |\tau|$ defined by $p_I(\overset{\circ}{t}_1, \ldots, \overset{\circ}{t}_n) =_{def.} \vee\{\mu \in |\tau| \mid p_\mu(t_1, \ldots, t_n) \in U\}$ (such function is well defined, so $p_\perp(t_1, \ldots, t_n) \in U$).

Moreover, define

$R =_{Def.} \{(U, U') \mid U/O \subseteq U'\}$

Lemma 5.2 For all propositional variable p and if U is a maximal non-trivial set of formulas, we have $p_{pI(t^\circ 1, \ldots, t^\circ n)}(t_1, \ldots, t_n) \in U$.

Proof It is a simple consequence of the previous theorem, item 5. o

Theorem 5.3 For any formula A and for any nontrivial maximal set U, we have $(K, U) \Vdash A$ iff $A \in U$.

Proof Let us suppose that A is $p_\lambda(t_1, \ldots, t_n)$ and $(K, U) \Vdash p_\lambda(t_1, \ldots, t_n)$. It is clear by previous lemma that $p_{pI(t^\circ 1, \ldots, t^\circ n)}(t_1, \ldots, t_n) \in U$. It follows also that $p_I(\overset{\circ}{t}_1, \ldots, \overset{\circ}{t}_n) \geq \lambda$. It is an axiom that $p_{pI(t^\circ 1, \ldots, t^\circ n)}(t_1, \ldots, t_n) \rightarrow p_\lambda(t_1, \ldots, t_n)$. Thus, $p_\lambda(t_1, \ldots, t_n) \in U$. Now, let us suppose that $p_\lambda(t_1, \ldots, t_n) \in U$. By previous lemma, $p_{pI(t^\circ 1, \ldots, t^\circ n)}(t_1, \ldots, t_n) \in U$. It follows that $p_I(\overset{\circ}{t}_1, \ldots, \overset{\circ}{t}_n) \geq \lambda$. Thus, by definition, $(K, U) \Vdash p_\lambda(t_1, \ldots, t_n)$. By theorem 5.1, $\neg^k p_\lambda(t_1, \ldots, t_n) \in U$ iff $\neg^{k-1} p_{\sim\lambda}(t_1, \ldots, t_n) \in U$. Thus, by definition 4.2, $(K, U) \Vdash \neg^k p_\lambda(t_1, \ldots, t_n)$ iff $(K, U) \Vdash \neg^{k-1} p_{\sim\lambda}(t_1, \ldots, t_n)$. So, by induction on k the assertion is true for hyper-literals.

The other cases, the proof is as in the classical case. o

Corollary 5.3.1 A is a provable formula of $D^*\tau$ iff $\Vdash A$.

6 Concluding remarks

Similarly, we can construct several other paraconsistent deontic systems analogous to the classical ones, even those with relative deontic operators.

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