The Harmonic Oscillator via the Discrete Path Approach

Adel F. Antippa
Département de Physique
Université du Québec à Trois-Rivières
Trois-Rivières, Québec, Canada, G9A 5H7
email: antippa@mailaps.org

Daniel M. Dubois
Centre for Hyperincursion and Anticipation in Ordered Systems,
CHAOS asbl, Institut de Mathématique, B37, Université de Liège
Grande Traverse 12, B-4000 Liège 1, Belgique
email: Daniel.Dubois@ulg.ac.be

Abstract The discrete path approach has recently been use to obtain a closed form solution for two simultaneous difference equations with variable coefficients. We apply this result to the solution of the discretized harmonic oscillator and recover the well known traditional solutions. In the process we learn how the enumerative discrete path solution transforms into a more convenient compact analytic closed form. The discrete path approach is specially adapted to problems with mixed boundary conditions like those arising in the modeling of anticipatory systems.
Keywords : discrete path, closed form solution, difference equations, harmonic oscillator, anticipatory systems.

1 Introduction

This is the first of a series of papers dealing with analytic solutions of difference equations arising in the modeling of anticipatory systems [4]. In this first paper, the simple harmonic oscillator is used as a mathematical model for gaining familiarity with the discrete path approach, as it applies to coupled difference equations, and in building up a compendium of related characteristics.

Every potential function, can in the neighborhood of its local minima be approximated by a parabolic potential. Furthermore, in Newton's second law of motion as in Schrödinger's non-relativistic wave equation, the harmonic potential has an exact, analytic and simple solution, which serves as starting point for perturbations that realistically model the physical system.

The usefulness of the harmonic oscillator as a physical model [8], combined with the simplicity of its mathematical solutions, has made it one of the favored proving grounds for mathematical methods, and one of the most frequently used models in physics, from mechanical springs, to Planck's photons, to the interactions of atoms in solids, to quantum field theory, to string theory.

The one-dimensional classical harmonic oscillator problem can be formulated in terms of difference equations by discretizing time. This leads to a three term recursion relation with constant coefficients, or equivalently to two coupled first order difference equations. The three term recursion relation has a solution in terms of powers of the roots of the corresponding characteristic equation. The coupled difference equations can be solved by the discrete path formalism [2].

The discrete path approach to the solution of difference equations is specially suited to anticipatory problems because it allows an arbitrary specification of the boundary conditions. Specifically it allows for the specification of mixed boundary conditions (partly initial and partly final) which are an inherent part of anticipatory problems. Furthermore, this approach can provide analytic solutions for anticipatory problems even when their complexity leads to difference equations with variable coefficients.

In this paper we apply the recently obtained (via the discrete path approach) closed form solution for two simultaneous difference equations with variable coefficients [1], to the problem of the discretized harmonic oscillator and recover the well known traditional solution. In the process we learn how the rather complex, and general, enumerative discrete path solution, compacts, in the case of a harmonic oscillator into a power solution (an exponential in the differential limit). The different mathematical results obtained along the way are useful as guides (as well as limiting test cases), in deriving analytic solutions for more complex anticipatory harmonic oscillator systems. Since every detail of the discrete path solution, as applied to the simple harmonic oscillator, will, in the case of more complex problems, inflate into an elaborate mathematical development proportional to the complexity of the problem to be solved, the usefulness of the present work resides, to a large extent, in the careful development of every detail in the derivation.

One important class of complex harmonic oscillators is that modeling anticipatory systems. As pointed out by one of us (D. Dubois) [5], anticipatory formulations of the discrete harmonic oscillator are crucial in solving the problem of energy non-conservation which is characteristic of the discrete harmonic oscillator. If this turns out to be the only way to solve the energy non-conservation problem, the result will have tremendous impact on the fundamental role of anticipation in the formulation of the laws of physics.

2 The Harmonic Oscillator

2.1 Continuum formulation

Newton's second law of motion for an invariant mass in a parabolic potential $U(x) = \frac{1}{2}kx^2$ (the one-dimensional non-relativistic harmonic oscillator) leads to the second order differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{k}{m}} \tag{1}$$

who's general solution is

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} \qquad v(t) = \frac{dx(t)}{dt} = i\omega\left(Ae^{i\omega t} - Be^{-i\omega t}\right) \tag{2}$$

or in terms of the initial conditions $x_0 = x(0) = A + B$ and $v_0 = v(0) = i\omega(A - B)$

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t \qquad v(t) = v_0\cos\omega t - x_0\,\omega\sin\omega t \tag{3}$$

or, alternatively

$$x(t) = C\sin(\omega t + \phi) \qquad v(t) = C\omega\cos(\omega t + \phi) \tag{4}$$

where $C = \sqrt{x_0^2 + (v_0/\omega)^2}$ and $\phi = \tan^{-1}(\omega x_0/v_0)$

2.2 Discrete Formulation

The finite difference equivalent of equation (1) is obtained via the transformations

$$d \to \Delta = (E - I) \qquad \text{and} \qquad dt \to \Delta t \tag{5}$$

where I is the identity operator and E (explicitly $\underset{\Delta t}{E}$) is Boole's displacement operator [7] defined by $Ef(x) = f(x + \Delta t)$. If we define $E^0 = I$, then Boole's displacement satisfies $E^n f(x) = f(x + n\Delta t)$ for all integer n, positive, zero or negative. For n nonnegative

$$d^n \to \Delta^n = (E - I)^n = \sum_{k=0}^{n}\binom{n}{k}E^n \tag{6}$$

and equation (1) transforms to:

$$\left(E^2 - 2E + I\right)x(t) + (\omega\Delta t)^2\,x(t) = 0 \tag{7}$$

Applying the displacement operators and making the change of variable $t \to t - 2\Delta t$, the above equation reduces to

$$x(t) - 2x(t - \Delta t) + \left[1 + (\omega\Delta t)^2\right]x(t - 2\Delta t) = 0 \tag{8}$$

The time variable is discretized according to

$$t \to t_n = t_0 + n\Delta t \qquad \text{where} \qquad n = \left\lfloor\frac{t - t_0}{\Delta t}\right\rfloor \tag{9}$$

and we use the notation $x_n = x(t_n)$ and $v_n = v(t_n)$, so the above difference equation (8) becomes

$$x_n - 2x_{n-1} + \left[1 + (\omega\Delta t)^2\right] x_{n-2} = 0 \tag{10}$$

Since the above difference equation has constant coefficients, it can be solved by the method of roots [6]. Setting $x_n = z^n$ leads to the characteristic equation

$$z^2 - 2z + \left[1 + (\omega\Delta t)^2\right] = 0 \tag{11}$$

with roots,

$$z_\pm = 1 \pm (i\omega\Delta t) \tag{12}$$

So the solution of equation (10) is given by

$$x_n = \bar{A}\left[1 + (i\omega\Delta t)\right]^n + \bar{B}\left[1 - (i\omega\Delta t)\right]^n \tag{13}$$

The discretized velocity is given by

$$v_n = \frac{(x_{n+1} - x_n)}{\Delta t} \tag{14}$$

leading to

$$\frac{v_n}{i\omega} = \bar{A}\left[1 + (i\omega\Delta t)\right]^n - \bar{B}\left[1 - (i\omega\Delta t)\right]^n \tag{15}$$

The coeffiients $\bar{A}$ and $\bar{B}$ are obtained from the initial conditions $x_0 = \bar{A} + \bar{B}$ and $v_0 = i\omega\left(\bar{A} - \bar{B}\right)$ as

$$\bar{A} = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right) \qquad \bar{B} = \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right) = \bar{A}^* \tag{16}$$

hence

$$x_n = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right)\left[1 + (i\omega\Delta t)\right]^n + \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right)\left[1 - (i\omega\Delta t)\right]^n \tag{17}$$

$$\frac{v_n}{i\omega} = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right)\left[1 + (i\omega\Delta t)\right]^n - \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right)\left[1 + (i\omega\Delta t)\right]^n \tag{18}$$

2.3 Interrelation

We define $S_n(t_n, t_0)$ by

$$S_n(t_n, t_0) = [1 + (i\omega\Delta t)]^n \qquad S_n^*(t_n, t_0) = [1 - (i\omega\Delta t)]^n \tag{19}$$

then

$$x_n = \bar{A}S_n(t_n, t_0) + \bar{A}^* S_n^*(t_n, t_0) \tag{20}$$

$$\frac{v_n}{i\omega} = \bar{A}S_n(t_n, t_0) - \bar{A}^* S_n^*(t_n, t_0) \tag{21}$$

Since $\Delta t = (t_n - t_0)/n$, then in the limit $n \to \infty$, $\Delta t \to dt$, and t_n can be brought as near as desired to any specific value of t . Hence, in the continous limit, $S_n(t_n, t_0) \to S(t, t_0)$, where $S(t, t_0) = \lim_{n\to\infty} S_n(t_n, t_0)\,|_{t_n=t}$, hence

$$S(t, t_0) = \lim_{n\to\infty}\left[1 + i\omega\left(\frac{t - t_0}{n}\right)\right]^n = e^{i\omega(t-t_0)} \tag{22}$$

leading to

$$x(t_n) \to x(t) = \lim_{n\to\infty} x(t_n)\,|_{t_n=t} = \bar{A}e^{i\omega(t-t_0)} + \bar{B}e^{-i\omega(t-t_0)} \tag{23}$$

$$v(t_n) \to v(t) = \lim_{n\to\infty} v(t_n)\,|_{t_n=t} = i\omega\left[\bar{A}e^{i\omega(t-t_0)} + \bar{B}e^{-i\omega(t-t_0)}\right] \tag{24}$$

and we recover the continuous case solution with $\bar{A} = Ae^{i\omega t_0}$ and $\bar{B} = Be^{-i\omega t_0}$.

2.4 Coupled Formulation

We can alternatively formulate the problem of the harmonic oscillator as two coupled first order differential equations,

$$\frac{dx(t)}{dt} - v(t) = 0 \quad \text{and} \quad \frac{dv(t)}{dt} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{k}{m}} \tag{25}$$

The corresponding discrete equations are, as before, obtained by the transformations $d \to \Delta = (E - I)$ and $dt \to \Delta t$, leading to the two coupled difference equations

$$x(t + \Delta t) = x(t) + \Delta t\; v(t) \qquad v(t + \Delta t) = v(t) - \Delta t\; \omega^2 x(t) \tag{26}$$

Making the change of variable $t \to t - \Delta t$, descritising time as before according to $t \to t_n = t_0 + n\Delta t$, and again setting $x_n = x(t_n)$ and $v_n = v(t_n)$ leads to

$$x_n = x_{n-1} + \Delta t v_{n-1} \qquad \text{and} \qquad v_n = v_{n-1} - \omega^2 \Delta t x_{n-1} \tag{27}$$

these coupled difference equations are the classical Euler discrete equations for the harmonic oscillator. They emerge when using Euler's method for the numerical integration of the second order differential equation (1). They will be solved by the discrete path formalism.

3 Simultaneous First Order Difference Equations

In this section we will present the general solution to a set of two coupled homogeneous linear difference equations with variable coefficients, and initially defined boundary conditions.

We write the two coupled linear homogeneous equations in the standard discrete path notation as:

$$R_{n,0} = f_{1,0}(n,0)\,R_{n-1,0} + f_{1,-1}(n,0)\,R_{n-1,1} \qquad n = 1,2,3,...,\infty \tag{28}$$

$$R_{n,1} = f_{1,0}(n,1)\,R_{n-1,1} + f_{1,1}(n,1)\,R_{n-1,0} \qquad n = 1,2,3,...,\infty \tag{29}$$

with the initial conditions given by $R_{0,0} = \lambda_{0,0}$ and $R_{0,1} = \lambda_{0,1}$.

The discrete path solution to the above coupled homogeneous equations is [1]

$$R_{n,k} = \sum_{k'=0}^{1} \sum_{q=0}^{q_{\max}(n,k,k')} \lambda_{0,k'}\Omega^{\alpha}_{n,k}(k',q\} \qquad n = 1,2,3,...,\infty \tag{30}$$

where

$$q_{\max}(n,k,k') = \left\lfloor \frac{(n-1+\delta_{kk'})}{2} \right\rfloor \tag{31}$$

$$\Omega_{n,k}(k',q) = \sum_{\substack{\ell_j\in\{0,1\} \\ \ell_1+...+\ell_n=2q+(1-\delta_{kk'})}} \prod_{j=1}^{n} F_{\ell_j}(j,k_j)\Big|_{\substack{k_j=h(k',m_j) \\ m_j=\ell_1+\ell_2+\cdots+\ell_j}} \tag{32}$$

$$h(k',m_j) = \frac{1}{2} + (-1)^{m_j}\left(k' - \frac{1}{2}\right) \tag{33}$$

and

$$F_\ell(n,k) = [f_{1,0}(n,k)]^{(1-\ell)}\,[f_{1,1}(n,1)]^{k\ell}\,[f_{1,-1}(n,0)]^{(1-k)\ell} \tag{34}$$

4 The Solution for the Harmonic Oscillator

In the case of the harmonic oscillator

$$R_{n,0} = x_n \qquad R_{n,1} = v_n \tag{35}$$

$$f_{1,0}(n,0) = 1 \quad f_{1,-1}(n,0) = \Delta t \quad f_{1,0}(n,1) = 1 \quad \frac{f_{1,1}(n,1)}{\omega^2} = -\Delta t \tag{36}$$

Hence

$$F_\ell(n,k) = \left(-\omega^2\Delta t\right)^{k\ell}(\Delta t)^{(1-k)\ell} = (i\omega)^{2k\ell}(\Delta t)^{\ell} \tag{37}$$

and

$$F_{\ell_j}(j,k_j) = (i\omega)^{2\left(k_j-\frac{1}{2}\right)\ell_j}(i\omega\Delta t)^{\ell_j} \tag{38}$$

4.1 Reduction of the Solution

Substituting the above expression (38) for $F_{\ell_j}(j, k_j)$ in equation (32), we obtain

$$\Omega_{n,k}(k',q) = \sum_{\substack{\ell_j \in \{0,1\} \\ \ell_1+\ldots+\ell_n = 2q+(1-\delta_{kk'})}} \prod_{j=1}^{n} (i\omega)^{2\left(k_j-\frac{1}{2}\right)\ell_j} (i\omega\Delta t)^{\ell_j} \Big|_{\substack{k_j=h(k',m_j) \\ m_j=\ell_1+\ell_2+\cdots+\ell_j}} \tag{39}$$

Du to equation (33)

$$\left(k_j - \frac{1}{2}\right)\Big|_{k_j=h(k',m_j)} = (-1)^{m_j}\left(k' - \frac{1}{2}\right) \tag{40}$$

Hence, imposing the auxiliary conditions in equation (39), we obtain

$$\Omega_{n,k}(k',q) = \sum_{\substack{\ell_j \in \{0,1\} \\ \ell_1+\ldots+\ell_n = 2q+(1-\delta_{kk'})}} \prod_{j=1}^{n} (i\omega)^{2(k'-1/2)\ell_j(-1)^{\ell_1+\ell_2+\cdots+\ell_j}} (i\omega\Delta t)^{\ell_j} \tag{41}$$

and executing the product, leads to

$$\Omega_{n,k}(k',q) = (i\omega\Delta t)^{2q+(1-\delta_{kk'})} \sum_{\substack{\ell_j \in \{0,1\} \\ \ell_1+\ldots+\ell_n = 2q+(1-\delta_{kk'})}} (i\omega)^{2(k'-1/2)\gamma_m^n(\ell_1,\ell_2,\cdots,\ell_n)} \tag{42}$$

where

$$\gamma_m^n(\ell_1,\ell_2,\cdots,\ell_n) = \sum_{j=1}^{n} \ell_j (-1)^{\ell_1+\ell_2+\cdots\ell_j}\Bigg|_{\substack{\ell_1+\ldots+\ell_n=m \\ \ell_j\in\{0,1\}}} \tag{43}$$

and we have made use of the fact that inside the summation, $\ell_1 + \ldots + \ell_n = 2q + (1 - \delta_{kk'})$.

Next we define the combinatorial structure functions $G_{\pm}^{n,m}(z)$ by

$$G_{\pm}^{n,m}(z) = \sum_{\substack{\ell_j \in \{0,1\} \\ \ell_1+\ldots+\ell_n = m}} (z)^{\pm\gamma_m^n(\ell_1,\ell_2,\cdots,\ell_n)} \tag{44}$$

so that equation (42) can be rewritten as

$$\Omega_{n,k}(k',q) = (i\omega\Delta t)^{2q+(1-\delta_{kk'})}\, G^{n,2q+(1-\delta_{kk'})}_{sign[2(k'-1/2)]}(i\omega) \tag{45}$$

Note that, to obtain (45) we made use of the fact that $k' \in \{0,1\}$, and hence $2(k'-1/2) = \pm 1$.

Subsituting for $\Omega_{n,k}(k',q)$ from (45) into solution (30) for $R_{n,k}$, and executing the sum over k' leads to

$$\begin{aligned} R_{n,k} = \lambda_{0,0} &\left\{ \sum_{q=0}^{q_{\max}(n,k,0)} (i\omega\Delta t)^{2q+(1-\delta_{k0})}\, G_{-}^{n,2q+(1-\delta_{k0})}(i\omega) \right\} \\ +\lambda_{0,1} &\left\{ \sum_{q=0}^{q_{\max}(n,k,1)} (i\omega\Delta t)^{2q+(1-\delta_{k1})}\, G_{+}^{n,2q+(1-\delta_{k1})}(i\omega) \right\} \end{aligned} \tag{46}$$

where $q_{\max}(n,k,0) = \lfloor (n-1+\delta_{k0})/2 \rfloor$ and $q_{\max}(n,k,1) = \lfloor (n-1+\delta_{k1})/2 \rfloor$. Since $k \in \{0,1\}$ we have the identities $(1-\delta_{k0}) = \delta_{1,k}$ and $(1-\delta_{k1}) = \delta_{0,k}$. Hence we finally obtain

$$\begin{aligned} R_{n,k} = \lambda_{0,0} \left\{ \sum_{q=0}^{\lfloor (n-\delta_{k1})/2 \rfloor} (i\omega\Delta t)^{2q+\delta_{k1}} G_{-}^{n,2q+\delta_{k1}}(i\omega) \right\} \\ +\lambda_{0,1} \left\{ \sum_{q=0}^{\lfloor (n-\delta_{k0})/2 \rfloor} (i\omega\Delta t)^{2q+\delta_{k0}} G_{+}^{n,2q+\delta_{k0}}(i\omega) \right\} \end{aligned} \tag{47}$$

4.2 The Structure Functions

We will now evaluate the combinatorial structure functions (The G functions) for the harmonic oscillator as defined by equation (44). Since the set of values $\{\ell_j\}$ is subject to the two constraints $\ell_j \in \{0,1\}$ and $\ell_1 + ... + \ell_n = m$, then there are exactly m ℓs which are equal to 1, and exactly $n-m$ ℓs which are equal to 0. Hence there are exactly m nonzero terms in $\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n)$. Let the k^{th} nonzero term be denoted by $\rho_m^n(k)$, so that

$$\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n) = \sum_{k=1}^{m} \rho_m^n(k) \tag{48}$$

It is easily seen that $\rho_m^n(1) = (-1)$ and $\rho_m^n(k) = (-1)\rho_m^n(k-1)$, hence $\rho_m^n(k) = (-1)^k$ and

$$\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n) = \sum_{k=1}^{m} (-1)^k = \left\{ \begin{array}{lll} 0 & for & m\ even \\ -1 & for & m\ odd \end{array} \right\} \tag{49}$$

Hence

$$\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n) = (\delta_{0, m \bmod 2} - 1) \tag{50}$$

Note that $\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n)$ is independent of the details of the set $\{\ell_j\}$, as long as $\{\ell_j\}$ obeys the two conditions $\ell_j \in \{0,1\}$ and $\ell_1 + ... + \ell_n = m$.

Substituting for $\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n)$ in the G-functions, we obtain

$$G_{\pm}^{n,m}(z) = \sum_{\substack{\ell_j \in \{0,1\} \\ \ell_1 + ... + \ell_n = m}} (z)^{\pm(\delta_{0, m \bmod 2} - 1)} \tag{51}$$

Since $(z)^{\pm(\delta_{0, m \bmod 2} - 1)}$ is a constant with respect to variable of summation, then the above expression can be rewritten as

$$G_{\pm}^{n,m}(z) = (z)^{\pm(\delta_{0, m \bmod 2} - 1)} \sum_{\substack{\ell_j \in \{0,1\} \\ \ell_1 + ... + \ell_n = m}} 1 \tag{52}$$

For every possible set of objects $\{\ell_j\}$ subject to the constraints $\ell_j \in \{0,1\}$ and $\ell_1 + ... + \ell_n = m$, there is a term of value unity in the sum. Hence the above sum is equal to the number of combinations of n objects taken m at a time [9], and is given by the binomial of m relative to n , leading to

$$G_{\pm}^{n,m}(z) = \binom{n}{m}(z)^{\mp\delta_{1,m \bmod 2}} \tag{53}$$

where we have also made use of the identity $(\delta_{0,m \bmod 2} - 1) = -\delta_{1,m \bmod 2}$. Hence

$$G_{-}^{n,2q+\delta_{k1}}(i\omega) = \binom{n}{2q+\delta_{k1}}(i\omega)^{+\delta_{1,(2q+\delta_{k1}) \bmod 2}} \tag{54}$$

and

$$G_{+}^{n,2q+\delta_{k0}}(i\omega) = \binom{n}{2q+\delta_{k0}}(i\omega)^{-\delta_{1,(2q+\delta_{k0}) \bmod 2}} \tag{55}$$

But $\delta_{1,(2q+\delta_{k1}) \bmod 2} = \delta_{1,\delta_{k1}} = \delta_{k1}$ and $\delta_{1,(2q+\delta_{k0}) \bmod 2} = \delta_{1,\delta_{k0}} = \delta_{k0}$. Hence

$$G_{-}^{n,2q+\delta_{k1}}(i\omega) = \binom{n}{2q+\delta_{k1}}(i\omega)^{+\delta_{k1}} \tag{56}$$

and

$$G_{+}^{n,2q+\delta_{k0}}(i\omega) = \binom{n}{2q+\delta_{k0}}(i\omega)^{-\delta_{k0}} \tag{57}$$

4.3 The Discrete Solution

Substituting for the G-functions their values from equations (56) and (57) into expression (47) for $R_{n,k}$ we obtain

$$\begin{aligned} R_{n,k} = \lambda_{0,0}(i\omega)^{\delta_{k1}} &\left\{\sum_{q=0}^{\lfloor (n-\delta_{k1})/2 \rfloor}(i\omega\Delta t)^{2q+\delta_{k1}}\binom{n}{2q+\delta_{k1}}\right\} \\ +\lambda_{0,1}(i\omega)^{-\delta_{k0}} &\left\{\sum_{q=0}^{\lfloor (n-\delta_{k0})/2 \rfloor}(i\omega\Delta t)^{2q+\delta_{k0}}\binom{n}{2q+\delta_{k0}}\right\} \end{aligned} \tag{58}$$

So $x_n(t)$ and $v_n(t)$, are given, for $n = 1, 2, 3, ..., \infty$ by

$$\begin{aligned} x_n = R_{n,0} = \lambda_{0,0} &\left\{\sum_{q=0}^{\lfloor n/2 \rfloor}(i\omega\Delta t)^{2q}\binom{n}{2q}\right\} \\ &+\lambda_{0,1}(i\omega)^{-1}\left\{\sum_{q=0}^{\lfloor (n-1)/2 \rfloor}(i\omega\Delta t)^{2q+1}\binom{n}{2q+1}\right\} \end{aligned} \tag{59}$$

$$\begin{aligned} v_n = R_{n,1} = \lambda_{0,0}(i\omega) &\left\{\sum_{q=0}^{\lfloor (n-1)/2 \rfloor}(i\omega\Delta t)^{2q+1}\binom{n}{2q+1}\right\} \\ &+\lambda_{0,1}\left\{\sum_{q=0}^{\lfloor n/2 \rfloor}(i\omega\Delta t)^{2q}\binom{n}{2q}\right\} \end{aligned} \tag{60}$$

which can alternatively be written as

$$x_n = \lambda_{0,0} \sum_{\substack{q=0 \\ q\ even}}^{n} \binom{n}{q} (i\omega\Delta t)^q + \lambda_{0,1} (i\omega)^{-1} \sum_{\substack{q=0 \\ q\ odd}}^{n} \binom{n}{q} (i\omega\Delta t)^q \tag{61}$$

$$v_n = \lambda_{0,0} (i\omega) \sum_{\substack{q=0 \\ q\ odd}}^{n} \binom{n}{q} (i\omega\Delta t)^q + \lambda_{0,1} \sum_{\substack{q=0 \\ q\ even}}^{n} \binom{n}{q} (i\omega\Delta t)^q \tag{62}$$

4.4 The Propagator Functions

Define the functions $S_{even}(n)$, $S_{odd}(n)$ and $S_{\pm}(n)$ by

$$S_{even}(n) = \sum_{\substack{q=0 \\ q\ even}}^{n} \binom{n}{q} (i\omega\Delta t)^q \qquad S_{odd}(n) = \sum_{\substack{q=0 \\ q\ odd}}^{n} \binom{n}{q} (i\omega\Delta t)^q \tag{63}$$

$$S_{\pm}(n) = S_{even}(n) \pm S_{odd}(n) \tag{64}$$

then

$$S_{even}(n) = \frac{1}{2}\left[S_{+}(n) + S_{-}(n)\right] \qquad S_{odd}(n) = \frac{1}{2}\left[S_{+}(n) - S_{-}(n)\right] \tag{65}$$

and equations (61) and (62) for $x_n(t)$ and $v_n(t)$ can be rewritten in terms of $S_{even}(n)$, $S_{odd}(n)$ as

$$x_n = \lambda_{0,0} S_{even}(n) + (i\omega)^{-1} \lambda_{0,1} S_{odd}(n) \tag{66}$$

$$v_n = i\omega\lambda_{0,0} S_{odd}(n) + \lambda_{0,1} S_{even}(n) \tag{67}$$

or, alternatively in terms of $S_{\pm}(n)$ as

$$x_n = \tfrac{S_+(n)}{2}\left[\lambda_{0,0} + (i\omega)^{-1}\lambda_{0,1}\right] + \tfrac{S_-(n)}{2}\left[\lambda_{0,0} - (i\omega)^{-1}\lambda_{0,1}\right] \tag{68}$$

$$v_n = \tfrac{S_+(n)}{2}\left[i\omega\lambda_{0,0} + \lambda_{0,1}\right] - \tfrac{S_-(n)}{2}\left[i\omega\lambda_{0,0} - \lambda_{0,1}\right] \tag{69}$$

Where $\lambda_{0,0}$ and $\lambda_{0,1}$ are given in terms of the initial conditions by $\lambda_{0,0} = x_0$ and $\lambda_{0,1} = v_0$. So that

$$x_n = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right) S_+(n) + \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right) S_-(n) \tag{70}$$

$$\frac{v_n}{i\omega} = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right) S_+(n) - \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right) S_-(n) \tag{71}$$

$S_{\pm}(n)$ are easily evaluated as

$$S_{\pm}(n) = \sum_{\substack{q=0 \\ q\ even}}^{n} \binom{n}{q} (i\omega\Delta t)^q \pm \sum_{\substack{q=0 \\ q\ odd}}^{n} \binom{n}{q} (i\omega\Delta t)^q \tag{72}$$

or

$$S_{\pm}(n) = \sum_{q=0}^{n} \binom{n}{q} (\pm i\omega\Delta t)^q = (1 \pm i\omega\Delta t)^n \tag{73}$$

So we finally have

$$x_n = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right)(1 + i\omega\Delta t)^n + \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right)(1 - i\omega\Delta t)^n \tag{74}$$

$$\frac{v_n}{i\omega} = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right)(1 + i\omega\Delta t)^n - \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right)(1 - i\omega\Delta t)^n \tag{75}$$

4.5 The Continuum Limit

Taking the limit as $n \to \infty$ while holding $t_n = t_0 + n\Delta t$ fixed at t, we have

$$S_{\pm}(t) = \lim_{n\to\infty} S_{\pm}(n)\,|_{t_n=t} = \lim_{n\to\infty}\left[1 \pm i\omega\left(\frac{t-t_0}{n}\right)\right]^n \tag{76}$$

or

$$S_{\pm}(t) = e^{\pm i\omega(t-t_0)} \tag{77}$$

Thus, making use of $x(t) = \lim_{n\to\infty} x(t_n)\,|_{t_n=t}$ and $v(t) = \lim_{n\to\infty} v(t_n)\,|_{t_n=t}$ we have

$$x(t) = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right)e^{i\omega(t-t_0)} + \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right)e^{-i\omega(t-t_0)} \tag{78}$$

$$\frac{v(t)}{i\omega} = \frac{1}{2}\left(x_0 + \frac{v_0}{i\omega}\right)e^{i\omega(t-t_0)} - \frac{1}{2}\left(x_0 - \frac{v_0}{i\omega}\right)e^{-i\omega(t-t_0)} \tag{79}$$

4.6 The Lowering and Raising Operators

The energy lowering and raising operators for the harmonic oscillator are given respectively by [3]

$$a(t) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}x(t) + i\frac{1}{\sqrt{m\hbar\omega}}p(t)\right] = \sqrt{\frac{m\omega}{2\hbar}}\left[x(t) - \frac{v(t)}{i\omega}\right] \tag{80}$$

and

$$a^*(t) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}x(t) - i\frac{1}{\sqrt{m\hbar\omega}}p(t)\right] = \sqrt{\frac{m\omega}{2\hbar}}\left[x(t) + \frac{v(t)}{i\omega}\right] \tag{81}$$

We define their discrete counterparts by

$$a_n = a(t_n) = \sqrt{\frac{m\omega}{2\hbar}}\left[x(t_n) - \frac{v(t_n)}{i\omega}\right] = \sqrt{\frac{m\omega}{2\hbar}}\left(x_n - \frac{v_n}{i\omega}\right) \tag{82}$$

and

$$a_n^* = a^*(t_n) = \sqrt{\frac{m\omega}{2\hbar}}\left[x(t_n) + \frac{v(t_n)}{i\omega}\right] = \sqrt{\frac{m\omega}{2\hbar}}\left(x_n + \frac{v_n}{i\omega}\right) \tag{83}$$

Their time development is given, via equations (74) and (75) by

$$a_n = (1 - i\omega\Delta t)^n a_0 = S_-(n)\, a_0 \tag{84}$$

$$a_n^* = (1 + i\omega\Delta t)^n a_0^* = S_+(n)\, a_0^* \tag{85}$$

$$a(t) = \lim_{n\to\infty} a(t_n)\,|_{t_n=t} = e^{-i\omega(t-t_0)}a_0 \tag{86}$$

$$a^*(t) = \lim_{n\to\infty} a^*(t_n)\,|_{t_n=t} = e^{i\omega(t-t_0)}a_0^* \tag{87}$$

5 Conclusion

The simplicity of the harmonic oscillator problem and its well known solutions for continuous as well as discrete time, have provided us with the framework to analyze and understand the structure of the discrete path solution as it applies to harmonic oscillator type problems. Four specific relevant mathematical functions emerge: these are the $F_\ell(n,k)$ functions as defined by equation (34); the G-functions as defined by equation (44); the $\gamma_m^n(\ell_1, \ell_2, \cdots, \ell_n)$ functions as defined by equation (43); and the $S_\pm(n)$ functions as defined by equations (63) and (64). In the case of the simple harmonic oscillator all these functions reduce to simple compact analytic forms. For more complex anticipatory problems, we should expect that the form of these functions will reflect the complexity of the problem being solved. But in all cases, the simple analytic results obtained here will serve as a guide and provide limiting test cases for the different parts of the solution.

References

[1] Antippa Adel F. (2002), "Exact Closed Form Solution For Two Coupled Inhomogeneous First Order Difference Equations With Variable Coefficients", *Journal of Difference Equations and Applications*, **8**, 255-276.

[2] Antippa Adel F. (1977), "Discrete Path Approach to Linear Recursion Relations", *Journal of Mathematical Physics*, **18**, 2214-2222.

[3] Cohen-Tannoudji Claude, Diu Bernard, et Lalo' Franck (1977), *Mecanique Quantique I*, Hermann, Paris, 486-501

[4] Dubois Daniel M. (2000), "Review of Incursive, Hyperincursive and Anticipatory Systems - Foundation of Anticipation in Electromagnetism, *Proceedings of the Third International Conference on Computing Anticipatory Systems*, Daniel M. Dubois, Ed., American Institute of Physics Conference Proceedings No. **517**, 3-30.

[5] Dubois Daniel M. (1998), "Introduction to Computing Anticipatory Systems", *International Journal of Computing Anticipatory Systems*, **2**, 3-14.

[6] Elaydi Saber N. (1999), *An Introduction to Difference Equations*, Springer-Verlag, New York, 63-70.

[7] Jordan Charles (1965), *Calculus of Finite Differences*, Chelsea Publishing Co., New York, 6-8.

[8] Moshinsky M. (1969), *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks*, Gordon and Breach, New York.

[9] Vilenkin N. Ya. (1971), *Combinatorics*, Academic press, New York, 26-27.