Ideas on Hyperincursive Proof Theory

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Abstract
This paper describes the possibility of incursive proof in classical formal theory.
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1 Introduction: Formal Theories in General

A *formal theory* $\mathbb{S}$ is defined when the following conditions are satisfied[1]:

1*) There is a set of symbols which is at most countable and which is the set of the symbols of $\mathbb{S}$. A finite sequence of symbols of $\mathbb{S}$ is called *expression* of $\mathbb{S}$.

2*) There is a subset of the expressions of $\mathbb{S}$ which is called set of the *well formed formulas*[2] (abbreviate with *wffs*, singular *wff*). Usually, there is an automatic procedure to decide if any expression of $\mathbb{S}$ is wff.

3*) There is a subset of the wffs of $\mathbb{S}$ which is called set of the *axioms*. $\mathbb{S}$ is called *axiomatic* iff there is an automatic procedure to decide if a wff of $\mathbb{S}$ is axiom.

4*) There is a finite set of relations $R_1,\dots,R_n$ among wffs that are called *inference rules*. For every R_i there is such a sole integer positive j that for every set of j wffs and for every wff $\mathcal{A}$ it can be decided if the j wffs are in relation R_i with $\mathcal{A}$. In this case $\mathcal{A}$ is called *direct consequence* of the j wffs by R_i.

A *proof* in $\mathbb{S}$ is a sequence $\mathcal{A}_1,\dots,\mathcal{A}_n$. of such wffs that for every i, $\mathcal{A}_i$ is axiom or direct consequence of a subset of previous wffs.

A *theorem* $\mathcal{A}$ in $\mathbb{S}$ is the last wff of one or more proofs. Such proofs are called *proofs of* $\mathcal{A}$.

$\mathbb{S}$ is called *decidable* iff there is an automatic procedure to decide if any wff of $\mathbb{S}$ is a theorem.

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[1] We consider only standard logic with non-contradiction principle in this work and we use Mendelson's formalism (1). This choice has been done for the wide circulation of such a logic and such a formalism. We think that the hyperincursion principles (see above) can be applied to any logic with any formalism but we do not prove this fact in this paper for space reasons. However, our convinction is based on the following achievements: Fuschino proved the reducibility of fuzzy logics to standard logic (2), Rutz proved the reducibility the many-valued logics to standard logic (3) (see also Grappone (4)), Malatesta proved that non-classical logics cannot take a step without a stock of tautologies belonging to classical one, which are laws of non-contradiction. Therefore either the set of laws of a non-classical logic is a proper subset of the classical one or there is an intersection between the sets of laws of classical logic and a non classical one, without which the last cannot work (5).

[2] Assume that a *well formed formula* is a symbol which means a given proposition in $\mathbb{S}$.

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A wff $\mathcal{A}$ of $\mathbb{S}$ is called *consequence* of a set Γ of wffs of $\mathbb{S}$ iff there is a sequence $\mathcal{A}_1,\ldots,\mathcal{A}_n$ of such wffs that for every *i*, $\mathcal{A}_i$ is axiom or direct consequence of a subset of previous wffs or $\mathcal{A}_i \in \Gamma$. Such a sequence $\mathcal{A}_1,\ldots,\mathcal{A}_n$ is called *proof* (or *deduction*) *of* $\mathcal{A}$ *from* Γ. The elements of Γ are called *hypotheses* or *premises* of $\mathcal{A}$. Read '$\Gamma \vdash \mathcal{A}$' "The wffs of the set Γ are premises of $\mathcal{A}$", in other words, "The wffs of the set Γ deduce $\mathcal{A}$". If $\Gamma \equiv \varnothing$, then $\Gamma \vdash \mathcal{A}$ iff $\mathcal{A}$ is a theorem. So, we can denote that $\mathcal{A}$ is a theorem with the expressions '$\varnothing \vdash \mathcal{A}$' and '$\vdash \mathcal{A}$'.

The concept of consequence has the following properties:

1**) If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \mathcal{A}$, then $\Delta \vdash \mathcal{A}$.
2**) $\Gamma \vdash \mathcal{A}$ iff there is such a finite set Δ that $\Delta \subseteq \Gamma$ and $\Delta \vdash \mathcal{A}$.
3**) If Γ deduces every wff of Δ and $\Delta \vdash \mathcal{A}$, then $\Gamma \vdash \mathcal{A}$.

The properties 1**), ..., 3**) of standard proof theory suggest the way to introduce hyperincursivity[3] in proof theory. As the (hyper)incursive processes have a

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[3] '*The recursion consists of the computation of the future value of the variable vector* X(t+1) *at time* t+1 *from the values of these variables at present and/or past times,* t, t-1,t-2, ... *by a recursive function:* X(t+1) = *f*(X(t), X(t-1),..., p) *where* p *is a command parameter vector. So, the past always determines the future, the present being the separation line between the past and the future. ... Starting from cellular automata the concept of fractal machines was proposed in which composition rules where propagated along paths in the machine frame. The computation is based on what I called 'INclusive reCURSION', i. e. INCURSION ... An incursive relation is defined by:* X(t+1) = *f*(..., X(t+1), X(t), X(t-1), ..., p) *which consists in the computation of the values of the vector* X(t+1) *at time* t+1 *from the values* X(t-i) *at time* t-i, i = 1,2,... *as a function of a command vector* p. *This incursive relation is not trivial because future values of the variable vector at time steps* t+1, t+2,... *must be known to compute them at the time step* t+1. ... *In a similar way to that in which we define hyper recursion when each recursive step generates multiple solutions, I define HYPERINCURSION. ... I have decided to do this for three reasons. First, in relativity theory space and time are considered as a four-vector where time plays a role similar to space. If time* t *is replaced by space* s *in the above definition of incursion, we obtain* X(s+1) = *f*(..., X(s+1), X(s), X(s-1), ..., p) *and nobody is astonished – a laplacean operator looks like this. Second, in control theory, the engineers control engineering systems by defining goals in the future to compute their present state, similarly to our human anticipative behaviour. ... Third, I wanted to try to do a generalisation of the recursive and sequential Turing machine in looking at space-time cellular automata where the order in which the computations are made is taken into account with an inclusive recursion*' (6) (see also (7)). This innovative mathematical perspective is relevant from a logical viewpoint. If a *proof* in $\mathbb{S}$ is a sequence $\mathcal{A}_1,\ldots,\mathcal{A}_n$ of such wffs that for every *i*, $\mathcal{A}_i$ is axiom or direct consequence of a subset of previous wffs, then we can write such a proof as $\mathcal{A}_1(t),\ldots,\mathcal{A}_n(t+n-1)$. The last formula of this proof, i e. $\mathcal{A}_n(t+n-1)$, is calculated in $\mathbb{S}$ by inference schemes of $\mathbb{S}$ and by axioms of $\mathbb{S}$. Assume the inference schemes of $\mathbb{S}$ as *n*–adic (with an opportune *n*) operations which have the deduced wff as achievement and the deducing wffs as arguments. If $\underbrace{\mathcal{A}_1(t),\ldots,\mathcal{A}_k(t+k-1)}_{k<n}$ are the axioms which appear in $\mathcal{A}_1(t),\ldots,\mathcal{A}_n(t+n-1)$ and f is an opportune combination of the inference schemes which appear in $\mathcal{A}_1(t),\ldots,\mathcal{A}_n(t+n-1)$ and that we assume as operations, then $\mathcal{A}_n(t+n-1) = \underbrace{f\big(\mathcal{A}_1(t),\ldots,\mathcal{A}_k(t+k-1)\big)}_{k<n}$. It is very easy to proof that f is a recursive function. But the most frequent logic problem is to decide if a given wff is a theorem in a formal theory $\mathbb{S}$, i. e. if it has a proof in $\mathbb{S}$. This fact means to return behind from the theorem to the axioms which deduce it. Given the previous considerations, such a decision corresponds to calculate functions as F, such that

set representation, we represent the wff in form of sets. Consider the countable set $\{\Gamma_1^{\mathcal{A}},\ldots,\Gamma_n^{\mathcal{A}},\ldots\}$ of all the possible sets that deduce $\mathcal{A}$. As $\{\mathcal{A}\}\vdash\mathcal{A}$ it is very easy to prove that $\{\mathcal{A}\}=\bigcap_{i=1}^{\infty}\Gamma_i^{\mathcal{A}}$. Thus every distinct wff has a distinct set representation. So, to develop any formal theory we can only consider set of wff and use an set language: e. g. we can replace $\Gamma\vdash\mathcal{A}$ with $\Gamma\supseteq\{\mathcal{A}\}$.

A first consequence of such an approach is an ensemblist definition of theorem and axiom: $\mathcal{A}$ is theorem or axiom iff $\{\mathcal{A}\}\equiv\varnothing$[4], i. e. $\mathcal{A}$ is theorem or axiom in any formal theory iff it is deducible from axioms without other premises.

Now we can consider the meaning of the set operation in a formal theory (8).

1***) If $\{\mathcal{C}\}\vdash\mathcal{A}$ and $\{\mathcal{D}\}\vdash\mathcal{B}$, then $\{\mathcal{C}\}\supseteq\{\mathcal{A}\}$ and $\{\mathcal{D}\}\supseteq\{\mathcal{B}\}$. Thus $\{\mathcal{C}\}\cup\{\mathcal{D}\}\supseteq\{\mathcal{A}\}\cup\{\mathcal{B}\}$ and $\{\mathcal{C}\}\cup\{\mathcal{D}\}\vdash\mathcal{A},\mathcal{B}$. Hence, if $\{\mathcal{C}\}\vdash\mathcal{A}$ and $\{\mathcal{D}\}\vdash\mathcal{B}$ then $\{\mathcal{C}\}\cup\{\mathcal{D}\}\vdash\mathcal{A},\mathcal{B}$. In particular, $\{\mathcal{A}\}\cup\{\mathcal{B}\}\vdash\mathcal{A},\mathcal{B}$.

2***) If $\{\mathcal{B}\}\vdash\mathcal{A}$ and $\{\mathcal{C}\}\vdash\mathcal{A}$, then $\{\mathcal{B}\}\supseteq\{\mathcal{A}\}$ and $\{\mathcal{C}\}\supseteq\{\mathcal{A}\}$. Thus $\{\mathcal{B}\}\cap\{\mathcal{C}\}\supseteq\{\mathcal{A}\}$ and $\{\mathcal{B}\}\cap\{\mathcal{C}\}\vdash\mathcal{A}$. Hence, if $\{\mathcal{B}\}\vdash\mathcal{A}$ and $\{\mathcal{C}\}\vdash\mathcal{A}$, then $\{\mathcal{B}\}\cap\{\mathcal{C}\}\vdash\mathcal{A}$.

3***) Let $\mathfrak{T}$ be the collection of all possible premise sets. Thus, for every $\{\mathcal{A}\}$, $\mathfrak{T}\supseteq\{\mathcal{A}\}$. Hence, for every $\{\mathcal{A}\}$, $\mathfrak{T}\vdash\mathcal{A}$.

4***) Let $\mathfrak{C}\{\mathcal{A}\}$ be $\mathfrak{T}\setminus\{\mathcal{A}\}$.

5***) Consider $\mathfrak{C}\{\mathcal{A}\}\cap\{\mathcal{B}\}$. It becomes $(\mathfrak{T}\setminus\{\mathcal{A}\})\cap\{\mathcal{B}\}$, i. e. $(\mathfrak{T}\cap\{\mathcal{B}\})\setminus(\{\mathcal{A}\}\cap\{\mathcal{B}\})$, i. e. $\{\mathcal{B}\}\setminus\{\mathcal{A}\}$. Also, if $\{\mathcal{A}\}\supseteq\{\mathcal{B}\}$ then $\{\mathcal{B}\}\setminus\{\mathcal{A}\}\equiv\varnothing$. Thus $\{\mathcal{A}\}\vdash\mathcal{B}$ is equivalent to $\{\mathcal{B}\}\setminus\{\mathcal{A}\}\equiv\varnothing$.

6***) $\{\mathcal{A}\}\setminus\{\mathcal{B}\}\equiv\varnothing$ and $\{\mathcal{B}\}\setminus\{\mathcal{A}\}\equiv\varnothing$ iff $\{\mathcal{A}\}\equiv\{\mathcal{B}\}$

7***) Given the Cartesian product of two sets '$\times$', consider the example $\{\alpha\}\times\{\beta\}\equiv[\alpha,\beta]$. If $[\alpha,\beta]$ is a sentence, then it can be interpretated as " Elements of α in first place deduce elements of β in second place and vice versa." Let $\{\pi_0,\ldots,\pi_n,\ldots\}$ be the place set[5]. Let $\frac{\{\alpha\}}{\pi_i}$ be the sentence " (There are) elements of

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$\mathcal{A}_{n-i}(t+n-1-i)=F\left(\mathcal{A}_n(t+n-1),\ldots,\mathcal{A}_{n-i+1}(t+n-1-i+1)\right)$. The incursive nature of this last equation is evident. Clearly, if f requests a parameter choice, then F is hyperincursive.

[4] This formula means that the set of the premixes which are necessary to a theorem or an axiom be true is void, i. e. any theorem or axiom is true alwais and not only under particular premixes.

[5] Grappone has proved that any polyadic predicate is equivalent to a predicate built only with monadic predicates and sentential connectives (9). He has given the formula $A_m^n(x_1,\ldots,x_n)\equiv\bigwedge_{i=1}^{n}\left(\supset_{s\,j=1}^{\;n}(A_{m_j}^1(x_j))A_{m_i}^1(x_i)\right)$ ($\supset_{s\,j=1}^{\;n}$ was introduced by Malatesta with the name of *polyadic sequence implication* (10)), where $(\supset_{s\,j=1}^{\;n}(A_{m_j}^1(x_j))A_{m_i}^1(x_i)$ means

$\{\alpha\}$ in first place" and, for a single element ε, let $\frac{\varepsilon}{\pi_i}$ be the sentence " (There are) ε in first place". Given any $\frac{\alpha}{\pi_i}$, apply the monadic operations to π_i. So the sentence $[\alpha,\beta]$ becomes "$\left\{\frac{\{\alpha\}}{\pi_0}\right\} \vdash \frac{\{\beta\}}{\pi_1}$ and $\left\{\frac{\{\beta\}}{\pi_1}\right\} \vdash \frac{\{\alpha\}}{\pi_0}$", i. e. $\left(\frac{\{\alpha\}}{\complement\pi_0} \cap \frac{\{\beta\}}{\pi_1}\right) \cup \left(\frac{\{\beta\}}{\complement\pi_1} \cap \frac{\{\alpha\}}{\pi_0}\right)$. In general if the poliadic Cartesian product $\alpha_1 \times \alpha_2 \times \ldots \times \alpha_n$ is the sentence composed by the terms $\alpha_1, \alpha_2, \ldots, \alpha_n$, then its premixe set is equivalent to $\bigcup_{i=0}^{n-1}\left(\frac{\alpha_{i+1}}{\pi_i} \cap \bigcap_{\substack{j=0 \\ j\neq i}}^{n-1} \frac{\alpha_{j+1}}{\complement\pi_j}\right)$. It is clear that, e. g., $\frac{\{\alpha\}}{\pi_0} \cup \frac{\{\alpha\}}{\pi_1} \equiv \frac{\{\alpha\}}{\pi_0 \cup \pi_1}$, and so on for every set operations.

2 From Gentzen's Natural Deduction to Anticipatory Deduction

The method of natural deduction is owed to Gerhard Gentzen (11). Gentzen listed the following rules that we describe in standard language in Table I.

Table 1: Gentzen's Rules

$\wedge$-Introduction	$\wedge$-Elimination	$\vee$-Introduction	$\vee$-Elimination
$\alpha,\beta \vdash \alpha\wedge\beta$	$\alpha\wedge\beta \vdash \alpha$. $\alpha\wedge\beta \vdash \beta$	$\alpha \vdash \alpha\vee\beta$. $\beta \vdash \alpha\vee\beta$	$\alpha\vee\beta, (\{\alpha\} \vdash \gamma), (\{\beta\} \vdash \gamma) \vdash \gamma$
$\supset$-Introduction	$\supset$-Elimination	$\sim$-Introduction	$\sim$-Elimination
$(\alpha \vdash \beta) \vdash \alpha \supset \beta$	$\alpha, \alpha\supset\beta \vdash \beta$	$(\alpha \vdash [\text{CONTR.}]) \vdash \sim\alpha$[6]	$(\alpha, \sim\alpha \vdash [\text{CONTR.}])$, $[\text{CONTR.}] \vdash \delta$

There are two other rules of public domain to simplify the equivalence calcule: (12)

$\sim A^1_{m_1}(x_1) \vee \ldots \vee \sim A^1_{m_{i-1}}(x_{i-1}) \vee A^1_{m_i}(x_i) \vee \sim A^1_{m_{i+1}}(x_{i+1}) \vee \sim A^1_{m_n}(x_n)$ (9). We can develop this previous achievement for this paper by putting $A''_m(x_1,\ldots,x_n) \equiv (x_0,x_1,\ldots,x_n) \equiv \bigwedge_{i=0}^{n}\left(\supset_{s}^{n}{}_{j=0}(\pi_{m_j}(x_j))\pi_{m_i}(x_i)\right)$ and finally: $\pi_{m_i}(x_j) \equiv \frac{x_j}{\pi_{m_i}}$. So, we can consider $\{\pi_1,\ldots,\pi_n\}$ as a place predicate set.

[6] $[\text{CONTR.}]$ is the logical contradiction, i. e. any sentence whose truth value is always equal to false.

Table 2: Public Domain Rules

≡-Introduction	≡-Elimination
$\alpha \supset \beta, \beta \supset \alpha \vdash \alpha \equiv \beta$	$\alpha \equiv \beta \vdash \alpha \supset \beta, \alpha \equiv \beta \vdash \beta \supset \alpha$

The rules of the tables I and II permit us to deduce various tautologies by Gentzen's method. E. g. we have:

1	(1)	$p \supset q$	Hypothesis
2	(2)	$q \supset r$	Hypothesis
3	(3)	p	Hypothesis
1,3	(4)	q	1,3 $\supset$-Elimination
1,2,3	(5)	r	3,5 $\supset$-Elimination
1,2	(6)	$p \supset r$	3,5 $\supset$-Introduction
1	(7)	$(q \supset r) \supset (p \supset r)$	2,6 $\supset$-Introduction
–	(8)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	1,7 $\supset$-Introduction

We start from some hypothesis and afterwards we apply Gentzen's rules untill the first column is void. Then, the formula in the same row and in third column is a tautology. (P1) is based on the calculus of premise sets: the premise set of a tautology is obviously empty because a tautology is a theorem. E. g.: $\{(p \supset q) \supset ((q \supset r) \supset (p \supset r))\} \equiv \varnothing$.

Consider the properties 1***), …,7***) of a formal theory. These and Gentzen's rules permit us to obtain reversible rules that define standard sentence logic as a formal theory and permit us to build an anticipatory deduction. We have:

1#) $\{\alpha\} \cup \{\beta\} \vdash \alpha, \beta$ by 1***) and $\alpha, \beta \vdash \alpha \wedge \beta$ by $\wedge$-Introduction, hence $\{\alpha\} \cup \{\beta\} \vdash \alpha \wedge \beta$ by 3**), thus $\{\alpha\} \cup \{\beta\} \supseteq \{\alpha \wedge \beta\}$. $\alpha \wedge \beta \vdash \alpha$ $\wedge$-Elimination and $\alpha \wedge \beta \vdash \beta$ $\wedge$-Elimination, hence $\{\alpha \wedge \beta\} \supseteq \{\alpha\}$ and $\{\alpha \wedge \beta\} \supseteq \{\beta\}$, so $\{\alpha \wedge \beta\} \supseteq \{\alpha\} \cup \{\beta\}$. Finally, from $\{\alpha \wedge \beta\} \supseteq \{\alpha\}$ and $\{\alpha \wedge \beta\} \supseteq \{\beta\}$:

$$\{\alpha \wedge \beta\} \equiv \{\alpha\} \cup \{\beta\} \qquad (1\#).$$

2#) $\alpha \vdash \alpha \vee \beta$ by $\vee$-Introduction and $\beta \vdash \alpha \vee \beta$ by $\vee$-Introduction, hence $\{\alpha\} \supseteq \{\alpha \vee \beta\}$ and $\{\beta\} \supseteq \{\alpha \vee \beta\}$, hence $\{\alpha\} \cap \{\beta\} \supseteq \{\alpha \vee \beta\}$. $\alpha \vee \beta, (\{\alpha\} \vdash \gamma), (\{\beta\} \vdash \gamma) \vdash \gamma$ by $\vee$-Elimination, so $(\{\alpha\} \vdash \gamma), (\{\beta\} \vdash \gamma) \vdash (\{\alpha \vee \beta\} \vdash \gamma)$, thus $(\{\alpha\} \cap \{\beta\} \vdash \gamma) \vdash (\{\alpha \vee \beta\} \vdash \gamma)$, hence $\{\alpha\} \cap \{\beta\} \supseteq \{\gamma\} \vdash \{\alpha \vee \beta\} \supseteq \{\gamma\}$, so $\{\alpha \vee \beta\} \supseteq \{\alpha\} \cap \{\beta\}$ by 1**),…,3**). Finally, from $\{\alpha\} \cap \{\beta\} \supseteq \{\alpha \vee \beta\}$ and $\{\alpha \vee \beta\} \supseteq \{\alpha\} \cap \{\beta\}$:

$$\{\alpha \vee \beta\} \equiv \{\alpha\} \cap \{\beta\} \qquad (2\#).$$

3#) $(\alpha \vdash \beta) \vdash \alpha \supset \beta$ by $\supset$-Introduction, so $\{\beta\} \setminus \{\alpha\} \equiv \varnothing \vdash \{\alpha \supset \beta\} \equiv \varnothing$ by 5***), thus $\{\beta\} \setminus \{\alpha\} \supseteq \{\alpha \supset \beta\}$. $\alpha, \alpha \supset \beta \vdash \beta$ by $\supset$-Elimination, so $\alpha \supset \beta \vdash (\alpha \vdash \beta)$, thus $\{\alpha \supset \beta\} \equiv \varnothing \vdash \{\beta\} \setminus \{\alpha\} \equiv \varnothing$, hence $\{\alpha \supset \beta\} \supseteq \{\beta\} \setminus \{\alpha\}$. But $\{\beta\} \setminus \{\alpha\} \equiv \mathfrak{C}\{\alpha\} \cap \{\beta\}$. Finally, from $\{\beta\} \setminus \{\alpha\} \supseteq \{\alpha \supset \beta\}$ and $\{\alpha \supset \beta\} \supseteq \{\beta\} \setminus \{\alpha\}$:

$$\{\alpha \supset \beta\} \equiv \{\beta\} \setminus \{\alpha\} \equiv \mathfrak{C}\{\alpha\} \cap \{\beta\} \qquad (3\#).$$

4#) $(\alpha \vdash [\text{CONTR.}]) \vdash \sim \alpha$ by $\sim$-Introduction. $[\text{CONTR.}]$ is the contradiction symbol. To represent $[\text{CONTR.}]$ in formal theories consider that a contradictory sentence deduces every sentence in standard sentence logic. (5) In other words, for every wff α, $\{[\text{CONTR.}]\} \supseteq \{\alpha\}$. But this relation is true iff $\{[\text{CONTR.}]\} \equiv \mathfrak{T}$, where $\mathfrak{T}$ is the set which includes all the possible premise sets. So we can write the $\sim$-Introduction: $(\alpha \vdash \mathfrak{T}) \vdash \sim \alpha$, so $\mathfrak{T} \setminus \{\alpha\} \equiv \varnothing \vdash \{\sim \alpha\} \equiv \varnothing$, thus $\mathfrak{C}\{\alpha\} \equiv \varnothing \vdash \{\sim \alpha\} \equiv \varnothing$, hence $\mathfrak{C}\{\alpha\} \supseteq \{\sim \alpha\}$. $(\alpha, \sim \alpha \vdash [\text{CONTR.}])$ and $[\text{CONTR.}] \vdash \delta$ by $\sim$-Elimination, so $\{\sim \alpha\} \vdash (\{\alpha\} \vdash \mathfrak{T})$, thus $\{\sim \alpha\} \equiv \varnothing \vdash \mathfrak{T} \pm \{\alpha\} \equiv \varnothing$, hence $\{\sim \alpha\} \equiv \varnothing \vdash \mathfrak{C}\{\alpha\} \equiv \varnothing$, so $\{\sim \alpha\} \supseteq \mathfrak{C}\{\alpha\}$. Finally, from $\mathfrak{C}\{\alpha\} \supseteq \{\sim \alpha\}$ and $\{\sim \alpha\} \supseteq \mathfrak{C}\{\alpha\}$:

$$\{\sim \alpha\} \equiv \mathfrak{C}\{\alpha\} \qquad (4\#).$$

5#) $\alpha \supset \beta, \beta \supset \alpha \vdash \alpha \equiv \beta$ by $\equiv$-Introduction, so $\{\alpha \supset \beta\} \cup \{\beta \supset \alpha\} \supseteq \{\alpha \equiv \beta\}$. $\alpha \equiv \beta \vdash \alpha \supset \beta$ and $\alpha \equiv \beta \vdash \beta \supset \alpha$ by $\equiv$-Elimination, so $\{\alpha \equiv \beta\} \supseteq \{\alpha \supset \beta\}$ and $\{\alpha \equiv \beta\} \supseteq \{\beta \supset \alpha\}$, thus $\{\alpha \equiv \beta\} \supseteq \{\alpha \supset \beta\} \cup \{\beta \supset \alpha\}$. Finally, from $\{\alpha \supset \beta\} \cup \{\beta \supset \alpha\} \supseteq \{\alpha \equiv \beta\}$ and $\{\alpha \equiv \beta\} \supseteq \{\alpha \supset \beta\} \cup \{\beta \supset \alpha\}$:

$$\{\alpha \equiv \beta\} \equiv \{\alpha \supset \beta\} \cup \{\beta \supset \alpha\} \qquad (5\#).$$

Rules (1#),…,(5#) permit us to obtain anticipatory proofs [7]. E. g., consider the proof (P1) in anticipatory form:

Start:

[7] A proof is anticipatory if it is built by starting from the theorem which has to be proved.

$\{(p \supset q) \supset ((q \supset r) \supset (p \supset r))\}$	(1)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	Hypothesis

By rule (3#):

$\{(q \supset r) \supset (p \supset r)\} \pm \{p \supset q\}$	(3)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	(1),(2) $\supset$-Introduction
$\{(q \supset r) \supset (p \supset r)\}$	(2)	$(q \supset r) \supset (p \supset r)$	Hypothesis
$\{p \supset q\}$	(1)	$p \supset q$	Hypothesis

By rule (3#):

$\{(q \supset r) \supset (p \supset r)\} \pm \{p \supset q\}$	(5)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	(1),(4) $\supset$-Introduction
$\{p \supset r\} \pm \{q \supset r\}$	(4)	$(q \supset r) \supset (p \supset r)$	(2),(3) $\supset$-Introduction
$\{p \supset r\}$	(3)	$p \supset r$	Hypothesis
$\{(q \supset r)\}$	(2)	$q \supset r$	Hypothesis
$\{p \supset q\}$	(1)	$p \supset q$	Hypothesis

By rule (3#):

$\{(q \supset r) \supset (p \supset r)\} \pm \{p \supset q\}$	(7)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	(1),(6) $\supset$-Introduction
$\{p \supset r\} \pm \{q \supset r\}$	(6)	$(q \supset r) \supset (p \supset r)$	(2),(5) $\supset$-Introduction
$\{r\} \pm \{p\}$	(5)	$p \supset r$	(3),(4) $\supset$-Introduction
$\{r\}$	(4)	r	Hypothesis
$\{p\}$	(3)	p	Hypothesis
$\{(q \supset r)\}$	(2)	$q \supset r$	Hypothesis
$\{p \supset q\}$	(1)	$p \supset q$	Hypothesis

By rules (1#) and (3#):

$\{(q \supset r) \supset (p \supset r)\} \pm \{p \supset q\}$	(8)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	(1),(7) $\supset$-Introduction
$\{p \supset r\} \pm \{q \supset r\}$	(7)	$(q \supset r) \supset (p \supset r)$	(2),(6) $\supset$-Introduction
$\{r\} \pm \{p\}$	(6)	$p \supset r$	(3),(5) $\supset$-Introduction
$\{r\}$	(5)	r	Hypothesis
$\{p\} \cup \{p \supset q\}$	(4)	q	(1),(3) $\supset$-Elimination
$\{p\}$	(3)	p	Hypothesis
$\{(q \supset r)\}$	(2)	$q \supset r$	Hypothesis
$\{p \supset q\}$	(1)	$p \supset q$	Hypothesis

By rules (1#) and (3#):

$\{(q \supset r) \supset (p \supset r)\} \pm \{p \supset q\}$	(8)	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	(1),(7) $\supset$-Introduction
$\{p \supset r\} \pm \{q \supset r\}$	(7)	$(q \supset r) \supset (p \supset r)$	(2),(6) $\supset$-Introduction
$\{r\} \pm \{p\}$	(6)	$p \supset r$	(3),(5) $\supset$-Introduction
$\{q\} \cup \{q \supset r\}$	(5)	r	(2),(4) $\supset$-Elimination
$\{p\} \cup \{p \supset q\}$	(4)	q	(1),(3) $\supset$-Elimination
$\{p\}$	(3)	p	Hypothesis
$\{(q \supset r)\}$	(2)	$q \supset r$	Hypothesis
$\{p \supset q\}$	(1)	$p \supset q$	Hypothesis

To verify if $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$ is a theorem we can consider the first column of its proof. We have: $\{(p \supset q) \supset ((q \supset r) \supset (p \supset r))\} \equiv \{(q \supset r) \supset (p \supset r)\} \pm \{p \supset q\} \equiv$ $(\{p \supset r\} \pm \{q \supset r\}) \pm \{p \supset q\} \equiv ((\{r\} \pm \{p\}) \pm \{q \supset r\}) \pm \{p \supset q\} \equiv$

$$\equiv (((\{q\} \cup \{q \supset r\}) \pm \{p\}) \pm \{q \supset r\}) \pm \{p \supset q\} \equiv$$

$$\equiv (((\{p\} \cup \{p \supset q\} \cup \{q \supset r\}) \pm \{p\}) \pm \{q \supset r\}) \pm \{p \supset q\} \equiv$$

$$\equiv (((\{p\} \cup (\mathbb{C}\{p\} \cap \{q\}) \cup (\mathbb{C}\{q\} \cap \{r\})) \cap \mathbb{C}\{p\}) \cap \mathbb{C}(\mathbb{C}\{q\} \cap \{r\})) \cap \mathbb{C}(\mathbb{C}\{p\} \cap \{q\}) \equiv$$

$$\equiv ((\{p\} \cap \{q\} \cap \{r\} \cap \mathbb{C}\{p\}) \cap \mathbb{C}(\mathbb{C}\{q\} \cap \{r\})) \cap \mathbb{C}(\mathbb{C}\{p\} \cap \{q\}) \equiv$$

$$\equiv (\varnothing \cap \mathbb{C}(\mathbb{C}\{q\} \cap \{r\})) \cap \mathbb{C}(\mathbb{C}\{p\} \cap \{q\}) \equiv \varnothing$$

3 Incursive Proofs in Standard Sentence Logic

To develop incursive proofs in standard sentence logic given below it is necessary that such inference schemes be completely reversible. A complete set of such schemes is:

$\vee^0 = 1$	(01°)	$\wedge^2(p,q) = p \vee q$	(14°)
$\vee^1(p) = p$	(02°)	$\wedge^n(p_1,\dots,p_n) = \wedge^{n-1}(p_1,\dots,p_{n-1}) \vee p_n$	(15°)
$\vee^2(p,q) = p \vee q$	(03°)	$\wedge^n(\dots,p_i,\dots,p_j,\dots) = \wedge^n(\dots,p_j,\dots,p_i,\dots)$	(16°)
$\vee^n(p_1,\dots,p_n) = \vee^{n-1}(p_1,\dots,p_{n-1}) \vee p_n$	(04°)	$\wedge^n(\dots,p_i,\dots,\sim p_i,\dots) = \sim 1$	(17°)
$\vee^n(\dots,p_i,\dots,p_j,\dots) =$ $= \vee^n(\dots,p_j,\dots,p_i,\dots)$	(05°)	$p_i \supset p_j \Rightarrow \wedge^n(\dots,p_i,\dots,p_j,\dots) =$ $= \wedge^{n-1}(\dots,p_i,\dots,,\dots)$	(18°)

$$\wedge^{n}(\vee^{m_1}(p_{1_1},\dots,p_{1_{m_1}}),\dots,\vee^{m_n}(p_{n_1},\dots,p_{n_{m_n}})) = \bigvee_{\substack{i_1=1\\ \dots\\ i_n=1}}^{\substack{m_n\\ \dots\\ m_1}} (\wedge^{n}(p_{1_{i_1}},\dots,p_{n_{i_n}})) \qquad (06°)$$

$$\wedge^{n}(\dots,\sim 1,\dots) = \sim 1 \qquad (07°)$$

$$\vee^{n}(\dots,p_i,\dots,\sim p_i,\dots) = 1 \qquad (08°)$$

$$\vee^{n}(\dots,1,\dots) = 1 \qquad (09°)$$

$$\vee^{n}(\dots,\sim 1,\dots) = \vee^{n-1}(\dots,,\dots) \qquad (10°)$$

$$\sim(p \vee q) = \sim p \wedge \sim q \qquad (11°)$$

$$\wedge^{0} = 1 \qquad (12°)$$

$$\wedge^{1}(p) = p \qquad (13°)$$

$$\vee^{n}(\wedge^{m_1}(p_{1_1},\dots,p_{1_{m_1}}),\dots,\wedge^{m_n}(p_{n_1},\dots,p_{n_{m_n}})) = \bigwedge_{\substack{i_1=1\\ \dots\\ i_n=1}}^{\substack{m_n\\ \dots\\ m_1}} (\vee^{n}(p_{1_{i_1}},\dots,p_{n_{i_n}})) \qquad (19°)$$

$$p_i \supset p_j \Rightarrow \vee^{n}(\dots,p_i,\dots,p_j,\dots) = \vee^{n-1}(\dots,,\dots,p_j,\dots) \qquad (20°)$$

$$\wedge^{n}(\dots,1,\dots) = \wedge^{n-1}(\dots,,\dots) \qquad (21°)$$

$$\sim(p \wedge q) = \sim p \vee \sim q \qquad (22°)$$

$$\sim\sim p = p \qquad (23°)$$

$$p \supset q = \sim p \vee q \qquad (24°)$$

$$p \equiv q = (p \supset q) \wedge (q \supset p) \qquad (25°)$$

An example of incursive proof with the previous inference schemes is given below:

$p \supset (q \supset p)$	
$\sim p \vee (\sim q \vee p)$	(24°)
$\vee^{3}(\sim p, q, p)$	(04°)
1	(08°)

This proof is completely reversible, in fact:

1	
$\vee^{3}(\sim p, q, p)$	(08°)
$\sim p \vee (\sim q \vee p)$	(04°)
$p \supset (q \supset p)$	(24°)

is also a proof. The hyperincusive nature of rules 1°),…,25°) is evident[8].

4 Conclusion

We have proved the possibility of a hyperincursive proof theory which is perfectly coherent with standard proof. The greatest advantage of a hyperincursive

[8] The incursive nature of the rules 1°),…,25°) is evident because they are totally reversible and therefore they generate proof totally reversible: thus anticipatory proofs (proofs which start from theorems which have to be proved) can also be generated.

proof theory is the possibility of building a proof by starting from the theorem instead of from the premixes. But this is the real application of the logic. In concrete development of sciences, we do want to generate theorems from premixes in a random way[9] which is the only way that natural deduction permits, but we want to verify if a given wff is a theorem in our mathematical theories. Hyperincursive proof theory permits this.

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[9] Observe that we cannot anticipate what theorem we will proof when we apply Gentzen's rules for natural deduction. Instead if we must prove a given *a priori* theorem, to find their proof is an anticipatory calculus.