Indecidability and Incompleteness In Formal Axiomatics as Questioned by Anticipatory Processes

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Abstract
Hilbert's conjecture that the whole of mathematics could be provided by a finite set of axioms (Hilbert, publ. 1980) was challenged in branches of mathematics, devoted to arithmetics and algorithmic computation, by Gödel (1931), Church (1936), Turing (1937), and Chaitin (1998). This questioned what can be expected from scientific knowledge, in particular through the mesh of mathematical certainty, in the assessment of what could be considered true about our universe, that is also on ourselves via self-evaluation possibility.

This study will thus revisit some current problems about the conditions required for allowing a measure of "something" likeky unknown, situated "somewhere", in terms of distances and dimensions. The debate will then focus on the scope of mathematical knowledge, with special regards to indecidability, incompleteness, and the fate of such mathematical realities claimed to escape the field of mathematics, like for Chaitin's 'omega number'. The formal involvement of anticipatory processes in finding solutions through biological self-evaluation will be analyzed in several steps.
Keywords: Availability-axiom; Functionality-Axiom; Compatible-intersection; Exploration-function ; Power-set of parts.

1 Introduction

A reasoning system is frequently identified with a logics, that is a collection of rules conventionally adopted for the assessment of the structure of a proposition, its method of deduction, and the proof of its validity (Carnap, 1958 ; Adamowicz and Zbierski, 1997, Malatesta, 2000, Tymoczko and Henle, 2000). The simple system of 'three valued' logics uses three possible issues : true, false and undecided. The output 'true' is eventually contained in a [0,1] interval in fuzzy logic (Zadeh and Kacprzyk, 1992), while 'false' nor 'undecided' differ from 'nonexisting' as analyzed by Bounias and Bonaly (1997a). A reasoning system is not a single nor a definitively fixed system (Kac and Ulam, 1992). However, an excess of symbols and abstract use of these symbols, as in Whitehead and Russell (1925), has been criticized (Jeffrey, 1967) and considered to be "bogging down" and "unilluminating" (Weisstein, 1999a). While Hadamard admitted that one who his 'well-reasoning' essentially feels the problem like him (cited in Gonseth, 1926), Lebesgue thought that some psychological factors lurk in the acceptance of a reasoning, while logic could only provide reasons for rejecting a reasoning (cited in Dugas, 1940). Thus, logic must use symbols with unambiguous meaning, and devoid of contradiction precluding the decidability of derived propositions, like for the usual theorems of predicate calculation. A demonstration can be direct, or *ad absusdum*, or recurrent, up to transfinite induction. However, the latter are rejected as 'not constructive' by some mathematical schools, like Brouwer's intuitionism (Largeault, 1992). Even the Bourbaki group has been charged with lacking of interest for the logic branch that it considered rather exterior to mathematics (Mashaal, 2000). Finally, truth might well be situated beyond these extremal positions.

In this study, the classical symbols of set theory will be adopted (regardless of some redundancies) : ($\wedge\vee,\forall,\exists,\neg,\cap,\cup,\supset\ \supseteq,\not\subset\ \subset\ \subseteq,\in\ \notin,\Leftrightarrow\ \Rightarrow$). The additional sign ($\angle$ X) will be used for : "given some X", or "in presence of X", in complement to $\forall$ and $\exists$. The concept of 'compatible-intersection' noted ($\bar{\cap}$) will be defined below (definition 2.1).

Axioms have been defined, from Euclid to Hilbert, as self-evident propositions, or primary truths accepted without proof, or propositions adopted conventionaly, even without need for

having any link with any reality. What an axiom should be has sometimes been nearly ignored even in articles dealing with axiomatics (Fraïssé, 1982).
An explicit **definition** contrasts with an implicit definition resulting from reciprocal relations in a system. A concept, or "definiens", involves a justified formulation by use of a "definiendum", composed of justifying concepts. Since a definition is also a kind of proposition, predicate or axiom, it should be given together with its domain of validity.
A **theorem** is a proposition or property which can be demonstrated to be true within a given system. The proof is usually considered as a sequence of signs and symbols connecting the objects involved as well as the axioms accepted. The concept of proof thus involves an ordered sequence of structures and rules applying to some sets.
A **law** or **combination rule** is usually noted ($\perp$, T, o,...). It can be internal, i.e. working within one given set (E), or external, i.e. extended to, or involving, outside sets in the contitution of the domain. Such combination rules as ($\cap,\cup,\supset\ \supseteq,\not\subset\ \subset\ \subseteq,\in\ \notin$) provide a set (X) with a structure, either directly or indirectly ; e.g. ($\subset$, $\not\subset$, or $\in\ \notin$) induce the complementarity property ($\mathbf{C}$). However, ($\not\subset\ \subset\ \subseteq,\in\ \notin$) can also appear as a given structure *per se* . A set (X) provided with a combination rule ($\perp$) is called a mathematical **magma**.
A **structure** $S(X,\perp)$ is usually provided on a set X by combination rule ($\perp$), or given as a primary property (Sx). Laws ($\subset\ \subseteq,\cup$) provide a set with parts of itself, owning the structure of a Boolean lattice. Algebraic structures come from defined combinations of both laws and structures. Laws ($\cap,\cup$) provide order relations and topological structures. Some relations ($\mathcal{R}$) can be considered as either structures or combination rules, and the distinction may be also vanishing for mappings. However, it is worth mentioning that a structure is provided by an operator (i.e. a combination rule), itself considered as a mathematical object, e.g. a group (Dieudonné, 1982). A set provided with a structure is called a mathematical **space**.

2 An extended form of the assessment of truth

Proposition 1. Parts of a magma (resp. a space) becomes a "**space of magmas**" iff the rule (resp. the structure) provided to the magma (resp. space) is explicitly intended to be operating (resp. applying) on these parts. This '**axiom of functionality**' will be noted Af.
Justification: spaces and magmas still are specifically and independently designated (Bourbaki, 1990; Chambadal, 1981); thus, sets of mathematical objects can exist as not-functional collections. The expression 'space of magmas' will then be sometimes used to denote functional associations of mathematical objects, e.g. $S=\{X, (\perp, Af)\}$.

Proposition 2.1. A logic system is a variety of space of magmas.
Proof. (i) Logics uses symbols equivalent to combination rules. (ii) Logic applies on objects that can be gathered in sets (X). If (X) is the collection of all elements that are compatible with a logic ($\mathcal{L}$), the corresponding reasoning system is valid for any $x \in X$ and illicit for a $y : (y \in Y \mid X \cap Y = \emptyset)$. Moreover, collections of signs and symbols, rules, definitions and theorems can be symbols, rules, definitions or theorems. An axiom can be composed of a group of axioms, provided the latter are independent from one another. Therefore, a logical system is a space. Since its combination rules applying to its own sets provide itself with a structure, a reasoning system is composed of spaces of magmas. A proof is a ordered sequence if any rule component is non-commutative.

Definition 1. A '**compatible-intersection**' of spaces of magmas is a space of magmas noted ($\bar{\cap}$) such that, given sets (X,Y), rules ($\perp$,T) : (i) members or parts of a set X or Y may obey members or parts of both of rules ($\perp$) and (T) ; or (ii) members or parts of a rule ($\perp$) or (T) may apply to members or parts of both sets (X) and (Y). Then a proposition or predicate P belongs to a compatible intersection of (X,Y, $\perp$,T) if :

$$P \in \{(\perp \bar{\cap} T)(X \bar{\cap} Y)\} \Leftrightarrow \{\exists\, x\in X, y\in Y, \perp(x)\wedge\perp(y)\neq\emptyset \text{ or } T(x)\wedge T(y)\neq\emptyset \} \text{ and:}$$
$$\{\exists\, \perp_i\in\perp, T_j\in T, \perp_i(x)\wedge T_j(x)\neq\emptyset \text{ or } \perp_i(y) \wedge T_j(y)\neq\emptyset \} \qquad (1)$$

Symbol $\bar{\cap}$ thus denotes shared properties rather than just common membering.

Proposition 2.2. A mathematical truth is included in the compatible-intersection of two spaces of magmas. It constitutes a collectivizing relation (Bourbaki, 1990a, II.3).

Proof. Let a logic system be depicted by $\mathcal{L}=\{X,\perp,S_{(X,\perp)}\}$, and a space of magmas denoted by $E = \{Y,T,S_{(Y,T)}\}$. Then, a proposition or property P must involve members of $X \,\overline{\cap}\, Y$ provided with rules included in $\perp \,\overline{\cap}\, T$ and providing the resulting space with a structure either belonging to $S_{(X,\perp)} \,\overline{\cap}\, S_{(Y,T)}$ or having a nonempty intersection with it.

$$P \in \mathcal{L} \,\overline{\cap}\, E \Rightarrow P \in \{X, \perp, S_{(X,\perp)}\} \,\overline{\cap}\, \{Y, T, S_{(Y,T)}\} \qquad (2.1)$$

However, it will be further argued that a proposition P infering from such an operation may be contained in neither $\mathcal{L}$ nor E. Hence :

$$\angle\, (\mathcal{L},E), \exists\, P : (P \notin \mathcal{L}) \wedge\vee\, (P \notin E) \qquad (2.2)$$

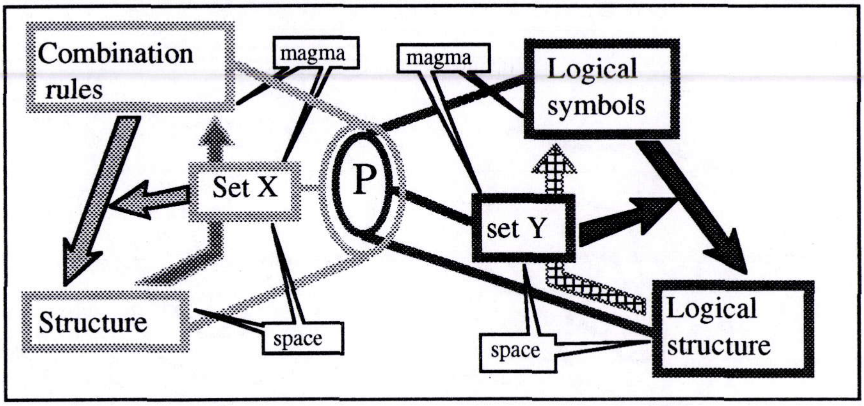

Figure 1: Illustration of the position of a proposition at the compatible-intersection of two spaces of magmas. A structure holding on a defined set implies back a combination rule to be involved such that the corresponding space provides the magma formerly considered.

The system, illustrated by Figure 1, shows that a proposition, that is also a theorem or a property, can appear in either the class of sets, structures or combination rules, or associations of them taken in appropriate combinations. A space like $\mathcal{L} \,\overline{\cap}\, E$ giving a validity to a proposition P has been called a **probationary space** (Bounias, 1997). In the classical sense, it would correspond with a 'truth-set' as "a set of all objects in the domain of a proposition P for which the value of P is a true proposition" (James and James, 1992, p. 433). Thus, there is no absolute truth : a proposition gets the status of truth once it has been shown to be compatible with the components of the space of a logical system (as a domain space), applied to a space of magmas (as the range). The identification of the probationary space fulfilling this compatibility is called a demonstration. Finally, there is no axiom out of identified members of the probationary space making true some proposition. Sequences of intersections thus hold instead of sequences of sentences. Accordingly: (i) two logical systems such that a theorem is true in one and indecidable in the other cannot be said "equivalent", in contrast with a statement by Kolata (1982) about the Paris-Harrington model; (ii) if sequences of sentences are indexed on time, then given axioms can be located in the past and the theorems to be proved in the future, as hypothesized in a former attempt to connect decidability with incursivity (Grappone, 1999); (iii) both axioms and theorems could be placed in either one or the other position, due to relativity of the whole system of spaces of magmas. Axioms may thus have to be identified rather than posed.

Corollary 1. An indecidable becomes decidable through involvement of an appropriate anticipatory process.

Preliminary proof. Let P a property true for a $L \,\overline{\cap}\, E$ system as described in the first section of this study. Then, an axiom or an axiomatic system (Ax) is anything else than P contained in $\{L \bar{\cap} E\}$. Would an axiom A(P) in the classical sense, be needed for proving P, this would mean that A(P) was not previously contained in P, that is :

$$A(P) \subseteq \{Ax = \mathbf{C}_{L \bar{\cap} P}(P)\} \qquad (3)$$

Thus, there can exist an unknown part $B \subset Ax$ such that $A(P) \not\subset B$ and $P \in B \bar{\cap} A(P)$: in effect, it suffices that the proof of P is needed to allow the identification of B (e.g. $p,q \in P$, $P(A) \Leftrightarrow p(A)$, $P(L \bar{\cap} E) \Leftrightarrow p(A) \vee q(B)$). Then, P cannot be found through a mechanical procedure. The solution comes from an anticipatory process by which the brain of the operator foresees inside a probe system different from the system studied how to identify B without proving P : this imposes that there exists L' and E' such that $(L' \bar{\cap} E') \supset (L \bar{\cap} E)$, and $A(P) \bar{\cap} \mathbf{C}_{L' \cap P'}(L \bar{\cap} E) \Rightarrow P$. (QED)

Consequently, P could be considered true as a predicate but incomplete as a axiomatic system in $(L \bar{\cap} E)$. A general configuration is given by Figure 2.

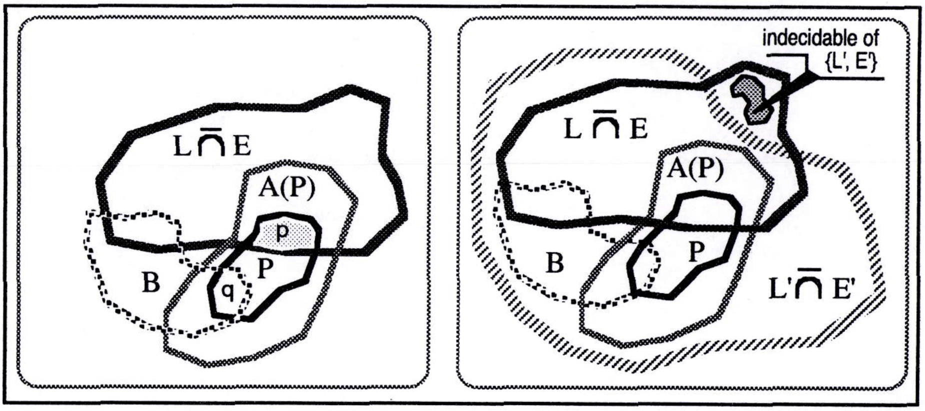

Figure 2: A formal space of magmas $\{L \bar{\cap} E\}$ includes a proposition $p \in P$ upon acceptance of an axiom A(P). However, for P to be decidable, one must know B which can be known only if the system is studied within a higher space $\{L' \bar{\cap} E'\} \supset \{L \bar{\cap} E\}$. This may occur if for instance mappings of q on p are surjective. An axiom or property may thus be at least only partly valid in another axiomatic system larger than the initial one.

Corollary 2. Extending a formal system from a component X to Y such that $Y \supset X$, and X does not imply Y by a recursive process, is a not computable, anticipation-dependent system.

This should not be confused with an extension of a consistent axiomatic system. In effect, extending some members of a $(L \bar{\cap} E)$ to $(L' \bar{\cap} E') \supset (L \bar{\cap} E)$ may lead to some properties of $(L' \bar{\cap} E')$ which are not valid for $(L \bar{\cap} E)$.

Examples are given by properties of a 4-manifold specifically not valid for n-manifolds ($n \neq 4$) (see Freedman and Yau, 1983 ; Donaldson, 1983). More simply, imagine we live in a 2-D universe (like in Abbott's "Flatland" or in Dewdney's "Planiverse"). Then there is no way for a continued arc (Γ) to escape the interior of a Jordan's curve (O) or a closed set (Θ) without intersecting (O) or the frontier of (Θ). The solutions need that a universe is conceived, other than the real one, in which the "flat programmers" would try to set up a mechanical probing system. Since this probing universe would need $n \geq 3$-D, not readily available for constructing an appropriate algorithmic device, then an anticipatory process would be required. The latter is supported in biological brains by mental images, and sometimes called "intuition", "imagination", or "creativity". After solving the immediate problem, it can then eventually drive back to a principle allowing a machine to be constructed to treat similar questions. This is analogous with the introduction of an anticipatory parameter in a computer, with a former mental anticipative operation further converted into a incursive computational form only if the incorporation of a higher

dimensional embedding space is compatible with a computer's capability. This imposes that ($n>1$) dimensions can be processed through a (1-D)-dimensioned system, which is made available below by relation (8.4). Conversely, a property may vanish through extension of the set of the space E of a ($L \bar{\cap} E$) system : e.g. the Hausdorff paradox states that for $n \geq 3$, there is no additive measure both finite and invariant of the group of translations in R^n.
All these situations involve qualitative-like jumps during a quantitative-like extension of a simple parameter of a system, which may be hardly managed on computer and likely provide a clue for indecidability problems.
The above argumentation supports and generalizes a "sentence-based" conjecture that the Gödel theorem could be invalid within incursive arithmetics (Grappone, 2000). A further argumentation will involve below the power set of parts of a formal system.

Proposition 3. An axiom is the ultimate undecidable of a formal axiomatic system.
Proof. Let ($\mathcal{L}$) a logical system of reasoning combined with a probationary space (E), and (A) an axiom in the classical sense (note that usually, logics is rather implicitly than explicitly included among the components of a classical formal axiomatic system). When (A) is added to $\{(\mathcal{L}) \bar{\cap} (E)\} = \mathsf{F}$, the extended system $\{\mathsf{F}\}\cup(A)$ leads to decidable propositions (P) and to some indecidable propositions (Q). Then, some like Chaitin (see above) suggest that Q should be included in A. Thus $\{\mathsf{F}\}\cup(A)$ leads to $(P)\cup((A\cup Q)=Q)$.
In effect : $P \subset [\{\mathsf{F}\}\cup(A)]$ and $Q \subset [\{\mathsf{F}\}\cup(A)\cup(Q)]$, thus: $Q \subset [\{\mathsf{F}\}\cup\{A,Q\}]$ (QED)
Therefore, elevating a so-called 'not-demonstrable truth' into an additional axiom of an axiomatic system (which then would get the status of a "metatheory", following Mirita, 2000) is a particular corollary of Proposition 3.

3 About self-evaluation and the concepts of 'measure'

A biological system gathers a set of perceptions from the outside world (through Jordan's points converted into fixed points standing for mental images with a perception) with a perception from its inside (through sets of Brouwer's fixed points also translated into mental images) (Bounias, 2000a). Direct or device-mediated brain measurement on a unknown space first implies the evaluation of distances and of dimensions.

3.1 Generalization of distances

Let E be a not totally ordered space, and A,B,C, ... subspaces in E. The symmetric difference $\Delta(A,B) = \complement_{A\cup B}(A\cap B)$ has been proved to be a true distance also holding for more than two sets (Bounias and Bonaly, 1996; Bounias, 1997,2000a). However, if $A\cap B=\emptyset$, this distance remains $\Delta = A \cup B$ regardless of the situation of A and B within an embedding space E such that $(A,B) \subset E$. Thus the following extensions, with X denoting the set of subparts of E having nonempty intersections with both A and B:

$$\Delta_E(A,B) \subseteq \Delta(A,X) \cup \Delta(B,X) \quad (4.1)$$

This form contains two components: $\Delta(A,B)$ called the **intrinsic distance** of A,B and

$$\Lambda_E(A,B) = E \cap \{\Delta(A,X) \cap \Delta(B,X)\} \quad (4.2)$$

called the **separating distance** of A,B. Thus (Fig.3):

$$\Delta_E(A,B) \subseteq \Delta(A,B) \cup \Lambda_E(A,B) \quad (4.3)$$

3.2 An exploration function as a general form of measure

Our approach aims at identifying distances that would be compatible with both the involved topologies and the scanning of objects not yet known in the studied spaces. No such configuration is believed to be an exception nor a general case.
A path $\varphi(x,y)$ such that $\varphi(0) = x$ and $\varphi(1) = y$ (Weisstein, 1999b), may provide a more general approach than Borel, Hausdorff, Fréchet, and other distances as reported by Choquet (1984), Tricot (1999b) and others (see Chambadal, 1981; James and James, 1992).

Proposition 4.1. A set can be scanned in unique way by the composition of a identity function with a difference function as below. Let $E = \{a,b,c,...\}$ a set having N members.
(i) An identity function Id maps any members of E into itself : $\forall x \in E$, $Id(x)=x$. Thus, given (a, or b, or c, ...) this provides one and only one response when applied to E.
(ii) A 'strict' difference function is f such that : $\forall x \in E$, $f^n(x) \neq x$.
The complete exploration function is a self-map M of E : $M : E \mapsto E$, $M = Id \perp f$:

$$\angle x \in E , M^1(x) = f(Id(x)), \forall n : M^n(x) = f^n(Id(x)) \neq x \tag{5.1}$$

Proof. (i) Suppose $M=Id(x)$: then, each trying maps a member of E to a fixed point and there is no possible scanning of E. (ii) Suppose one poses just not strictly $f(Id(x)) \neq x$: then, given $f(Id(x)) \neq x$, say $f^1(Id(x) = y$, since $y \neq x$, one may have again $f^2(Id(x) = x$. Therefore, there can be a loop without further scanning of E, with probability $(N-1)^{-1}$. (iii) Suppose one poses $M = f$, such that $f^n(x) \neq x$. Then, since $f^0(x) \neq x$, there can be no start of the scanning process. If in contrast one accepts as a property of the function f that $\angle x$, $f^0(x) = x$, this again stops the exploration process, since then $f^0 \in Id$.
The sequence of functions $M^n(x) = f^n(Id(x)) \neq x$, $\forall n$, is thus necessary and sufficient to provide a measure of E which scans N-1 parts and/or members of E. The sequence stops at the Nth iterate if in addition :

$$f^n(Id(x)) \neq \{f^i(Id(x))\}_{(\forall i \in [1,N])} \tag{5.2}$$

The described sequence thus represents an example of a path as described above in more general terms. (QED)

Proposition 4.2. A general distance between spaces A,B within their common embedding space E is provided by the intersection of a path-set $\varphi(A,B)$ joining each member of A to each member of B with the complementary of A and B in E, such that :
$\varphi(A,B)$ is a continued sequence of (iterations of a) function(s) f of a gauge (J) belonging to the ultrafilter of topologies on {E, A, B, ...}.
The path $\varphi(A,B)$ is a set defined as follows on a sequence interval $[0, f^n(x)]$, $x \in E$:

$$\varphi(A,B) = \bigcup_{a \in A,\, b \in B} \varphi(a,b) \mid \Lambda_E(A,B) \subseteq \varphi(A,B) \tag{6}$$

Remarks. (i) Let E° be the interior of E: $\min\{\varphi(A,B) \cap E^\circ\}$ is a geodesic of space E connecting A to B, and $\max\{\varphi(A,B) \cap \complement_{E^\circ}(A \cup B)\}$ is a tessellation of E out of A and B.
(ii) A gauge must belong to the ultrafilter of the embedding space. In effect, filtering conditions encompass intersection properties (work in progress).
(iii) The measure of a open subspace C is achievable iff the dimension $Dim(\varphi)$ of the path differs from Dim(C). $Dim(\varphi)>Dim(C)$ corresponds to a outer Lebesgue measure (here, a point or singleton $\{x\}$ would be measurable even if it is in the configuration of a open), and $Dim(\varphi)<Dim(C)$ to a inner (e.g. Jordan's) measure. $Dim(\varphi)=Dim(C)$ meets Borel's conditions.
(iv) Particular case of a totally ordered space: let A and B be disjoint segments in space E. Let E be ordered by the classical relations: $A \subset B \Leftrightarrow A \prec B$ and $(A,B) \subset E \Leftrightarrow E \succ A, E \succ B$. Then, E is totally ordered if any segment owns a infimum and a supremum. Therefore, a distance (d) between A and B is represented by the following relation :

$$d(A,B) \subseteq \text{dist}(\inf A, \inf B) \cap \text{dist}(\sup A, \sup B) \tag{7}$$

with the distance (dist) evaluated through either classical forms or even the set-distance $\Delta(A,B)$ which itself implies that $\{A,B\}$ is a partly ordered space.

Corollary 3. The former Bonaly's conjecture (Bonaly, 1992) that a physical space should be topologically closed can be tentatively extended to the following: 'An abstract space gets the status of a physical space if any of its parts can be scanned by using a measure whose intersection with this part, be it open or closed, is a closed structure'.

This extended conception would encompass topologically open parts eventually lurking in our spacetime and thus escaping 3-D probes: this would provide a status to the still missing "dark matter" (Maiani, 2000).

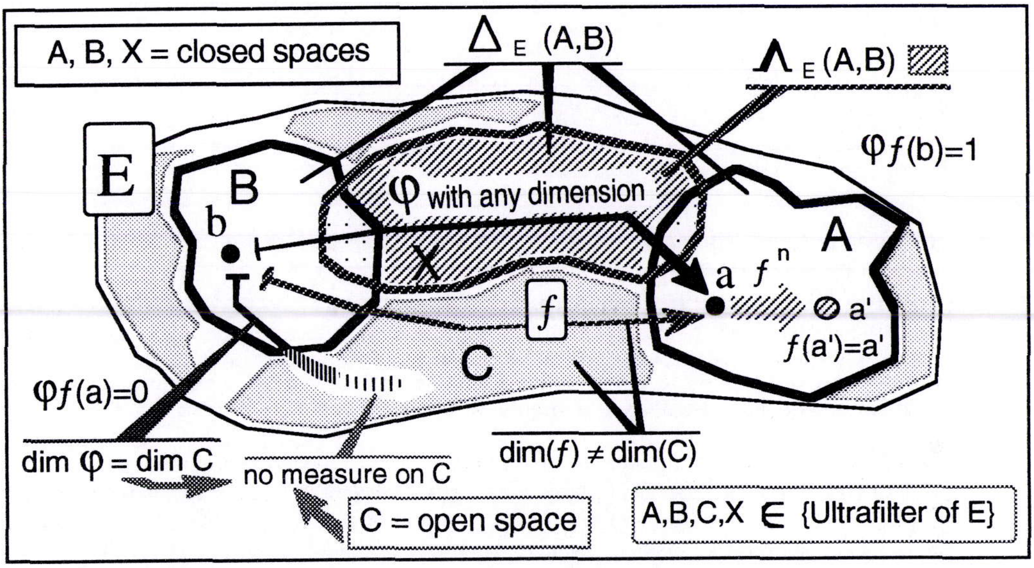

Figure 3: Topological constraints on measuring a distance in a complex space, through the function of perception (here the perceiver is A with f^n a neuronal chain).

3.3 Dimensional assessment

A generalization of the triangular inequality holds for a space X being a N-object:

$$M(A^k_{max}) \prec \bigcup_{i=1}^{N-1} \{M(A^k_i)\} \Leftrightarrow Dim(X)=d>k \qquad (8.1)$$

with N = number of vertices, i.e. eventually of members in X, $k \geq (d-1) = N-2$, and A^k_{max} the k-face with maximum size in X (still the former K-simplex itself), (A^k_i) its complementaries in the whole simplex.
Fulfilment of the nonequality condition for the probe and the scanned space can be achieved with a 1-D simplicial gauge composed as follows for D>1 spaces.
A n-simplex is composed with a set (E_{n+1}) and a combination rule $(\perp)$ providing it with the structure of dimension n. Hence : $S^n = \{(E_{n+1}), (\perp n)\}$. This confers a simplex the status of a space of magma. Space $S_n^d = \{(E_{n+1}), (\perp d)\}$ becomes a simplex iff d=n. Since it represents the smaller space allowing a given dimension to be assessed, it will be called a **'simplicial canonical space'** (S_n^n) whose rule $(\perp n)$ is defined as follows.

Definitions 2. A n-(simplex-ball) is a connected topological unit ball circumscribed to a n-simplex. For any one of its points, there exists n other points in it forming a n-simplex.
Let the set (E_n) be the component of a simplex $S_{n-1}{}^{n-1}$.
A **'loop distance'** is defined as the sum of distances between all vertices $\{x_1, x_2, ..., x_n\}$ of (E_n) from the first point back to itself :
$\mathbf{L_n{}^1} = L^1(E_n{}^n) = dist(x_1, x_2)+dist(x_2, x_3)+...+dist(x_{n-1}, x_n)+dist(x_n, x_1)$
or more generally :

$$\mathbf{L}_n{}^1 = \bigcup_{\substack{i=1 \to n-1 \\ j=2 \to n}} \{dist(x_i, x_j), dist(x_n, x_1)\} \tag{8.2}$$

Let a n-simplex $S^n = \{(E_{n+1}), (\perp n)\}$: the **'starred distance'** $¥^1{}_{n+1}$ is defined by the sum of distances of the last point x_{n+1} called the 'central', to each of the other points :
$¥^1{}_{n+1} = \{dist(x_{n+1},x_1)+dist(x_{n+1},x_2)+...+dist(x_{n+1},x_n)\}$ that is also:

$$¥^1{}_{n+1} = \bigcup_{i=1 \to n} \{dist\ x_{n+1} \to x_i)\} \tag{8.3}$$

Conjecture 1. An assessment of the dimension of a space $S^d{}_n = \{(E_{n+1}), (\perp d)\}$ can be provided by a relation of the following type :

$$(\perp d) = \{(¥^1{}_{n+1} > k(n).\mathbf{L}_n{}^1 \Leftrightarrow d>n-1)\}_{n=2\to d} \tag{8.4}$$

Tentatively, for n>1 : k(2)=1/2, for n>2 : k(3)=2/3, while for n>3, k(4) and its generalization to $k(n) \propto (n-1)/n$ is under consideration. This approach conjecturally encompasses the Lebesgue cover-dimension through iteration of n-conditions over (n = $2 \to d$) for a sequence of emboxed n-simplices.

Corollary 4. Consider a number (n) of sets $\{A_i\}_{(i=1\to n)}$ embedded within a set E. Let $E \supseteq \{A_i\} \cup (D)$ where (4.1-2) : $\Lambda_E(\{A_i\}) = E \cap \{\cap\Delta(\{A_i\},D)\}$. Then E is comparable with a simplicial canonical ball ($\mathcal{B}^n$) where$\{A_i\}$represents the loop of a (n-1)-simplex and $\Delta(\{A_i\},D)$ a starred-like distance. Hence, $\Lambda_E(\{A_i\})$ is analogous with $\Lambda_{\mathcal{B}^n}$ $(L^1{}_n)$.

3.4 Simplest cases of incompleteness and indecidability

Let set E = {a,b,c} be analyzed with the *f*(Id) process of measure described above, which needs a formal system with essentially the following components :

$Ax = \{X=(E,[1,2,3]), \perp = \{f\colon x \mapsto f(x), =, \neq, \in\}\}$.

The result will be either {b,c} or {a,c} or {a,b}. Without a substantial supplement of both the definition set and the combination rules, not formally predictable by recurrence in Ax, the result is thus necessarily incomplete. For allowing completeness, one should accept to include in Ax $f(Id(x)) \neq x$ instead of $f^n(Id(x)) \neq x$. However, in this case there is a sequence of probabilities (here p_i=0.5) that the system stops before having scanned all the members of E : $f(a)=b$, $f(b)=a$, and the system may fall into a loop and thus give a false response. Thus classical indecidability may fulfill the recurrent demonstration principle: if the impossibility of finding all theorems of a system is true, it may be true for a small as well as for a large number of axioms. The case of a small system emphasizes the role of anticipation: here there should be a outer set F from which upon connectivity of an embedding space, each member of E would be surjectively mapped as image of a member of F. But the structure of F should involve a number of members larger than in E, which is not predictable from the system Ax since it cannot explore E with certainty. Indeed, the set of parts of E fulfills the qualification for F, but while it exists, it cannot be known if E is not elucidated before. Thus, indecidability is again proved in a recursive system. A further argument would be that the content of E is assessed by CardE=(n+1). However, this is precluded by the fact of whatever the first term (x) taken by Id(x) in f^n(Id(x), the internal structure of (x) is not known. Not only it is not possible to assess what are the parts composing (x) (this would just repeat for (x) the same operation as for E), but if (x) is a singleton {x}, the problem becomes even more unsolvable, as it will be shown below. Again some imaginative anticipation would be needed in order to provide a general solution. Similarly, the above system cannot verify whether the obtained path f^n(Id(x)) is actually a geodesic, excepted through an anticipatory process.
In a impressive effort, Chaitin (1998, p.20) has deviced a monstruous system as a diophantine equation of 17,000 variables written over 200 pages, which shown typical

indecidability. Now consider the above set E={a,b,c} reduced to E'={a,b}. Apply the f^n(Id) method : it scans one member. Let us examine the case (a=b) : then the system cannot return a solution although it includes just no more, no less than its initial axioms. Incidentally, it is even more strictly incapable of deciding whether $f^n(Id(x))$ is a geodesic. This minimal formal case is illustrated by the example of tossing a strictly symmetrical coin, i.e. without head nor tail engraved on its sides.

Remark. The complexity of a computer program, in algorithmic information theory, has been identified with the shortest program giving the expected output (Chaitin, 1998-1999). The distance proposed here also applies to such cases. {p} is a set of N members written say in binary digits. Hence : G = {(0,1), p} is a set E={0,1} provided with combination rule p : thus, G is a magma. If p provides it with some kind of structure, G candidates for a space of magma. Its diameter can be derived from relation (4.2) :

$$\max\{(li,lj) \subset p \mid m \in \{0,1\}, \Lambda_E(li, lj)\} = \text{length } L(p) \tag{9.1}$$

Let inf (L(p)) the shortest program. Then, in terms of a scanning function, one has:

$$\inf. (L(p) = \min\{\varphi(li,lj) \cap p^{\circ}\} \text{ is a geodesic of space (p)} \tag{9.2}$$

Therefore, an elegant program in the sense of Chaitin is a geodesic in the sense of our generalized distance.

4 Axiomatic limits on knowability revisited

4.1 About some paradoxes and their formulation

Following the metamathematical results of Gödel (1931), Church (1936), and Turing (1937), Chaitin (1998) more recently claimed he had "exploded Hilbert's dream" of finding "a small finite set of axioms and rules of inference that we could all agree on, from which all the infinite mathematical truth would follow". In its algorithmic information theory, Chaitin (1999, p. 84) argues that there are irreducible mathematical facts that can only be deduced by adding them as axioms, and he even adds " So, not only Hilbert's faith in the axiomatic method was wrong, in some cases it was **completely** wrong (Chaitin's emphasis). Because to say that some mathematical truths are irreducible means that they cannot be compressed into axioms at all, they cannot be deduced from any principles simpler than they are". Here, "simpler" may be a key-word of the sentence.
The Russell and Berry's paradoxes play with ordinary sentences that they confuse with formal definitions of constructibility (see Chaitin, 1999, p.9, or Grappone, 2000 for recent examples). "The first ordinal that cannot reside within less than 'twelve' words" is addressed in this sentence with eleven words (or even less if some unnecessary components of the sentence are neglected). However, what is given is a property of something not well defined, neither in the concerned number nor by the sentence words. Since numbers 'ten', 'fifty' or 'one' all take the same number of words (here: one), the system is biased by lack of correct foundation of order relations and mappings.
Gödel's completeness theorem states that given (T) an axiomatic system in 1st order language, and P a proposition valid for any structure consistent with (T), then P can be formally deduced in a definite way from (T) (Weisstein, 1999c). Here, the concept of axiom needs to be revisited as in relation (1) : $P \in \mathcal{L} \,\bar{\cap}\, E \Rightarrow P \in \{X, \perp, S(X,\perp)\} \,\bar{\cap}\, \{Y, T, S(Y,T)\}$. If $M \subseteq \mathcal{L} \,\bar{\cap}\, E$, then since logic $\mathcal{L}$ and space E are defined, for any property $P' \in M$, parts of $\mathcal{L}$ and E involved in the assessment of P' are contained in ($\mathcal{L} \,\bar{\cap}\, E$), that is: $\{\mathcal{L}' \,\bar{\cap}\, E'\} \subseteq \{\mathcal{L} \,\bar{\cap}\, E\}$. This meets the property of a gauge which must belong to the ultrafilter of an embedding space E if one expects that a measure will work for the evaluation of the distance of A and B within E.
Some other paradoxical sentences deserve attention, such as : P = "I am a liar" (Chaitin, 1998, p. 12). One analysis of this situation is a loop, e.g. : P ⇒ I lie in saying I am a liar, ⇒ thus I am not a liar ⇒ thus I did not lie in saying I am a liar ⇒ thus I told a truth in saying I am a liar ⇒ thus I lied in further saying I am not a liar ⇒ thus I am a liar ⇒ start of a second loop, etc.

In an algorithmic system, if the computer program halts when an output apparently confirms the proposition, then the program eventually halts before the second loop starts (e.g. "I told a truth..."), excepted if it takes the output as a statement to be evaluated.

4.2 Anticipatory computation versus spontaneous self-anticipation

The opposite propositions cited above just came alternatively. This can be compared with Turing's system : if a subprogram H of P checks for the validity of P, P must evaluate H for H can in turn evaluate P and the program goes into an infinite loop. If H says that P will not halt, but since $H \subset P$, H must not halt. Then, Chaitin (1999, p. 16-17) wrote that "there is no algorithm, no mechanical procedure, no computer program that can determine *in advance* if **another** program will ever halt". What Chaitin emphasized in bold character was "another": but the expression "in advance" (my emphasis) deserves even more attention since it means that such an algorithm is not an anticipatory system. This may conflict with the concept of computed anticipatory processes, where an instruction about p(t+1, t+2, ...) can be inserted in the definition of p(t). However, while such an instruction makes the system anticipation-dependent, an instruction at step (t_0) contains an instruction for (t_0+kt) which has been introduced before: thus since the anticipation has been programmed, it becomes preexisting. This contrasts with a mental image of the future, spontaneously foreseen by the brain, while the reality (whose successive images are equivalent to the computer iterations) is progressively compared with the anticipatory image. Therefore the following :

Remark. An anticipatory system could be considered strictly anticipatory, or spontaneously self-anticipatory iff the initiative of inserting an anticipated instruction or its basic justification comes from the system itself and was not already present nor extemporaneously introduced in a former step of the process.
A strong anticipatory process in the sense of Dubois (2000) describes how a decision first arising in the brain of the programmer as a self-organized system, is further incorporated in a computer program, where it turns into a incursive or hyperincursive process. Hence, anticipation would arise as a primary step, distinguishable from and resulting in incursive modelization. Order relations on strong and strict forms deserve further examination.

4.3 The case of claimed "strongest unknowability" reexamined

Refering to its "omega number" Chaitin (1998, p. 80) wrote : " Ω really shows that some areas of mathematics have no structure, have no pattern at all. (...) the bits of Ω are mathematical facts that are true for no reason, they're accidental!" The famous Ω-number is a computer program obtained by getting all of its successive bits (written in binary numbers) at random, by "tossing a coin", Chaitin explains (1998, p. 12). Here, both the system of construction and the deductions that were derived deserve some attention.

4.3.1 The construction of omega as a mathematical object.
Omega is a magma : in effect, it owns a domain set (X) = {0,1}, and a combination rule (W) = "tossing a coin for randomly getting each bit of the program until the program halts on the output at the p^{th} step". The obtained result, if any, would represent one among all of the combinations of the set of p bits that is one out of all members of the set of parts, i.e. $(1/2)^p$. Omega is finally presented as contained in]0,1[because it is claimed to be a probability that p halts:

$$0 < \{\Omega = \sum_{p\ \text{halts}} 2^{-|p|}\} < 1 \in \{X,W\} \qquad (10)$$

In fact, equation (10) could reflect a so-called probability *a priori* (or mathematical probability) $P(e)_{a\ priori} = n(e) / N$, would N be a known number of observation of event (e) (here, $P(e)_{a\ priori} = 1 / \Omega$ with (e) = p halts). It would become a true probability as a limit of the former for $N \rightarrow \infty$. Since N-1 attempts fail and one succeeds, the *a posteriori* probability would be $P(e)_{a\ posteriori} = 1 / (N-1 + 1) = 1/N$. Thus it is the experimental assessment of success and failing through N tryings that estimates the expectancy of a success at the N+1 trying. Unfortunately, since N cannot be known, none of these probabilities can be actually calculable. This is a putative probability rather than a true probability. However, there lurks another problem in the status of omega.

4.3.2 Omega as a practically inconstructible object.
Experiments on large numbers (5,000 tryings) showed that the operation tossing a coin unfortunately does not practically generate a random distribution (Beltrami, 1999). This is due to heterogeneity of the coin as well as of the movement. Suppose that the movement is perfect random. Since one side of the coin is head and the other is tail, there is a difference in the engraving of the coin which results in nonrandom behavior. Even engraving just 0 and 1 (i.e. the definition set) would also result in heterogeneity, at least since a bar and a circle are not homeomorphic. Therefore, trying to solve the question so as to be able to run the protocol randomly needs that nothing should be engraved on the coin. Thus, the result could be perfect random, but after tossing, it would become impossible to know which side (one for zero, the other for one) has actually appeared though one really has. Hence, the result is that not only Ω is the most uncomputable number, the most random one, the most incompressible one (dixit the author) but its very existence also seems questionable, not only in the Brouwer's constructivity sense, but even in principle. However, it has been proved (Bounias and Bonaly, 1997b) that so-called 'unexisting' objects cannot 'mathematically' exist. So, omega can hardly really escape the field of mathematics.

4.4 Generalized formalism and the power of anticipatory mathematics

Recall relation (1) : $P \in \{X, \perp, S_{(X,\perp)}\} \bar{\cap} \{Y, T, S_{(Y,T)}\}$ in which the actual axiomatic system is a space of magmas composed with the intersection of definition sets, combination rules and the relevant structures of a reasoning system and a probationary space :

$$\{Ax\} = \{(X \bar{\cap} Y), (\perp \bar{\cap} T), (S_{(X,\perp)} \bar{\cap} S_{(Y,T)}\} \quad (11.1)$$
$$=\{\text{set } F, \text{ combination rule } C, \text{ structure } S\} = \{F,C,S\}.$$

Recall that $(\perp \bar{\cap} T)$ may result in new combinations of rules pertaining to neither of $(\perp)$ and (T). Now, consider that F, C and S are members of the set of axioms : since all three categories of concepts can be axioms, their combination is also an axiom, just more complicated (F,C,S also are sets of mathematical objects, so that their unions still are sets of such objects). Let us make some anticipation (perhaps as strictly as defined above) and consider the set of parts of this set :

$$P(Ax) = \{F, C, S, \{F,C\}, \{F, S\}, \{C,S\}, \{Ax\}\} \quad (11.2)$$

Among these sets, predicted by the basic set theory, one denoted {F,C} is strictly a magma composed of a set and a combination rule : therefore it has no structure holding on its members (at least unless the additional axiom of functionality allows the rule to be operating on the set, which would lead to a new object different from just part {F,C}). Compare with $\Omega \in \{X,W\}$, then : X=F, W = C.
Therefore, even Chaitin's number, said to have no structure, and even not existing, is predicted by a more general conception of an axiomatic system as depicted in the first part of this study. This is further confirmed by the fact that would Ω exist, it would belong to X, not represented in {F,C}.

4.5 The 'power set of parts' and Hilbert's dream reconsidered

Suppose that the proposition that all knowledge could effectively be derived from a finite number of axioms, is true. Then, any new concept must belong to some combination of the primary set of axiomatic elements, that is to the set of parts of the set of parts of ... of the set of parts of the initial axiomatic system. Therefore, there could not exist any concept pertaining to one mathematical branch that would be totally independent of some other mathematical branch. This provides a clue for revisiting the validity of the proposition.
Consider, now, the alternative hypothesis that a definite part of the whole of possible mathematical knowledge arises with absolutely no link with some other knowable branch.

This implies that there could exist two systems having empty intersections of their respective sets, rules and or logics. In other words, there could be a set that could not be defined through similar principles used for other sets, rules with absolutely no link with others, and logics that would be described with no common concept or expression. This raises the alternative of either a construction through combinations of combined parts, or a descent by decomposition into elementary founding parts, both indexable on ongoing time.

4.5.1 Definition and some properties of the power set of parts

Proposition 5.1. $\angle$ (Ax) an axiomatic system, and $\mathcal{P}$(Ax) the set of parts of (Ax). Then, the set of parts of $\mathcal{P}$(Ax) is $\mathcal{P}(\mathcal{P}(Ax)) = \mathcal{P}^2(Ax)$. Running the same process n times provides : $\mathcal{P}(\mathcal{P}...(\mathcal{P}(Ax))...) = \mathcal{P}^n(Ax)$ which represents a countable sequence of axiomatic systems, with $n \in \mathbb{N}$. Each member of the sequence $\{\mathcal{P}^i(Ax)\}_i$ contains its predecessor and is contained in its successor. Thus, the power set of parts is ordered, but it may not necessarily have boundaries, as this will be considered below (relation 12).

Corollary 5.1. Denote by $\cup\mathbf{K}$ the totality of possible knowledge contained in a power set of parts $\{\mathcal{P}^n(Ax)\}_{n\in \mathbb{N}}$ and k a member of this set : hence, $k \in \mathcal{P}^n(Ax)$.

Then, for any other axiomatic system $Bx \neq Ax$, one necessarily has: $Bx \in \mathcal{P}^n(Ax) \Rightarrow$ $Ax \subseteq (Bx)$ or $Bx \subseteq (Ax)$. If $Bx \subset (Ax)$, then there exists $\mathcal{P}$(Bx) and there also exists a particular integer m such that: $(Ax) \subseteq \mathcal{P}^m(Bx)$. Therefore, (Ax) reduces to (Bx).

Proposition 5.2. The Hilbert's conjecture implies that there would exist no theorem (that is no part of the parts of a axiomatic system) which would not belong to the power set of parts of any axiomatic system.
Given $Ax(X,\mathcal{L}) \supseteq \{(X=\{E,\perp,S(E\perp)\} \mathrel{\bar{\cap}} (F,T,S(FT))\}$, with NE, N$\perp$, ... , the number of member objects of each subset of X and $\mathcal{L}$, then the power set of parts of Ax includes the combinations of the power sets of parts $\mathcal{P}^n(X)$ and $\mathcal{P}^n(\mathcal{L})$, so that :

$$\mathrm{Card}\mathcal{P}^n(Ax) \propto \underbrace{2\exp\{2\exp...\{2\exp\{2^{(N_E+N_F+N_T+N_T+N_{(SE)}+N_{(SFT)})}\}\}\}}_{n \text{ times}} \qquad (12)$$

From finite numbers $N_{(...)}$ one obtains a infinitely denumerable combinations of intersections, and thus of predicates. This may be not enough to encompass the whole of mathematical knowledge, but enough for knowledge of a finite world. Then, from denumerable sets of parts $\mathcal{P}^n(X)$ and $\mathcal{P}^n(\mathcal{L})$, a power of continuum could emerge for the resulting axiomatic power : whether this would realize Hilbert's dream therefore remains a open question, since a not denumerable axiomatic system could actually arise from a finite set of primary axioms.

Corollary 5.2. For any two axiomatic systems(Ax) and (Bx) as remote from one another as possible, one should have at least the following property, since parts of Ax may not be an axiom for(Bx) and conversely : $\mathcal{P}^n(Ax) \mathrel{\bar{\cap}} \mathcal{P}^m(Bx) \neq \emptyset$

4.6 A strange convergence problem

A set like $\mathcal{P}^n(Ax)$ is the union of the members of the sequence, that is :
$\mathcal{P}^n(Ax) = \{Ax, Ax+1, Ax+2, ..., Ai, , ..., Ax+n\}$
However, Ax could itself be the last term of a descent, so that it can be decomposed into :
$\{Ax\}_x = \{A1, ..., Ak, ..., A(x-1)\}$

If the sequence $\{Ax\}_x$ is convergent to some (Ao), then (Ao) stands for an axiomatic system fulfilling the condition holding in Hilbert's dream. Here, $\mathcal{P}^n(Ax)$ would have a lower boundary.If however the sequence does not converge, two cases deserve attention. (i) (condition C-1): $\emptyset \subset A_o \subset A_k$: then, the axiomatic system can be defined by existing, though not identifiable boundaries. (ii) (condition C-2): $A_o \subset \emptyset$ which indicates that the members of A_o should be defined by anti-existence properties, that is in a anti-constructive way. In this case, only what A_o is not, could be stated, not what A_o is.

Lemma 5.3. The boundaries of a sequence $\{\mathcal{P}^i(Ax)\}_i$ are not necessarily the limits of the sequence. In effect, the space of $\{\mathcal{P}^i(Ax)\}_i$ is discrete but not necessarily finite. It can thus be only locally compact within finite subparts. Thus, $\{\mathcal{P}^i(Ax)\}_i$ is bounded by a - not necessarily knowable - lower boundary which is not necessarily a limit.

Remark. Even Chaitin's omega number does not fulfill condition (C-2) although it might be the known object closest to condition (C-1). The conditions stated by Gödel, Chürch or Turing are valid for some axiomatic systems not devoid of connections with more complete axiomatic systems on which they no longer hold : their underlying reasoning systems fall into a category such that $\mathcal{P}^n(Ax) \,\bar{\cap}\, \mathcal{P}^m(Bx) \neq \emptyset$.

Corollary 5.3. For any given (Ax), one has $(Ax) \in \mathcal{P}^m(Axo)$ eventually not knowable. However, if $\mathcal{P}^m(Axo) \subset \mathcal{P}^n(Ax)$, then $\mathcal{P}^m(Axo)$ contains both some elements of (Ax) and some elements of $\mathcal{P}^n(Ax)$ which is larger than $\mathcal{P}(Ax)$. Then, (Ax) is knowable. More generally, an axiomatic system $Ax=\{Ax_i\}_i$ is founded on the union of the founding axioms of each of its components: $\cup_i\{Ax_{io}\}_i$, which is likely to be generally larger than Ax itself. Consequently, the answer to Hilbert's dream may be that a "finite small number of axioms" may not be a "shortest" nor a "simple" one since, owing to relation (12), the whole of knowledge actually available from even a finite small number of axioms may not be practically expressed within a finite lapse of biological time or human generations.

4.7 Mental imaging and anticipatory solution

Theorem 1. The formal assessment of an exception to the Hilbert's dream potentially results from a mental anticipatory process.
Preliminary proof. Suppose that there exists a yet unknown system (Yx) such that, contrary to Hilbert's conjecture, one has $\mathcal{P}^h(Yx) \,\bar{\cap}\, \mathcal{P}^n(Ax) = \emptyset$. Since (Ax) is known, $\mathcal{P}^n(Ax)$ is at least knowable too. Thus, in the neuronal system of a skilled mathematician, there exists a structure of mental images $Im\{\mathcal{P}^m(Ax)\}$ reflecting $\mathcal{P}^n(Ax)$. Then, (Yx) cannot infer from $Im\{\mathcal{P}^m(Ax)\}$ which stands for the past and for recursive processes. Consequently, (Yx) can infer only from the prior elaboration of a set of mental images Im(Ax) : this provides the knowledge of (Yx) with the characteristics of an anticipatory process, which completes the argumentation supporting the theorem in former parts of this study.
This further precludes that any knowledge could be "in principle" unreachable. Incidentally, part of a set used for self-evaluation of the set gives raise to a power set of these parts which will actually allow a surjective mapping to apply on the complete set.

4.8 Anticipation between actual and potential axiomatic parts

Preliminary statement. While the set of parts of a set exists if the set exists, its availability to further reasoning must be stated as either a structure or a combination rule in consistency with the other members of the same axiomatic system. Hence :

Lemma 5.4. If some elements of a demonstration holding on a set $E_n = \{e_1, ..., e_n\}$ are not explicitly available unless some additional condition must be fulfilled, then this condition must be incorporated to the basal axiomatic system as 'axiom of availability'.

Proof. Let the set $E_2 = \{a,b\}$, and suppose one needs to use its Cartesian product without assuming that (E_2) is available in duplicate. Then, the availability of the members of the cartesian product is not even immediately valid from the set of parts of E_2 . In effect, at least the second iterate of the power set is required :

$\mathcal{P}(E_2) = \{a,b, (a,b)\} \cup \emptyset$.

$\mathcal{P}^2(E_2) = \{a,b, (a,b), [a,(a,b)], [b,(a,b)], (a,b), [\emptyset,a,b, (a,b)]\} \cup \emptyset$.

The unordered pair (a,b) is contained, as a member, once in $\mathcal{P}(E_2)$ and twice in $\mathcal{P}^2(E_2)$: thus only $\mathcal{P}^2(E_2)$ allows the cartesian product to be obtained from $(a,b) \times (a,b)$. However:

Corollary 5.4. A complete subset of the rational numbers is not necessarily provided in all bases by the Cartesian product of a segment of natural numbers.

Proof. Let $E_n = \{1, ..., n\}$ with $n<9$, and let two integers, $p,q \in E_n$. Then, the pair $(p,q) \in (E_n^2) = E_n \times E_n$. Usually, (p,q) accounts for the ratio p/q, so that the set of pairs (p,q) is equipotent to the set of rational numbers noted as fractions : $(e_i.d_1d_2...d_i$)where e_i.stands for the entire part and $d_1d_2...d_i$ ... for the decimal part.

Let n=4, and take p=1, q=4. Then the ordered pair (1,4) stands for the ratio 1/4. However, writing 1/4 = 0.25 needs digit 5 to be available, whereas one has just 1,2,3,4 available, not 5. Therefore, in this system, since digit 5 does not exist unless the additional axiom of the addition is introduced, the mapping of ordered pairs to the writing in base 10 of the corresponding rational numbers is not valid.

In his book "The Limits of Mathematics", Chaitin writes a k^{th} computer program $p_k = 0. d_{k1}d_{k2...}d_{ki...}d_{kn}$. However, he uses the binary system, that is an axiomatic system in which only digit (0) and (1) are available. Therefore, he cannot write the program unless he has posed as an additional rule that he gets the power set of parts of founding set $\{0,1\}$, that is : $\{\mathcal{P}^N(0,1)\}_N$. Therefore, in his intellectual processing and subsequent computed realization, Chaitin has implicitly used an anticipatory process.

The use of such members of sets not previously made available implies that such an implicit use should be considered each time a member is implicitly introduced as a new axiom, which will belong to a class of 'axioms of availability'. Hence, some indecidable propositions may become even more indecidable within the frame of classical set theory, and the axioms of availability should be further examined with respect to the foundations of set theory. In particular, an extended axiom of availability would be needed to construct a denumerable set from founding members. Eventually, this would further help in strengthening the construction of nonwellfounded sets.

Each time a reasoning implies a step in which not all of the used elements have been first introduced by former deductions, this reasoning is at best an **implicitly anticipatory** one. Its validity is established iff it can be *a posteriori* verified that the availability of the postulated elements could have been demonstrated from the former steps of the reasoning process used. Otherwise this remains at best a speculation.

5 Conclusions

(i) The reality of Chaitin 'Omega' number as the strongest case of axiomatic indecidability claimed so far, is questioned by a collection of arguments involving anticipatory components. (ii) A formal axiomatic system extended to a reasoning system based on combined spaces of magmas turns into a new system with eventually no axiom at all. Interestingly, the infering process leading to deduction of an axiom rather than arbitrarily posing it seems to have been already considered by Aristotle, as argued by Malatesta and Grappone (2000). (iii) Brain anticipatory processes can jump over barriers forbiding a computed system to reach a result not directly infering from its program. A biological brain results from a chain of self-organized systems and its anticipatory mental imaging capability confers self-evaluation properties on it.

Then, upon self-understanding combined with interpretation of outside measurements, biological brains have started building computers to which they have provided neural-like and strong anticipatory properties.

This may appear as a generalization of the completeness theorems, involving the role of anticipatory processes in the search for the identification of hidden probationary spaces.

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