Remarks on the Classical Probability of Bifuzzy Events

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Abstract
The present paper discusses the conception of the cardinality of a bifuzzy set in two versions - as a real number and as a bifuzzy set. The notions introduced here serve one to calculate the probability of a bifuzzy event defined in a finite space of elementary events and are a generalization of Laplace's approach in the Kolmogorov probability calculus. The paper refers to works of Gerstenkorn and Mańko (1998) and Mańko (1992) and is illustrated by an example.
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1 Introduction

In paper of Mańko (1992), attention was drawn, for the first time, to the possibility of constructing a probability measure for bifuzzy events. This construction consists in making use of a probability space $(E, \mathfrak{I}, P)$ in the ordinary sense, i.e. in the one that $\mathfrak{I}$ denotes the σ-field of subsets of a set E and P - a probability measure on $\mathfrak{I}$, and applying it to the intuitionistic fuzzy set theory of Atanassov (1986) (in our terminology: bifuzzy set theory). Therefore, in the family IFS(E) of bifuzzy sets

$$\{(x, \mu(x), \nu(x)) \colon x \in E \text{ and } \mu, \nu \colon E \to \langle 0,1 \rangle \text{ and } 0 \le \mu(x) + \nu(x) \le 1\}$$

we consider only those sets for which membership functions μ and non-membership functions ν are measurable. Such sets are called bifuzzy events, and the family of them is denoted by IM(E). Then the probability of a bifuzzy event $A \in$ IM(E) is defined by Mańko (1992)

$$P(A) = \int_E \frac{\mu_A(x) + 1 - \nu_A(x)}{2} P(dx). \tag{1}$$

The function so defined satisfies the probability properties consistent with Kolmogorov's axiomatics and, thereby, a number of other characteristic probability properties. In the special case when $\nu_A(x) = 1 - \mu_A(x)$, that is, in the case of the Zadeh fuzziness, formula (1) reduces to the formula for probability of a fuzzy event, defined in Zadeh (1968) as

$$P(A) = \int_E \mu_A(x) P(dx)\,. \qquad (2)$$

Gerstenkorn and Mańko (1998) presented a conception of the probability of a bifuzzy event in the form of a bifuzzy set, and not, as in Zadeh (1968), of a real number. This conception agrees with Yager's proposal of a fuzzy probability and, analogously to it, describes the so-called bifuzzy probability of a bifuzzy event. The concept is based on the notion of an (α, β)-level of a bifuzzy set and the so-called extension principle (Stojanova, 1990) for bifuzzy sets. Their ideas are explained below:

For numbers $\alpha, \beta \in \langle 0,1 \rangle$ such that $\alpha + \beta \leq 1$, by the product of a pair (α, β) and a bifuzzy set $A \in \mathrm{IFS}(E)$ we mean the bifuzzy set $(\alpha, \beta) * A$ of the form

$$(\alpha, \beta) * A = \left\{ (x, \alpha \cdot \mu_A(x), \beta + (1-\beta) \cdot \nu_A(x)) : x \in E \right\}. \qquad (3)$$

Then the set

$$A_{\alpha,\beta} = \{(x \in E : \mu_A(x) \geq \alpha \wedge \nu_A(x) \leq \beta\} \qquad (4)$$

is called an (α, β)-level of the set $A \in \mathrm{IFS}(E)$, whereas the set

$$N_{\alpha,\beta}(A) = \left\{ (x,1,0) : x \in A_{\alpha,\beta} \right\} \qquad (5)$$

is called a bifuzzy analogue of the (α, β)-level of the set A. With that, in virtue of formula (3), we have

$$(\alpha, \beta) * N_{\alpha,\beta}(A) = \left\{ (x, \alpha\ \beta) :\ x \in A_{\alpha,\beta} \right\}. \qquad (6)$$

Stojanova (1990) proved the theorem on the decomposition of an arbitrary bifuzzy set A into its (α, β)-level in the form

$$A = \bigcup_{\alpha,\beta} (\alpha, \beta) * N_{\alpha,\beta}(A) \qquad (7)$$

where the symbol $\bigcup$ denotes the union in the bifuzzy sense (Atanassov, 1986) and extends to all $\alpha, \beta \in \langle 0,1 \rangle$, $\alpha + \beta \leq 1$.

Let now $f : E \to L$ be any function in the ordinary sense and let E, L be arbitrary non-fuzzy sets. Let $A \in \mathrm{IFS}(E)$. The formula

$$f(A) = \left\{ (f(x), \mu_A(x), \nu_A(x)) : f(x) \in L \right\} \qquad (8)$$

defines the image of the set A under the mapping f, that is, extends the action range of the function f from sets in the ordinary sense to sets in the bifuzzy sense. It is convenient to write (8) down in the form

$$f(A) = \bigcup_{\alpha,\beta} (\alpha, \beta) * f(N_{\alpha,\beta}(A))\,. \qquad (9)$$

Formulae (8) and (9) are the contents of so-called bifuzzy extension principle (Stojanova, 1990).

Now, using the probability function P in formula (9) and keeping the assumptions about the function P and the set A as in (1), we obtain the so-called bifuzzy probability $\widetilde{P}(A)$ of a bifuzzy event $A \in \mathrm{IM}(E)$ in the form

$$\widetilde{P}(A) = \bigcup_{\alpha,\beta} (\alpha,\beta) * P(N_{\alpha,\beta}(A)) . \tag{10}$$

The above procedure is a generalization of that proposed by Yager (1979) and was discussed by Gerstenkorn and Mańko (1998).

2 Numerical example

Let $E = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of five persons engaged professionally in economy. Consider for them a set A of „good economists". Adopt arbitrarily:

$$A = \{(x_1, 0.3, 0.6), (x_2, 1, 0), (x_3, 0.8, 0.1), (x_4, 0.5, 0.4), (x_5, 0.4, 0.5)\}.$$

Assume that, as a result of some random experiment, individual persons are being chosen with the probability equal to $\frac{1}{5}$ for each of them.

Then, according to formula (1), the probability of the event A of choosing a „good economist" from the set E is equal to

$$P(A) = \frac{1}{5} \cdot \frac{1}{2}((0.3 + 1 - 0.6) + (1 + 1 - 0) + (0.8 + 1 - 0.1) + \\ + (0.5 + 1 - 0.4) + (0.4 + 1 - 0.5)) = 0.64.$$

In accordance with procedure (3)-(10), we then have in succession:

$$A_{0.3,0.6} = \{x_1, x_2, x_3, x_4, x_5\},$$
$$A_{0.4,0.5} = \{x_2, x_3, x_4, x_5\},$$
$$A_{0.5,0.4} = \{x_2, x_3, x_4\},$$
$$A_{0.8,0.1} = \{x_2, x_3\},$$
$$A_{1,0} = \{x_2\}.$$

Other (α, β)-levels reduce to the above ones.
In consequence, we further have:

$$N_{0.3,0.6}(A) = \{(x_1, 1, 0), (x_2, 1, 0), (x_3, 1, 0), (x_4, 1, 0), (x_5, 1, 0)\},$$
$$N_{0.4,0.5}(A) = \{(x_2, 1, 0), (x_3, 1, 0), (x_4, 1, 0), (x_5, 1, 0)\},$$

$N_{0.5,0.4}(A) = \{(x_2,1,0),(x_3,1,0),(x_4,1,0)\}$,
$N_{0.8,0.1}(A) = \{(x_2,1,0),(x_3,1,0)\}$,
$N_{1,0}(A) = \{(x_2,1,0)\}$.

Then:

$$P(N_{0.3,0.6}(A)) = 1, \quad P(N_{0.4,0.5}(A)) = \frac{4}{5},$$
$$P(N_{0.5,0.4}(A)) = \frac{3}{5}, \quad P(N_{0.8,0.1}(A)) = \frac{2}{5},$$
$$P(N_{1,0}(A)) = \frac{1}{5}$$

or, in the bifuzzy notation:

$$P(N_{0.3,0.6}(A)) = \{(1,1,0)\}, \quad P(N_{0.4,0.5}(A)) = \left\{\left(\frac{4}{5},1,0\right)\right\},$$
$$P(N_{0.5,0.4}(A)) = \left\{\left(\frac{3}{5},1,0\right)\right\}, \quad P(N_{0.8,0.1}(A)) = \left\{\left(\frac{2}{5},1,0\right)\right\},$$
$$P(N_{1,0}(A)) = \left\{\left(\frac{1}{5},1,0\right)\right\}.$$

Then formula (10) yields the result:

$$\widetilde{P}(A) = (0.3,0.6)*\{(1,1,0)\} \cup (0.4,0.5)*\left\{\left(\frac{4}{5},1,0\right)\right\} \cup (0.5,0.4)*\left\{\left(\frac{3}{5},1,0\right)\right\}$$
$$\cup (0.8,0.1)*\left\{\left(\frac{2}{5},1,0\right)\right\} \cup (1,0)*\left\{\left(\frac{1}{5},1,0\right)\right\} =$$
$$= \left\{(1,0.3,0.6),\left(\frac{4}{5},0.4,0.5\right),\left(\frac{3}{5},0.5,0.4\right),\left(\frac{2}{5},0.3,0.1\right),\left(\frac{1}{5},1,0\right)\right\}.$$

3 The cardinality of a bifuzzy set and the classical probability

Let E stand for a finite non-fuzzy set, that is, $E = \{x_1, x_2, x_3, \ldots, x_n\}$. Let a bifuzzy set $A \in \mathrm{IFS}(E)$ be given, described by a membership function $\mu_A(x_i)$ and non-membership function $v_A(x_i)$ $i = 1,2,3,\ldots,n$. We adopt

DEFINITION 1. By the cardinality of a set A we mean a non-negative real number *card A* defined to be

$$card\ A = \sum_{i=1}^{n} \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}. \tag{11}$$

Note that formula (11) is a generalization of the notion of the cardinality of a fuzzy set, proposed by de Luca and Termini (1972) and, in the case of the reduction of bifuzziness to the Zadeh fuzziness, reduces to the formula $|A| = \sum_{i=1}^{n} \mu_A(x_i)$.

Extending formula (9) to a function $f = card$, we are in a position to give a definition of the so-called bifuzzy cardinality of a bifuzzy set:

DEFINITION 2. By the bifuzzy cardinality of a set $A \in \mathrm{IFS}(E)$ we mean a bifuzzy set $card_f A$ defined by the formula

$$card_f A = \bigcup_{\alpha,\beta} (\alpha,\beta) * card\ N_{\alpha,\beta}(A) \tag{12}$$

where the bifuzzy union $\bigcup$ extends to all numbers $\alpha, \beta \in \langle 0,1 \rangle$ such that $\alpha + \beta \leq 1$.

Let us now notice that formula (12) is dual to the notion of the fuzzy cardinality of the fuzzy set defined by Zadeh (1997) as $|A|_f = \bigcup_{\alpha} \alpha * |A_\alpha|$.

Now, assume that we have defined in E a probability function according to the Laplace assumption, i.e. that each $x_i \in E$ is equally privileged and appears, as a result of some random experiment, with the probability $\frac{1}{n}$ for any $i \in \{1,2,\ldots,n\}$. In conformity with the so-called classical Laplace definition of probability, we then have, for $A \in \mathrm{IFS}(E)$,

$$\begin{aligned} P(A) &= \frac{card\,A}{card\,E} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \left(\mu_A(x_i) + 1 - \nu_A(x_i) \right) = \\ &= \sum_{i=1}^{n} \frac{1}{2} \left(\mu_A(x_i) + 1 - \nu_A(x_i) \right) \cdot p(x_i) \end{aligned} \tag{13}$$

which is a special case of formula (1) and, at the same time, agrees with the classical version of probability calculus.

Using now Definition 2 of the bifuzzy cardinality of a bifuzzy set, we have

$$\begin{aligned} \widetilde{P}(A) &= \frac{card_f\ A}{card\ E} = \frac{1}{card\ E} \bigcup_{\alpha,\beta} (\alpha,\beta) * card\ N_{\alpha,\beta}(A) = \\ &= \bigcup_{\alpha,\beta} (\alpha,\beta) * \frac{card\ N_{\alpha,\beta}(A)}{card\ E} = \bigcup_{\alpha,\beta} (\alpha,\beta) * P(N_{\alpha,\beta}(A)) \end{aligned} \tag{14}$$

which, in turn, agrees with formula (10).

4 Example (cont.)

Coming back to the set $A \in \mathrm{IFS}(E)$ of „good economists", considered in Section 2, we then have, according to (11),

$$card\, A = \frac{1}{2}((0.3+1-0.6)+(1+1-1)+(0.8+1-0.1)+$$
$$+(0.5+1-0.4)+(0.4+1-0.5)) = 3.2$$

whence

$$P(A) = \frac{card\ A}{card\ E} = \frac{3.2}{5} = 0.64\,.$$

Whereas using (12), we have and, in consequence,

$$card_f\ A = \{(5,\, 0.3,\, 0.6),\, (4,\, 0.4,\, 0.5),\, (3,\, 0.5,\, 0.4),\, (2,\, 0.8,\, 0.1), (1,\, 1,\, 0)\}\,.$$

Then

$$\widetilde{P}(A) = \frac{card_f\ A}{card\ E} = \left\{(1{,}0.3{,}0.6), \left(\frac{4}{5}{,}0.3{,}0.5\right), \left(\frac{3}{5},\, 0.5,\, 0.4\right), \left(\frac{2}{5},\, 0.8\ 0.1\right), \left(\frac{1}{5},\, 1,\, 0\right)\right\}$$

which is identical with the result obtained in Section 2.

5 Discussion

Various conceptions of the cardinality of a bifuzzy set, proposed here, are generalizations of the concepts of a fuzzy set (De Luca, Termini (1972) and Yager (1979)) and, like the latter ones, can be applied in probabilistic computations for finite bifuzzy events. Their unquestionable advantage is the simplification of calculations in comparison with Yager (1979) and Zadeh (1997) when the support of a bifuzzy event is finite and each of its elements is equally possible in some random experiment.

6 Conclusion

The „numerical" conception, though much easier in calculations, may arouse doubts concerning its adequacy to the bifuzzy situation. In turn, the bifuzzy conception is more toilsome in computations but, at the same time, more intuitive. The choice of a conception depends on the situation.

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