The Transcendental Role of the Gödelian Non-decidable Propositions in the Diachronic Inclusion of Axiomatic Theories and Metatheories

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Abstract

The theorems established by the logicians show that the truth value of a proposition, constructed in a logical system S, cannot be enunciated in the system ***S*** itself, but in metasystem ***S'***, which refers to the proposions of the system S therefore it is not apt be confused with ***S'***.

In the paper, according to Kant's transcendental methodology, where it is asserted that any rational knowledge is either a piece of knowledge from concepts, or a piece of knowledge from the construction of concepts, the transcendental role of the non-decidable Gödelian propositions is considered in the diachronic inclusion of the axiomatic theories and metatheories.

Key words: transcendental, Gödelian, Non-decidable, metatheory, axiomatic.

1 The Fundamental Notions of Axiomatic and Formalization

1.1 The Axiomatic Method

Any scientific theory comprises a body of concepts and a set of assertions [6].

The definition of explanation or a concept is given on grounds of other concepts.

The justification of the truth of certain assertions, as well as of the reasons we have to believe them to be true, is given by showing that the assertion can be, or is to be inferred from other accepted assertions.

If we endlessly search for definitions or deductions, we shall move in circles, using in our answers concepts and assertions whose sense and justification we have initially set out to explain, or, at a certain stage, we shall refuse to bring forth any other definitions or deductions, saying that we have already used in our answer fundamental concepts and assertions which we adopt as valid.

When the problem consist of understanding the sense of a concept in seeing if a proposition is true, there is no fundamental objection to the circular procedures.

However, when we are able to start only from a small number of primitive ideas and propositions, the linear approach exerts a certain attraction and a special fascination, as the problems of significance and truth are concentrated in those initial primitive elements, to which certain typical methods of definition and deduction are added.

Primitive propositions are called axioms or postulates.

When the concepts and propositions of a theory are placed in accordance with definability and deductibility links, then we have an axiomatic system of the theory.

In axiomatic systems, a basic inference logic is adopted (the theory of quantification and the calculation of predicates), which uses the logical invariant expression "if-then", "no", "all" "some", "or", "and", "if and only if".

During the evolution of axiomatic systems, from axioms has resulted a precise formalization criterion, according not to sense and concepts, but to features connected with the notations of terms and formulae [2].

1.2 The Formalization Process

Within a given science, there is a body of asserted and non-asserted propositions. From the asserted propositions, certain propositions are selected, which have the role of axioms and from which other propositions can be deduced.

In order to be adequate, the axioms should express all the relevant properties of the undefined technical terms, so that the deductions will be achieved even if the technical terms are considered senseless words or symbols.

The principles which determine the sense of the logical particles or of the non-technical words that govern their use are explicitated.

As a result, it is possible to recognize the axioms and demonstrations in a notational model.

Systems are axiomatic or formal only when they meet the following criteria: there is a mechanical way of determining if a given notational model is a symbol within the system and if a combination of such symbols is a well-formed formula (an enunciation that has sense), an axiom or a demonstration of the system.

1.3 The Metatheory of an Axiomatic Theory

The analysis of an axiomatic theory is named metatheory.

In the metatheoretical analysis of an axiomatic theory T, we are interested in the properties of the theory, as well as in its structure and its relation with their theories [3].

Metatheoretical analysis relies on two points of view: syntactic and semantic.

The analysis of the way in which theory T is constructed is the syntactic point of view.

The analysis of theory T through interpretations and models is the semantic point of view. Its application leads to the semantics of theory T.

The interpretation of an axiomatic theory T is the association (representation) to primary notions and relations T of primary notions and relations from a theory T' (possibly non-formalized). In theory T' primary notations can be the elements of a given set.

A model of the axiomatic theory T is an interpretation T' in which the axioms of T are represented by truths from T'. We note by T a certain axiomatic theory.

The following elements are important for the metatheory of T:

- non-contradiction;
- independence;
- completeness and categoricness.

Definition 1 °

T is non-contradictory if it does not contain any proposition p so that p and negation p should be deduced from axioms.

Definition 2 °

An axiom A from T is independent from the other axioms in T if A cannot be logically deduced from them.

The same definition holds good for the independence of notions and the primary relations of T.

The system T is independent or minimal if each notion, relation or axiom in it is independent from the others.

Definition 3 °

An axiomatic theory T is complete if for any correctly constructed proposition p (non-contradictory) p or $\bar{p}$ can be deduced from axioms.

If T is complete, the system of axioms in T is complete.

Two models of a theory T are isomorphic if, between them there is a biunivocal correspondence that preserves their true properties.

Definition 4 °

An axiomatic theory T is categoric if all its models are isomorphic among themselves.

Definition 5 °

A mathematical structure is a set that relies on certain relations, so that its elements and given relations satisfy a system of axioms.

1.4 The Semiotic Aspect of the Proposition and the Notion

In any proposition there are three factors:

- the individual in whose conscience the proposition exists;
- the state of fact it reflects;
- the linguistic form under which it exists.

These three factors also hold good for notions.

The notion is a dialectic unit consisting of intension and extension. Intension reflects certain characteristics, namely properties and relations.

Extension reflects a certain class of objects.

The dialectic unity between the intension and extension of a notion is characterize as it follows:

- the extension of a notion is the reflection in our mind of the class of objects whose invariable characteristics are reflected through the intension of that notion;

- the intention of a notion is the reflection of the invariable characteristics within the class of objects reflected by the extension of the notion.

Any notion is obtained by abstractizing concrete objects and phenomena.

The relation between the three factors is dealt with by Pragmatics. Semantics neglects the first factor and analyzes only the relation between other two.

Syntax deals only with the third factor.

Pragmatics, Semantics and Syntax form Semiotics.

1.5 The Theory of Semantic Steps

The theory of semantic steps is an important notion of Semiotics.

There are object, properties and relations that belong to objective reality and are not signs of language.

A language of the second step or metalanguage comprises all the necessary signs to characterize the signs of the object language.

These objects form step zero. The signs used to note the objects of step zero belong to an object language or the first step language.

A language of the second step or metalanguage comprises all the necessary signs to characterize the signs of the object language.

The analysis of a metalanguage is made in a third step language and so on.

The difference between languages has its origin in the differentiation in object-theories, metatheories and second step theories, i.e. in a hierarchy of theories.

The objects of step zero form the basis of the whole sequence of steps of human knowledge, representing state of facts and the relation between them.

The expression

$$p \Leftrightarrow q\,;\ (p \rightarrow q) \wedge (q \rightarrow p) \tag{1}$$

where p and q are propositions, belong to the object-language.

The properties of such expressions, as well as the relations between them and the relation of these properties and relations with the sense of the expressions are studied within the framework of a metalogic and formulated in a metalanguage.

1.6 The Theory of the Logic Types and the Types of Truth

The theory of logic types derives from the principle of the vicious circle. Bertrand Russell in Principia Mathematica, shows that paradoxes appear because it is supposed that a collection of objects can contain members that can be defined only by considering the collection as a whole [1],[3].

The example of the paradox of the class of classes that do not contain themselves as an element is explained by Russell as it follows: a class is an object that derives from a propositional function $\varphi(x)$, but a class cannot be the argument of the function which defines it, i.e. if we not by $\hat{z}(\varphi_z)$ the class defined by the function $\varphi(z)$, the symbol $\varphi(\hat{z}(\varphi_z))$ should be regarded as senseless, on grounds of the vicious circle principle.

Consequently, the argument of a function cannot have arbitrary values, and the values that can be attributed to it are limited.

Russell establishes a logic hierarchy of concepts, namely: the individual, logic objects of t_0 type, the properties of the individuals, the concepts of t_1 type; the properties of the individuals, the concepts of t_2 type etc.

The theory of types establishes that a concept of type n could be applied only to a concept of type n-1. In other words, in the propositional functions $\varphi(x)$ the argument can have only values of type n-1, if φ is of type n.

Expressions of the form $\alpha \in \alpha$ or non $\alpha \in \alpha$ or in comprehension $\varphi(\varphi)$ or non $\varphi(\varphi)$ not observing the theory of types, have no sense and are excluded from the logical symbolism; the proposition "abstract is abstract" or "concret is abstract" or "class α contains class α as an element" are not possible.

Russell also established different types of truth when we assert the truth or the falseness of a proposition p, for example "p is false", we use a certain type of truth; when we say that the proposition "p is false" is false, we use another type of truth.

The theorems established by logicians show that the truth value of a proposition constructed in a logic system S cannot be enunciated in the system S, but in a metasystem S', that refers to the propositions of system S.

Logicians have reached the conclusion that any deductive formalism has its own limits and logic deductibility cannot surpass certain limits. Therefore, any logic symbolism, and any mathematical symbolism in general is limited.

1.7 The Structural Method

The structural method develops a number of methodological principles and a body of rules that, by generalization, can be named structural analysis rules.

For instance, the Rule of immanence states that the analysis is placed exclusively within the investigated system, temporarily closing the respective system of signs, for methodological reasons.

By explaining the internal organization of a totality structural analysis established regularities not on grounds of resemblance, but of differences, grouping and arranging differences, especially binary oppositions (where the relations between the elements are of complementarity).

In the methodological strategy of structuralism, a special role is played by the Rule of diachronic variation, which explains the variation of the system, by structural invariants. Starting from the distinction between synchronic (the relation between coexisting terms) and diachronic (the relation between successive terms), structural analysis starts with the study of synchronic relations.

1.8 The Incompleteness of Axiomatic Theories and Metatheories

Aristotle was the first to formulate the ideas on the deductive methods of logic [3].

By the term of deductive science, in the sense of present concepts, Aristotle understands a system S of notions and propositions made up in such a way that:

1° - all the propositions in system S should refer to one and the same domain of objects and relations among objects;

2° - any proposition in the system S should be a true proposition;

3° - if certain propositions belong to the system S, then all the propositions that can be deduced from them according to the laws of logic have to be belong to the system S;

4° - a finite set of notions must be given in the system S so that their meaning sould not need any explanation, and the meaning of the remaining notions in the system S could be defined by means of this first group of notions;

5° - a finite number of propositions must be given in system S and they have to be made up in such a way so that their truth should be obvious and any other propositions could be deduced from these propositions according to the laws of logic.

Gödel, in the work "On propositions about which one cannot decide in the system "Principia Mathematica" showed that the syntax of mathematics theories and of mathematics logic can be arithmetized, that is mathematics models can be given for the respective theories.

Supposing that we have managed to formalize elementary arithmetic within the framework of deductive theory T, then there will certainly be an accurate definition of what we should understand by the expression H.

To each expression H, there corresponds a Gödel number G(T, H). Within this theory, arithmetic predicates P(T) can be defined which should be made up in such a way as to correspond to a natural number n if and only n is Gödel's number G(T, H) of the theorem H in the theory T. An arithmetic predicate called critical predicate - $K_r(T)$, can be thus defined: the natural number n has the property $K_r(T)$, that is we have $K_r(T)$ only when n is Gödel's number of an expression

$$n = Gö(H) \tag{2}$$

The property $K_r(T)$ can formally be expressed in theory T.

The results reached by Gödel have a great significance for dialectic logic. The fact that in the theory T there are true propositions which cannot be demonstrated within this theory can also be expressed by the finding that theory T is not complete.

This theory can be completed by broadening the axiom system, by raising to axiom rank the non-demonstrable proposition, which is also true, by adding it to the other axioms. A modification in the axioms of an axiom system entails a change of the predicate meaning and of the relations in the respective theory.

By modifying the axiom system, the meaning of the critic predicate K_r (T) changes, and this time, too, a proposition can be formulated, which is true but it is not demonstrable in the new axiom system. The new theory is more comprehensive, but it is incomplete, this time, as well.

Each of these theories represents only a relative truth, which can be replaced by a more comprehensive relative truth ranking a step higher.

Thus, a succession of theories can be built.

Within dialectic logic, it is maintained that truth would be a process. The assertion finds a parallel in the finitist and constructivist position of some trends in mathematics. The finitist-constructivist current accepts the mathematic truth not in theorems and in the proof result but in the demonstration process itself.

The forerunner of this conceptions is Kant, who in his transcendental methodology says that any piece of rational knowledge is ether knowledge from concepts or knowledge *the construction of concepts*.

2 The Transcendental Role of Non-decidable, Gödelian Propositions

The paper approaches Kant's transcendental methodology accepting Gödels assertions that in any axiom system one proposition can be constructed, at least, which is true if and only if it cannot be demonstrated by the given axioms.

Starting from the fact that there are objects, properties and relations, which in the theory of semantic stages belong to the zero stage, and the signs by which objects are denoted belong to the object language or to the first stage language the languages can become differentiated into object language, second order language or metalaguage, third stage language a.s.o.

The differentiation between languages enables to differentiate between object theories, metatheories or second stage theories a.s.o. that is differentiating between a theory hierarchy.

According to the type theory, a logical concept hierarchy is established that is a concept of type n can be applied only to concept of type n-1.

In accordance with the typification of the truth theory we show that the truth value of a proposition constructed in a logic system cannot be enunciated in the system itself but in a metasystem referring to the system propositions.

According to dialectic logic, by raising the non-demonstrable but true proposition to axiom rank and by adding it to the axiom system a new theory is obtained.

The presence of a true proposition which cannot be demonstrated in the new theory enables to reiterate the process, a sequence of theories being obtained.

Each of these theories represents a relative truth, which can always be replaced by a more comprehensive relative truth, situated one higher step, obtaining an ascending hierarchical structure of truth, convergent to A_{00} truth termed *axiomatic truth* in paper (Figure 1) [4], [5].

Consequently, the true but non-demonstrable, non-decidable, Gödelian proposition that raised to axiom rank play a transcendental role in the diachronic inclusion of axiomatic theories and metatheories and allow the hierarchic ascending proposition structuring, convergent toward the proposition of maximum generality P_{00} (Figure 2) [4].

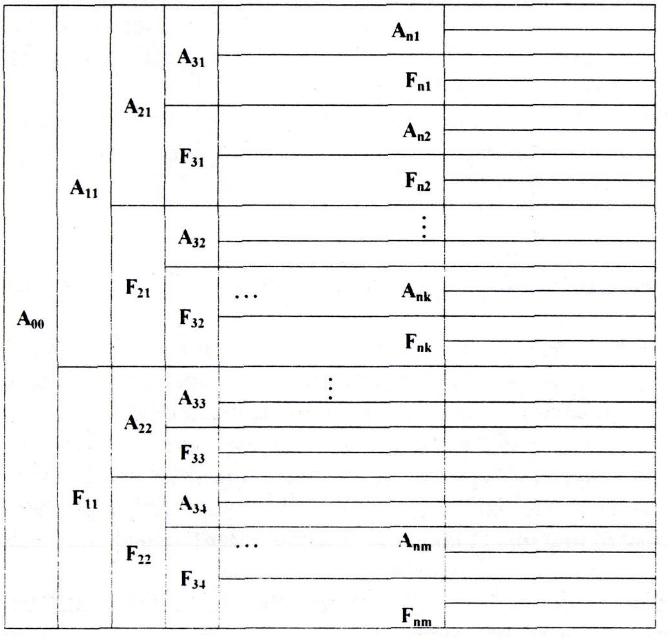

Fig.1. The descendant hierarchic structure of the propositions truth

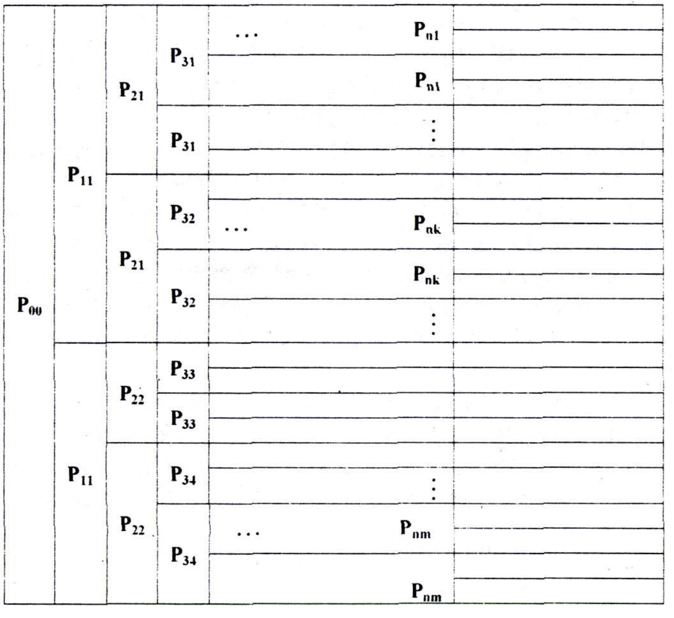

Fig.2. The hierarchic descendant structure of the propositions theories

3 Conclusions

The existence in an axiomatic theory of a true proposition, which cannot be demonstrated within that theory, non-decidable, means that the theory is incomplete.

By raising a non-demonstrable but true proposition in an axiomatic system to axiom rank, a new incomplete theory is obtained. Reiterating the process of raising to axiom rank of non-decidable Gödelian propositions enables to construct a sequence of incomplet theories.

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