Signal Transmission Along Singularity Free Gradient Fields and Quantization Caused by Internal Degrees of Freedom

Ernst Binz* and Walter Schempp**

* Lehrstuhl für Mathematik I, Universität Mannheim, D-68131 Mannheim, Germany. binz@math.uni-mannheim.de

** Lehrstuhl für Mathematik I, Universität Siegen, D-57068 Siegen, Germany. schempp@mathematik.uni-siegen.de

Abstract: Any singularity free vector field X defined on an open set in a three-dimensional Euclidean space with curl $X = 0$ admits a complex line bundle F^a with a fibre-wise defined symplectic structure, a principal bundle $\mathcal{P}^a$ and a Heisenberg group bundle. For $X = \text{const.}$ the geometry of $\mathcal{P}^a$ defines the Schrödinger representation of any fibre of the Heisenberg group bundle and a quantization procedure for homogeneous quadratic polynomials on the real line visualised as a transport along field lines of internal degrees of freedom in F^a. This is related to signal transmission.

Keywords: Complex line bundles, Schrödinger representation, signal transmission, quantization

> *Wenn man das leisten könnte, was man meines Erachtens wird leisten können, auch den Begriff des dreidimensionalen Raums aus der abstrakten Quantentheorie herzuleiten - und damit auch die Begriffe von Feld und Teilchen aus der abstrakten Quantentheorie - dann ist Quantentheorie primär nicht eine Theorie über Materie, sondern über Information, genauer "über Bits in der Zeit".*
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> - C. F. v. Weizsäcker, Heisenberg als Physiker und Philosoph, Rede zum Tod von Werner Heisenberg anlässlich der Gedenkfeier des Max-Planck-Instituts am 12. Mai 1976 in München.

1 Introduction

It is well-known that the quadratic approximation in optics yields in a natural way a quantization procedure of quadratic homogeneous polynomials on the real line (Guillemin, Sternberg, 1991). A basic ingredient on which this quantization relies is the three-dimensional Heisenberg group. Moreover, in the treatment of radar signals as done in (Schempp, 1986) and in the mathematical description of MRI as presented in (Schempp, 1998), the same sort of quantization procedure appears. The fundamental ingredient in both of these procedures is again the three-dimensional Heisenberg group with its harmonic analysis (Schempp, 1986), (Folland, 1989). The metaplectic group involved in the quantization is the symmetry group of the coadjoint orbits of the Heisenberg group mentioned above.

The aim of this note is to show a natural relation between the transmission of a signal along a special vector field and the quantization procedure of homogeneous quadratic polynomials on the real line. The key ingredient in the construction of this quantization is the three-dimensional Heisenberg group again.

The relation mentioned above is based on a natural type of internal degrees of freedom associated with a singularity free vector field X defined on an open part O of a three-dimensional Euclidean space E. This kind of internal degrees of freedom is directly related to the points of (non-trivial) coadjoint orbits of Heisenberg groups, constructed from the vector field X, as we will see.

If the vector field X is a gradient field with principal part a, say, then there are natural bundles over O such as a complex line bundle F^a (of integral Chern class) with a fibre-wise defined symplectic form ω^a, a Heisenberg group bundle G^a and a four-dimensional principal bundle $\mathcal{P}^a$ with the structure group $U(1)$. (Fibres over O are indicated by a lower index x.) For any $x \in O$ the fibre F_x^a is the orthogonal complement of $a(x)$ formed in E and encodes internal degrees of freedom at x. The elements of F_x^a are called the internal variables. The fibre of x is, moreover, identified as a coadjoint orbit of G_x^a. The principal bundle $\mathcal{P}^a$, a subbundle of the fibre bundle F^a (associated to $\mathcal{P}^a$), is equipped with a natural connection form α^a, encoding the vector field in terms of the geometry of the local level surfaces: The field X can be reconstructed from α^a. The collection of all internal variables in F^a provides all tangent vectors to all locally given level surfaces. The curvature Ω^a of α^a describes the geometry of the level surfaces of the gradient field in terms of ω^a and the Gaussian curvature. For central symmetric vector fields $\mathcal{P}^a|_{S^2}$ is diffeomorphic to the orthogonal frame bundle of S^2.

To demonstrate the mechanism we have in mind, the principal part a of the vector field X is assumed to be constant (for simplicity only). Thus the solution curves, i.e. the field lines, are straight lines. Fixing some $x \in O$ and a solution curve β passing through $x \in O$, we consider the collection of all geodesics on the restriction of the principal bundle $\mathcal{P}^a$ to β. This restriction is a cylinder. Each of these geodesics, called a periodic lift of β, has the same speed as β and passes through a common initial point $v_x \in \mathcal{P}_x^a$, say. This collection of periodic lifts of β defines a unitary representation of the Heisenberg group G_x^a, the Schrödinger representation (Guillemin, Sternberg, 1991).

The automorphism group of G_x^a is the symplectic group $Sp(F_x^a)$ of the symplectic complex line F_x^a. Therefore, representations of G_x^a yield projective representations of $Sp(F_x^a)$, due to the theorem of Stone-von Neumann. This projective representation is resolved to a unitary representation of the metaplectic group $Mp(F_x^a)$ in the usual way. Associated with this kind of representation is a geometric construction of a transport of all internal degrees of freedom in F_x^a. This transport visualises the metaplectic representations in geometric terms. Its infinitesimal representation of the Lie algebra $mp(F_x^a)$ of $Mp(F_x^a)$ yields the quantization procedure on all homogeneous polynomials defined on the real line. Of course, this is in analogy to the quantization procedure mentioned above emanating from the quadratic approximation in optics.

The link to the transmission of signals is made as follows: The choice of $v_x \in \mathcal{P}_x^a$ turns F_x^a into a field of complex numbers. The real axis corresponds to the q-axis on which the complex-valued signals are defined. A geodesic γ given on $\mathcal{P}^a|_\beta$ transmits this signal ψ to some F^a_y with y on the curve β. The values of the signal ψ are in F^a_y which is also turned into a field isomorphic to the complex numbers via the horizontal lift of β passing through v_x. Detecting this signal means to bring the transmitted signal into resonance with the reference signal being the transmission of ψ along a horizontal lift of β. This reference signal has the same spatial initial condition v_x on $\mathcal{P}^a_x$ as γ. The channel of information transmission is the field line β. This kind of transmission yields a Schrödinger type of equation for a linear subspace of signals made time dependent via the Schrödinger representation. This procedure is then extended to data on all of F^a_x again reflecting and visualising geometrically the metaplectic representation mentioned above.

We point out here that the complex number i appearing in the formalism of quantum mechanics corresponds to the unit vector $\frac{a}{|a|}$ of the channel of information. This unit vector $\frac{a}{|a|}$ is the imaginary unit in a commutative subfield of the skew field of quaternions. This subfield is isomorphic to $\mathbb{C}$. Hence $\frac{a}{|a|}$ realizes i.

2 The Complex Line Bundle Associated with a Singularity Free Gradient Field in Euclidean Space

Let O be an open subset not containing the zero vector 0 in a three-dimensional oriented $\mathbb{R}$-vector space E with scalar product $< . >$. The orientation on the Euclidean space E shall be represented by the Euclidean volume form μ_E.

Our setting relies on a smooth, singularity free vector field $X : O \longrightarrow O \times E$ with principal part $a : O \longrightarrow E$, say.

Moreover, let $\mathbb{H} := \mathbb{R} \cdot e \oplus E$ be the skew field of quaternions where e is the multiplicative unit element. The scalar product $< . >$ on E extends to all of $\mathbb{H}$ such that $e \in \mathbb{H}$ is a unit vector and the above splitting of $\mathbb{H}$ is orthogonal. The unit sphere S^3, i.e. $Spin(E)$, is naturally isomorphic to $SU(2)$ and covers $SO(E)$ twice (Greub, 1975, 1978).

Given any $x \in O$, the orthogonal complement F^a_x of $a(x) \in E$ is a complex line as can be seen as follows: Let $\mathbb{C}^a_x \subset \mathbb{H}$ be the orthogonal complement of F^a_x. Hence the field of quaternions $\mathbb{H}$ splits orthogonally into

$$\mathbb{H} = \mathbb{C}^a_x \oplus F^a_x. \tag{1}$$

As it is easily observed,

$$\mathbb{C}^a_x = \mathbb{R} \cdot e \oplus \mathbb{R} \cdot \frac{a(x)}{|a(x)|}$$

is a commutative subfield of $\mathbb{H}$ naturally isomorphic to $\mathbb{C}$ due to

$$\left(\frac{a(x)}{|a(x)|}\right)^2 = -e \qquad \forall x \in O,$$

where $|\cdot|$ denotes the norm defined by $< . >$. Obviously $\frac{a(x)}{|a(x)|}$ realizes $i \in \mathbb{C}$. This isomorphism shall be called $i^a_x : \mathbb{C} \longrightarrow \mathbb{C}^a_x$; it maps 1 to e and i to $\frac{a(x)}{|a(x)|}$. The multiplicative group on the unit circle of $\mathbb{C}^a_x$ is denoted by $U^a_x(1)$. It is a subgroup of $SU(2) \subset \mathbb{H}$ and hence a group of spins. Obviously $\frac{a(x)}{|a(x)|}$ generates the Lie algebra of $U^a_x(1)$.

F^a_x is a $\mathbb{C}^a_x$-linear space under the (right) multiplication of $\mathbb{H}$ and hence is a $\mathbb{C}$-linear space, a complex line. Moreover, $\mathbb{H}$ is the Clifford algebra of F^a_x equipped with $-<\,,\,>$ (Greub, 1978). The topological subspace $F^a := \bigcup_{x\in O}\{x\} \times F^a_x$ of $O \times E$ is a $\mathbb{C}$-vector subbundle of $O \times E$, if curl $X = 0$, as it is easily seen. In this case F^a is a complex line bundle (Sniatycki, 1980), the complex line bundle associated with X. Let $\mathrm{pr}^a : F^a \longrightarrow O$ be its projection. Accordingly there is a bundle of fields $\mathbb{C}^a \longrightarrow O$ with fibre $\mathbb{C}^a_x$ at each $x \in O$. Clearly,

$$O \times \mathbb{H} = \mathbb{C}^a \times F^a,$$

as vector bundles over O.

We therefore assume that curl $X = 0$ from now on.

Due to this assumption there is a locally given real-valued function V, a potential of a, such that $a = \mathrm{grad}\, V$. Each (locally given) level surface S of V obviously satisfies $TS = F^a|_S$. Here $F^a|_S = \bigcup_{x\in S}\{x\} \times F^a_x$. Each fibre F^a_x of F^a is oriented by its Euclidean volume form $\mu_E(\frac{a(x)}{|a(x)|}, \ldots\ldots)$. For any level surface the scalar product yields a Riemannian metric g_S on S given by

$$g_S(x; v_x, w_x) := < v_x, w_x > \qquad \forall x \in O \text{ and } \forall v_x, w_x \in T_xS.$$

For any vector field Y on S, any $x \in O$ and any $v_x \in T_xS$, the covariant derivative ∇^S of Levi-Cività determined by g_S satisfies

$$\nabla^S_{v_x} Y(x) \;:=\; dY(x; v_x) + < Y(x), W^a_x(v_x) > .$$

Here $W^a_x : T_xS \longrightarrow T_xS$ is the Weingarten map of S assigning to each $w_x \in T_xS$ the vector $d\frac{a}{|a|}(x; w_x)$, the differential of $\frac{a}{a}$ at x evaluated at w_x.

A simple, but fundamental observation in our setting is that each fibre $F^a_x \subset F^a$ carries a natural symplectic structure ω^a defined by

$$\omega^a(x; h, k) := < h \times a(x), k > = < h \cdot a(x), k > \qquad \forall h, k \in F^a_x.$$

where $\times$ is the cross product in E, being here identical with the product in $\mathbb{H}$.

Let $\kappa(x) := \det W^a_x$ for all $x \in S$, the Gaussian curvature of S. The relation between the Riemannian curvature $\dot{R}$ and ω is given by

$$\dot{R}(x; v_x, w_x : u_x, y_x) = \frac{\kappa(x)}{|a(x)|} \cdot \omega^a(x; u_x, y_x) \qquad \forall x \in S \quad \text{and} \quad \forall u_x, y_x \in T_xS. \tag{2}$$

provided v_x, w_x is an orthonormal basis of T_xS.

3 The Natural Principal Bundle $\mathcal{P}^a$ Associated with X

We recall that the vector field X on O has the form $X = (\mathrm{id}\,, a)$. Let $K_x^a \subset F_x^a$ be the circle centred at zero with radius $|a(x)|^{-\frac{1}{2}}$ for any $x \in O$. Then

$$\mathcal{P}^a := \bigcup_{x\in O} \{x\} \times K_x^a$$

equipped with the topology induced by F^a is a four-dimensional submanifold of F^a. It inherits its smooth fibre-wise orientation from F^a. Moreover, $\mathcal{P}^a$ is a $U(1)$-principal bundle. $U(1)$ acts from the right on the fibre $\mathcal{P}_x^a$ of $\mathcal{P}^a$ via $i_x^a : U(1) \longrightarrow U_x^a(1)$ for any $x \in O$. The reason for choosing the radius of K_x^a to be $|a(x)|^{-\frac{1}{2}}$ will be made apparent below.

Both F^a and $\mathcal{P}^a$ consist of internal variables and both are constructed from X, of course. Clearly, the vector bundle F^a is associated with $\mathcal{P}^a$. Apparently the vector field X can be reconstructed from the smooth fibre-wise oriented principal bundle $\mathcal{P}^a$ as follows: The vector field X admits a characteristic geometric object, namely the smoothly fibre-wise oriented principal bundle $\mathcal{P}^a$ on which all properties of X can be reformulated in geometric terms. Vice versa, all geometric properties of $\mathcal{P}^a$ reflect characteristics of X in particular of a.

The fibre-wise orientation can be implemented in a more elegant way by introducing a connection form, α^a, say, which is in fact much more powerful. This will be our next task.

Since $\mathcal{P}^a \subset O \times E$, any tangent vector $\xi \in T_{v_x}\mathcal{P}^a$ can be represented as a quadruple

$$\xi = (x, v_x, h, \zeta_{v_x}) \in O \times E \times E \times E$$

with $x \in O$, $v_x \in \mathcal{P}_x^a$ and $h, \zeta_{v_x} \in E \subset \mathbb{H}$ with the following restrictions, expressing the fact that ξ is tangent to $\mathcal{P}^a$:

Given a curve $\sigma = (\sigma_1, \sigma_2)$ on $\mathcal{P}^a$ with $\sigma_1(s) \in O$ and $\sigma_2(s) \in \mathcal{P}^a_{\sigma_1(s)}$ for all s, then

$$< \sigma_2(s), a(\sigma_1(s)) >= 0 \quad \text{and} \quad |\sigma_2(s)|^2 = \frac{1}{|a(\sigma_1(s))|} \qquad \forall s.$$

Each $\zeta \in T_{v_x}\mathcal{P}^a$ given by $\zeta = \dot{\sigma}_2\,(0)$ is expressed as

$$\zeta = r_1 \cdot \frac{a(x)}{|a(x)|} + r_2 \cdot \frac{v_x}{|v_x|} + r \cdot \frac{v_x \times a(x)}{|v_x| \cdot |a(x)|} \tag{3}$$

with

$$r_1 = - < W_x^a(v_x), h > \quad , \quad r_2 = -\frac{|v_x|}{2} \cdot d\ln|a|(x; h)$$

and a free parameter $r \in \mathbb{R}$. The Weingarten map W_x^a is directly determined by the differential da of a which is of the form

$$da(x;k) = |a(x)| \cdot W_x^a(k) + a(x) \cdot d\ln|a|(x;k) \qquad \forall x \in O\,, \forall k \in E, \tag{4}$$

where we set $W_x^a(a(x)) = 0$ for all $x \in O$. With these preparations we define the one-form $\alpha^a : T\mathcal{P}^a \longrightarrow \mathbb{R}$ for each $\xi \in T\mathcal{P}^a$ with $\xi = (x, v_x, h, \zeta)$ to be

$$\alpha^a(v_x, \xi) := < v_x \times a(x), \zeta > .$$

One easily shows that α^a is a connection form (cf. (Greub et al., 1973) and for the field theoretic aspect (Binz et al., 1988)). To match the requirement of a connection form, the size of the radius of $\mathcal{P}^a_x$ for any $x \in O$ is crucial. The connection form provides $\mathcal{P}^a$ with a smooth fibre-wise orientation among other qualities, of course.

Thus the principal bundle $\mathcal{P}^a$ together with the connection form α^a characterises the vector field X, and vice versa. In other words the geometry of the collection of all internal variables in $\mathcal{P}^a$ characterizes X uniquely.

Next we will determine the curvature Ω^a of α^a. It is defined to be the exterior covariant derivative of α^a. To this end the horizontal bundles in $T\mathcal{P}^a$ will be characterised. Given $v_x \in \mathcal{P}^a$, the horizontal subspace $Hor_{v_x} \subset T\mathcal{P}^a$ is defined by

$$Hor_{v_x} := ker\ \alpha^a(v_x; ..).$$

A vector $\xi_{v_x} \in Hor_{v_x}$, being orthogonal to $v_x \times a(x)$, has the form $(x, v_x, h, \zeta^{hor}) \in O \times E \times E \times E$ where h varies in O and ζ^{hor} satisfies

$$\zeta^{hor} = - < W^a_x(v_x), h > \cdot \frac{a(x)}{|a(x)|} - \frac{|v_x|}{2} \cdot d \ln |a|(x; h) \cdot \frac{v_x}{|v_x|}.$$

The exterior covariant derivative $d^{hor}\alpha^a$ is defined by

$$d^{hor}\alpha^a(v_x, \xi_0, \xi_1) := d\alpha^a(v_x; \xi_0^{hor}, \xi_1^{hor}) \qquad \forall\, \xi_0, \xi_1 \in T_{v_x}\mathcal{P}^a\ ,\ \forall\, v_x \in \mathcal{P}^a_x \text{ and } x \in O$$

(Binz et al., 1988), (Sniatycki, 1980). The curvature $\Omega^a := d^{hor}\alpha^a$ is sensitive in particular to the geometry of the (locally given) level surfaces. The following is easily proved:

Proposition 1 *Let X be a smooth singularity free vector field on O with principal part a. The curvature Ω^a of the connection form α^a is*

$$\Omega^a = \frac{\kappa}{|a|} \cdot \omega^a$$

where $\kappa : O \longrightarrow I\!R$ is the leaf-wise defined Gaussian curvature on the foliation of O given by the collection of all level surfaces of the locally determined potential V. The curvature Ω^a vanishes if one argument at any x is a multiple of $a(x)$; in particular it vanishes along field lines of X.

The curvature Ω^a is integral since $\frac{1}{|a|} \cdot \omega^a$ is the Riemannian curvature and hence by the theorem of Gauss-Bonnet

$$\frac{1}{4\pi} \int_S \Omega^a = \chi(S)$$

where $\chi(S)$ denotes the Euler characteristic of S, which is an integer.

The fact that the curvature Ω^a vanishes along field lines plays a crucial role in our set-up. From section five on we will assume $a = \text{const.}$ in order to demonstrate on a simple model the relation between the transmission of internal variables along field lines of X and the quantization of homogeneous quadratic polynomials on the real line. Clearly, the field lines of X are straight lines in this special case.

4 An Example

As an example of a principal bundle $\mathcal{P}^a$ associated with a singularity free vector field let us consider a central symmetric field $X = \text{grad } V$ on E with the only singularity at the origin. In particular, this is to say that the principal part a is invariant under $SO(E)$. The field lines of X are thus straight lines emerging from $0 \in E$. Restricting X to $O := E\backslash\{0\}$ yields a singularity free vector field with level surfaces S_r^2, i.e. spheres of radius r centered at $O \subset E$. Obviously $a(x) = \pm|a(x)| \cdot \frac{x}{|x|}$ for all $x \in O$. The Gaussian curvature $\kappa_{S_r^2}(x)$ of S_r^2 satisfies $\kappa_{S_r^2}(x) = \frac{1}{r^2}$ at any $x \in S_r^2$ and for all $r > 0$. In fact the following holds true:

The well-known Hopf projection pr_H of S^3 to S^2 extends to all of $\dot{\mathbb{H}}$ by setting

$$\text{pr}_H(r \cdot u) = r \cdot \tau_u(x_0) \qquad \forall\, r \in \dot{\mathbb{R}} \text{ and } \forall\, u \in SU(2)$$

where τ_u is the inner automorphism of $\mathbb{H}$ given by $u \in SU(2)$. Hence $\dot{\mathbb{H}}$ is a $U(1)$–principal bundle with $\text{pr}_H : \dot{\mathbb{H}} \longrightarrow O$ as its projection, here called the extended Hopf fibration. Therefore, we state:

Proposition 2 *The extended Hopf fibration of $\dot{\mathbb{H}}$ over $O := E\backslash\{0\}$ defined by the projection pr_H is the two-fold covering of the principal bundle $\mathcal{P}^a$ of any central symmetric gradient field in E with the only singularity at the origin.*

The above proposition visualises the geometry of the level surfaces and the field lines in terms of the principal bundle $\mathcal{P}^a$. The field strength of the vector field is encoded in the connection form α^a, of course. The Hopf fibration plays an important role in teleportation, planetary motion and the treatment of the magnetic monopole (Binz. Schempp, 1999, 2000a), (Greub. Petry, 1975).

5 Horizontal Lifts and Periodic Lifts of β

Let $a = \text{const.}$ from now on. Since $\Omega^a \neq 0$, in general, the horizontal distribution is not integrable along level surfaces. However, Ω^a vanishes along field lines of X. Let us look at $\mathcal{P}^a|_\beta$ where β is a field line of X. Due to $a = \text{const.}$, β is a straight line and hence $\mathcal{P}^a|_\beta$ is diffeomorphic to a cylinder.

A horizontal lift of β is a curve β^{hor} in $Hor_\beta = \ker \alpha^a$ which satisfies $T\text{pr}^a \dot{\beta}^{hor} = \dot{\beta} = a$ and obeys an initial condition in $T\mathcal{P}^a|_\beta$. Hence there is a unique curve $\beta^{hor}_{v_{\beta(0)}}$ passing through $v_{\beta(0)} \in \mathcal{P}^a_{\beta(0)}$, say, called horizontal lift of β. This is nothing else but a meridian of the cylinder $\mathcal{P}^a|_\beta$ containing $v_{\beta(0)}$. Let $\beta(0) = x$.

Obviously, a horizontal lift is a geodesic on $\mathcal{P}^a|_\beta$ equipped with the metric g_{Hor_β}, say, induced by the scalar product $<\,,\,>$ on E.

Here a curve γ on $\mathcal{P}^a|_\beta$ is called a periodic lift of β through v_x iff it is of the form

$$\gamma(s) = \beta^{hor}_{v_x}(s) \cdot e^{p \cdot s \cdot \frac{a}{|a|}} \in \mathcal{P}^a_{\beta(s)} \qquad \forall\, s, \tag{5}$$

where p is a fixed real.

Clearly, γ is a horizontal lift through v_x iff $\gamma = \beta^{hor}_{v_x}$, i.e. iff $p = 0$. In fact any periodic lift γ of β is a geodesic on $\mathcal{P}^a|_\beta$.

Due to the $U(1)$–symmetry of $\mathcal{P}^a|_\beta$, a geodesic σ on $\mathcal{P}^a|_\beta$ is of the form

$$\sigma(s) = \beta_{v_x}^{hor}(\theta \cdot s) \cdot e^{p \cdot \theta \cdot s \cdot \frac{a}{|a|}} \qquad \forall\, s$$

as it is easily verified. Here p and θ denote reals. θ determines the speed of the geodesic. Thus σ and β have accordant speeds if $\theta = 1$. as is easily seen from

$$\dot{\gamma}(s) = p \cdot \beta_{v_x}^{hor}(s) \cdot \frac{a}{|a|} \cdot e^{p \cdot s \cdot \frac{a}{|a|}} + \dot{\beta}_{v_x}^{hor}(s) \in E \qquad \forall\, s$$

and in particular from

$$\dot{\gamma}(0) = p \cdot v_x \cdot \frac{a}{|a|} + \dot{\beta}_{v_x}^{hor}(0).$$

The real number p determines the frequency of the periodic lift γ due to $\frac{2 \cdot \pi}{T} = \frac{p}{|v_x|}$. The frequency of γ counts the number of revolutions around $\mathcal{P}^a|_\beta$ per unit time and is determined by the F_x^a-component of the initial velocity. On the other hand p is the F_x^a–component of the momentum of the motion γ, due to the $U(1)$–symmetry of the cylinder $\mathcal{P}^a|_\beta$.

Any periodic lift γ of β through v_x is uniquely determined by the $U^a(1)$-valued map

$$s \longmapsto e^{p \cdot s \cdot \frac{a}{|a|}}.$$

here called an elementary periodic function. Therefore, we can state:

Proposition 3 *Let $x = \beta(0)$. Under the hypothesis $a = const.$ made above there is a one-to-one correspondence between all elementary periodic $U^a(1)$-valued functions and all periodic lifts of β passing through a given $v_x \in \mathcal{P}_x^a$.*

$U^a(1) \subset SU(2) = Spin(E)$ (Greub, 1978) is a one-parameter group of spins generated by the unit vector $\frac{a}{|a|}$: it acts on F_x^a. Thus a periodic lift on $\mathcal{P}^a|_\beta$ can be interpreted as the evolution of an initial spin v_x in F_x^a along a geodesic on $\mathcal{P}^a|_\beta$. In this context the line $\mathbb{R} \cdot \frac{a}{|a|}$ is the channel of information transmission.

On the other hand an internal variable of X can be interpreted as a piece of information. Thus the fibres F_x^a and $\mathcal{P}_x^a$ can be regarded as a collection of pieces of information at x and the periodic lifts of β on $\mathcal{P}^a|_\beta$ describe the evolution of pieces of information of $\mathcal{P}^a|_\beta$ along β. We shall show later how the information in F_x^a evolves along β. This evalution will be linked to the quantization procedure of homogeneous quadratic polynomials on the real line.

6 The Heisenberg Group Bundle Associated with the Singularity Free Vector Field

Associated with the $(2+1)$–splitting of the Euclidean space E caused by the vector field X there is a natural Heisenberg group bundle G^a with ω^a as symplectic form. The bundle G^a allows to reconstruct X as well. Heisenberg groups play a central role in signal theory (Schempp, 1986, 1998).

Given $x \in O$ the vector $a(x)$ determines F_x^a and $\mathbb{C}_x^a$ which decompose $\mathbb{H}$ according to (1).

The submanifold $G^a_x := U^a_x(1) \oplus F^a_x$ of $I\!H$ carries a Heisenberg group structure the multiplication of which is defined by

$$(z_1 + h_1) \cdot (z_2 + h_2) := z_1 \cdot z_2 \cdot e^{\frac{1}{2} \cdot \omega^a(x; h_1, h_2) \cdot \frac{a}{|a|}} + h_1 + h_2$$

for any two $z_1, z_2 \in U^a_x(1)$ and any pair $h_1, h_2 \in F^a_x$. The collection

$$G^a := \bigcup_{x \in O} \{x\} \times G^a_x$$

can be made into a group bundle associated with the principal bundle $\mathcal{P}^a$. Clearly $F^a \subset G^a$ as fibre bundles. Due to the assumption $a = \text{const.}$ on O, all the bundles, the complex line bundle F^a, the principal bundle $\mathcal{P}^a$ and G^a are trivial. This means that the field X can be described in only one Heisenberg group. This is the standpoint adopted in (Schempp, 1998).

The Lie algebra $\mathcal{G}^a_x$ of G^a_x is

$$\mathcal{G}^a_x := I\!R \cdot \frac{a}{|a|} \oplus F^a_x$$

together with the operation

$$[\vartheta_1 \cdot \frac{a}{|a|} + h_1, \vartheta_2 \cdot \frac{a}{|a|} + h_2] := \omega^a(x; h_1, h_2) \cdot \frac{a}{|a|}$$

for any $\vartheta_1, \vartheta_2 \in I\!R$ and any $h_1, h_2 \in F^a_x$. The exponential map $\exp_{G^a_x} : \mathcal{G}^a_x \longrightarrow G^a_x$ is surjective. In the case under consideration, the Lie algebra bundle $\mathcal{G}^a$, frequently called the adjoint bundle of G^a, is trivial as well. Obviously, X can be reconstructed from both G^a and $\mathcal{G}^a$. The algebras $\mathcal{G}^a_{\beta(0)}$ and $\mathcal{G}^a_{\beta(\tau)}$ are naturally isomorphic, as is easily seen.

Given $x \in O$ the adjoint operation $Ad^a : G^a_x \times \mathcal{G}^a_x \longrightarrow \mathcal{G}^a_x$ acts by

$$Ad^a_{z+h}(\vartheta \cdot \frac{a}{|a|} + h_1) = (\vartheta + \omega^a(x; h, h_1)) \cdot \frac{a}{|a|} + h_1$$

for any $z + h \in G^a_x$ and any $\vartheta \cdot \frac{a}{|a|} + h_1 \in \mathcal{G}^a_x$. The coadjoint action $Ad^{a^*} : G^a_x \times \mathcal{G}^{a^*}_x \longrightarrow \mathcal{G}^{a^*}_x$ sends each $< \vartheta \cdot \frac{a}{|a|} + h_1, .. > \in \mathcal{G}^{a^*}_x$ applied to $(\vartheta' \cdot \frac{a}{|a|} + k) \in \mathcal{G}^a_x$ into

$$\begin{aligned} Ad^{a^*}_{(z+h)^{-1}}(< \vartheta \cdot \frac{a}{|a|} + h_1, .. >)(\vartheta' \cdot \frac{a}{|a|} + k) &= < \vartheta \cdot \frac{a}{|a|} + h_1, (\vartheta' - \omega^a(x; h_1, k)) \cdot \frac{a}{|a|} + k > \\ &= \vartheta \cdot (\vartheta' - \omega^a(x; h, k)) + < h_1, k > . \end{aligned}$$

Therefore, the coadjoint orbit of Ad^{a^*} passing through $< \vartheta \cdot \frac{a}{|a|} + h_1, .. >$ with $\vartheta \neq 0$ is

$$\vartheta \cdot \frac{a}{|a|} \oplus F^a_x. \tag{6}$$

This orbit is a real affine space the underlying vector space of which is a complex line and a symplectic manifold, of course. The symplectic structure is $\omega^a(x; \ldots, \ldots)$. This establishes the relation between our geometric setting and the use of coadjoint orbits of the Heisenberg group in (Schempp, 1986, 1998). In our setting these coadjoint orbits serve as screens.

7 Representation of the Heisenberg Group Associated with Periodic Lifts of β on $\mathcal{P}^a|_\beta$

Throughout this section β denotes a field line of X, a straight line, with initial condition $\beta(0) = x$. Again $\beta^{hor}_{v_x}$ is the horizontal lift of β through $v_x \in \mathcal{P}^a_x$. There is a unique periodic lift γ of β passing through $v_x = \gamma(0)$ with prescribed velocity $\dot\gamma(0)$. At first we will associate with $\dot\gamma(0)$ a well-defined unitary linear operator on a Hilbert space as follows:

The specification of $v_x \in \mathcal{P}^a_x$ turns F^a_x into a field $\hat F^a_x$ isomorphic to $\mathbb{C}$, since $\frac{v_x}{|v_x|}\cdot\mathbb{C} = F^a_x$. The real axis is $\mathbb{R}\cdot\frac{v_x}{|v_x|}$ and the imaginary one is $\mathbb{R}\cdot\frac{v_x}{|v_x|}\times\frac{a}{|a|}$. We rename these axes by q-axis carried by the unit vector $\bar q_x$ and by p-axis carried by the unit vector $\bar p_x$, respectively. Clearly, $\bar p_x = \bar q_x \cdot i^a_x(i)$. Any $h \in F^a_x$ is thus of the form $h = (q,p)$.

The Schrödinger representation ρ_x of G^a_x acts on each $\psi \in \mathcal{S}(\mathbb{R}_q) \subset L^2(\mathbb{R}_{q_x})$ by

$$\rho_x(z+h)(\psi(\tau)) := z^{-1}\cdot e^{|a|\cdot p\cdot\tau\cdot i}\cdot e^{-|a|\cdot\frac{p\cdot q}{2}\cdot i}\cdot\psi(\tau - q) \qquad \forall\tau\in\mathbb{R} \tag{7}$$

for all $z + h \in G^a_x$ with $h = (p,q)$ (Guillemin, Sternberg, 1991), (Schempp, 1986). Here $\mathcal{S}(\mathbb{R}_q)$ is the Schwartz space of $\mathbb{R}_q$. Clearly,

$$-p\cdot q\cdot i = \omega^a_x((p,0),(0,q))\cdot i \quad\text{and}\quad z^{-1} = e^{-\vartheta\cdot\frac{a}{|a|}}$$

for some $\vartheta \in \mathbb{R}$. By the Stone-von Neumann theorem ρ_x is irreducible (Schempp, 1986).

Setting $z = 1$ and $q = |v_x|$, for any $p \in \mathbb{R}$, equation (7) turns into

$$\rho(1+(p,|v_x|))\psi(\tau+\frac{|v_x|}{2}) = e^{|a|\cdot p\cdot\tau\cdot i}\cdot\psi(\tau-\frac{|v_x|}{2}) \qquad \forall\tau\in\mathbb{R}. \tag{8}$$

Operators of this form generate $\rho(G^a_x)$, of course.

On the other hand the $U^a_x(1)$-valued function $\tau \longrightarrow e^{p\cdot\tau\cdot\frac{a}{|a|}}$ entirely describes the periodic lift γ passing through v_x as expressed in (5). Thus γ is characterised by the unitary linear transformation $\rho(1+(p,|v_x|))$ on $L^2(\mathbb{R}_{\bar q_x})$.

Therefore, we state:

Theorem 4 *Any periodic lift γ of β on $\mathcal{P}^a|_\beta$ with initial conditions $\gamma(0) = v_x$ and momentum p is characterised by the unitary linear transformation $\rho_x(1+(p,|v_x|))$ of $L^2(\mathbb{R}_{\bar q_x})$ with $(1+(p,|v_x|)) \in G^a_x$. Vice versa any element $(1+(p,|v_x|))$ characterises via $\rho_x(1+(p,|v_x|))$ a unique periodic lift γ of β with spatial initial conditions v_x and momentum p. Thus $v_x \in \mathcal{P}^a_x$ determines a unitary representation ρ on $L^2(\mathbb{R}\cdot q_x)$ characterising the collection $C^a_{v_x}$ of all periodic lifts of β passing through v_x.*

We may reinterpret theorem 4 as follows: Any curve $\gamma \in C^a_{v_x}$ on $\mathcal{P}^a|_\beta$ with prescribed momentum p is an evolution of the spin v_x over β. The piece of information v_x is thus transported by γ with a velocity determined by the momentum p. Hence $\rho_x(1+(p,|v_x|))$ describes this transport of a piece of information along the field line of β of X in terms of a unitary linear operator. Since these operators generate the representation of G^a_x we may state the following:

Corollary 5 *The Schrödinger representation of G^a_x describes the transport of any piece of information $(p, |v_x|) \in T_{(v_x,0)}\mathcal{P}^a|_\beta$ along the field line β, with $I\!R \cdot a$ as information transmission channel.*

8 Time Dependent Signals Along β, their Evolution Equation and Detection

To construct a signal related to the internal degrees of freedom along the straight field line β, we first need to construct the bundle $\hat{F}^a_{\gamma_{v_x}}$ of complex numbers determined by a periodic lift γ_{v_x} of β. We will proceed according to section seven.

To do so, let γ_{v_x} be a periodic lift of β with initial condition $v_x \in \mathcal{P}^a_x \subset F^a_x$ and momentum p. For any τ the vector $\gamma_{v_x}(\tau) \in F^a_{\beta(\tau)}$ yields the orbit $\gamma_{v_x}(\tau) \cdot \mathbb{C}^a_{\beta(\tau)} = F^a_{\beta(\tau)}$. Thus $F^a_{\beta(\tau)}$ is turned into a field of complex numbers $\hat{F}^a_{\gamma_{v_x}(\tau)}$ as well. Its real axis is $I\!R \cdot \frac{\gamma_{v_x}(\tau)}{|\gamma_{v_x}(\tau)|}$ and the imaginary one is $I\!R \cdot \frac{\gamma_{v_x}(\tau)}{|\gamma_{v_x}(\tau)|} \times \frac{a}{|a|}$. It is evident that

$$\hat{F}^a_{\gamma_{v_x}} := \bigcup_\tau \{x\} \times \hat{F}^a_{\gamma_{v_x}(\tau)}$$

is a naturally given trivial bundle of fields of complex numbers determined by γ. A bundle of this sort will allow us to describe the mechanism of signal detection along the field line β. The role of the screen is taken over by $F^a_{\beta(\tau)}$.

In order to implement time dependence into a smooth signal $\psi \in \mathcal{S}(I\!R \cdot \bar{q}_x) \subset L^2(I\!R \cdot \bar{q}_x)$, let

$$\hat{\psi}_{\gamma_{v_x}}(\tau, t) := \frac{\gamma_{v_x}(\tau)}{|\gamma_{v_x}(\tau)|} \cdot e^{\nu \cdot t \cdot \frac{a}{a}} \cdot i^a_{(\beta(\tau))}(\psi(\tau)) \qquad \forall t \text{ and } \forall \tau.$$

an element of $F^a_{\beta(\tau)}$. Hence

$$\hat{\psi}_{\gamma_{v_x}}(\tau, t) = \frac{\beta^{hor}_{v_x}(\tau)}{|\beta^{hor}_{v_x}(\tau)|} \cdot e^{p \cdot \tau \cdot \frac{a}{a}} \cdot e^{\nu \cdot t \cdot \frac{a}{|a|}} \cdot i^a_{(\beta(\tau))}(\psi(\tau)) \qquad \forall t \text{ and } \forall \tau$$

or, expressed in terms of the Schrödinger representation,

$$\hat{\psi}(\tau, t) := i^a_{(\beta(\tau))}(\rho(e^{-\nu \cdot t \cdot \frac{a}{a}} + (p, |v_x|)) \cdot \psi(\tau + |v_x|)).$$

In fact $i^a_{(\beta(\tau))}$ does not vary with τ due to $a = $ const. However, $i^a_{\gamma_{v_x}(\tau)} : \mathbb{C} \longrightarrow \hat{F}^a_{\gamma_{v_x}(\tau)}$ does depend on τ. We call $\hat{\psi}_{\gamma_{v_x}}$ the internal transmission of the signal ψ associated with the periodic lift γ_{v_x} of β having the momentum p and the frequency ν.

The above construction of time dependence refers to the information transmission channel. Let us justify this construction in terms of a Minkowski metric on $I\!H$. To do so we reconstruct $I\!H$ from the oriented Euclidean space E with its scalar product by setting $I\!H = I\!R \cdot e \oplus E$ and extend $<\,,\,>$ to all of $I\!H$ by requiring $< e, e >= 1$. The operations on $I\!H$ are then defined by

$$(\lambda_1 \cdot e + h_1) \cdot (\lambda_2 \cdot e + h_2) := \lambda_1 \cdot \lambda_2 \cdot e + \lambda_1 \cdot h_2 + \lambda_2 \cdot h_1 + h_1 \cdot h_2$$

where $h_1 \cdot h_2 = h_1 \times h_2 - < h_1, h_2 > \cdot e$ for all $\lambda_1, \lambda_2 \in I\!R$ and all $h_1, h_2 \in E$. The Heisenberg algebra $\mathcal{G}_x^a = I\!R \cdot \frac{a}{|a|} \oplus F_x^a$ has $I\!R \cdot \frac{a}{|a|}$ as its centre. The above construction of time dependence implements time on this axis. This suggests to multiply $\mathcal{G}_x^a$ by $-a$ which yields

$$-a \cdot \mathcal{G}_x^a = +I\!R \cdot |a| \cdot e \oplus F_x^a,$$

for $\nu = 1$. The Minkowski metric $(\ ,\ )$ on $I\!H$ we have in mind is given by

$$(t \cdot e + h, t \cdot e + h) := -t^2 \cdot c^2 + < h, h > \qquad \forall t \in I\!R \quad \text{and} \quad h \in E$$

with a positive real c. Time is thus implemented by the imaginary quaternion $c \cdot t \cdot \frac{a}{|a|}$. Setting $c := |a|$ in particular yields

$$|t \cdot c|^2 = -t^2 \cdot |a|^2 = t^2 \cdot a^2 = (-t \cdot a)^2 \qquad \forall t \in I\!R.$$

$c = |a|$ is correlated with the speed of β since $|\dot{\beta}(t)|^2 = |a|^2$.

Thus the Minkowski metric on $I\!H$ reflects the speed of the horizontal transport of information along β.

Next we turn to the evolution of $\hat{\psi}$. The values of $\hat{\psi}_{\gamma_{v_x}}$ are in the screen $\hat{F}^a_{\gamma_{v_x}(\tau)}$. Hence

$$-\left(\frac{a}{|a|}\right)^{-1} \cdot \frac{\partial \hat{\psi}_{\gamma_{v_x}}}{\partial t} = \nu \cdot \hat{\psi}_{\gamma_{v_x}} \tag{9}$$

is the evolution equation of $\hat{\psi}_{\gamma_{v_x}}$. If, for example, the signal ψ is a constant map, i.e. $\psi(\tau) = \text{const.}$ for all $\tau \in I\!R$, then

$$-\left(\frac{a}{|a|}\right)^{-1} \cdot \frac{\partial \hat{\psi}_{\gamma_{v_x}}}{\partial t} = \Delta \hat{\psi}_{\gamma_{v_x}},$$

an equation of Schrödinger type, provided $p^2 = \nu$. Clearly, there is a linear subspace in $L^2(I\!R \cdot q_x)$ of signals ψ for which $\hat{\psi}_{\gamma_{v_x}}$ satisfies a Schrödinger type of equation as an equation of evolution.

To compute the effect of $d\rho$ for the Schrödinger representation ρ, we write q_x and p_x in G_x as

$$q_x = \exp_{G_x^a} \vartheta \cdot Q \quad \text{and} \quad p_x = \exp_{G_x^a} \vartheta' \cdot P,$$

respectively, where Q and P are unit vectors in $\mathcal{G}_x^a$. Hence

$$d\rho(Q)(\psi)(\tau) = \frac{d}{d\vartheta}(\exp_{G_x^a} \vartheta \cdot Q)|_{\vartheta=0}(\psi)(\tau) = -\frac{d}{d\tau}\psi(\tau) \qquad \forall \tau \in I\!R$$

and

$$d\rho(P)(\psi)(\tau) = \frac{d}{d\vartheta'}(\exp_{G_x^a} \vartheta' \cdot P)|_{\vartheta'=0}(\psi)(\tau) = i \cdot \tau \cdot \psi(\tau) \qquad \forall \tau \in I\!R$$

with

$$d\rho([(Q,0),(0,P)]) = -i \cdot \mathrm{id}_{L^2(I\!R \cdot \bar{q}_x)} = \omega_x^a(Q,P) \cdot d\rho(\frac{a}{|a|})$$

yielding $\omega_x^a(Q,P) = 1$. This means that P and Q are orthogonal.

Now let us turn to signal detection in the most elementary form. Comparing the signals $\hat{\psi}_{\gamma_{v_x}}$ with the reference signal $\hat{\psi}_{\beta^{hor}_{v_x}}$ yields

$$\hat{\psi}_{\gamma_{v_x}}(\tau,t) = \hat{\psi}_{\beta^{hor}_{v_x}} \cdot e^{p \cdot t \cdot \frac{a}{|a|}}.$$

Hence $\hat{\psi}_{\gamma_{v_x}}$ can be detected in $\hat{F}^a_{\beta^{hor}_{v_x}(t)}$ iff the field $\hat{F}^a_{\beta^{hor}_{v_x}(t)}$ is rotated backwards about $e^{-p\cdot\tau\cdot\frac{a}{|a|}}$. In this case the signals $\hat{\psi}_{\gamma_{v_x}}$ and the reference signal $\hat{\psi}_{\beta^{hor}_{v_x}}$ are brought into resonance; in other words $\hat{\psi}_{\gamma_{v_x}}$ is detected in $\hat{F}^a_{\beta^{hor}_{v_x}(t)}$. In this context the filter $\hat{\psi}_{\beta^{hor}_{x}}$ generates a filter bank. For more advanced signal detection and processing techniques as used in magnet resonance imaging we refer for instance to (Schempp, 1998), in particular to section 2.8.

9 Periodic Lifts of β on $\mathcal{P}^a|_\beta$, the Metaplectic Group $Mp(F_x^a)$ and the Projective Metaplectic Representation

Let a be constant again. Given $v_x \in \mathcal{P}_x^a$ and $\dot{\gamma}_{v_x}(0)$ of a periodic lift γ_{v_x} of β,

$$\dot{\gamma}_{v_x}(0) = \dot{\gamma}_{v_x}(0)^{F_x^a} + \dot{\beta}^{hor}_{v_x}(0)$$

is an orthogonal splitting of the velocity of γ_{v_x} at 0. Clearly the F_x^a–component of $\dot{\gamma}_{v_x}(0)$ is $\dot{\gamma}_{v_x}(0)^{F_x^a} = p \cdot \bar{p}_x$, where p is the momentum. Thus the momenta of periodic lifts of β passing through v_x are in a one-to-one correspondence with elements in $T_{v_x}\mathcal{P}_x^a$.

Therefore, the collection $\bar{C}_x^a$ of all periodic lifts of β on $\mathcal{P}^a|_\beta$ is in a one-to-one correspondence with $T\mathcal{P}_x^a$ (being diffeomorphic to a cylinder) via a map $f : \bar{C}_x^a \longrightarrow T\mathcal{P}_x^a$, say. Let

$$c : T\mathcal{P}_x^a|_\beta \longrightarrow F_x^a$$

be given by $c := -T\tilde{c}$ where $\tilde{c} : \mathcal{P}_x^a \longrightarrow \mathcal{P}_x^a$ is the antipodal map. Thus

$$c(w_x,\lambda) = c(w_{-x},\lambda) = \lambda \qquad \forall\,(w_x,\lambda) \in T_{w_x}\mathcal{P}_x^a\,,\ \forall\, w_x \in \mathcal{P}_x^a \quad \text{and} \quad \forall\,\lambda \in I\!R.$$

Clearly, c is two-to-one. Setting $\dot{F}_x^a = F_x^a \backslash \{0\}$, the map

$$c \circ f : \bar{C}_x^a \longrightarrow \dot{F}_x^a$$

is two-to-one, turning $\bar{C}_x^a$ into a two-fold covering of $\dot{F}_x^a$. It describes the correspondence between periodic lifts in $\bar{C}_x^a$ and their momenta.

The symplectic group $Sp(F_x^a)$ acts transitively on F_x^a equipped with ω^a as symplectic structure. Therefore, the metaplectic group $Mp(F_x^a)$, which is the two-fold covering of $Sp(F_x^a)$, acts transitively on $T\mathcal{P}_x^a$.

Thus given $u \in F_x^a$, there is a smooth map

$$\theta : Sp(F_x^a) \longrightarrow F_x^a$$

given by $\theta(A) = A(u)$ for all $A \in Sp(F_x^a)$. Since $cof(u_{w_x}) = cof(u_{-w_x})$ for all $u_{w_x} \in T\mathcal{P}^a|_{\beta(0)}$, the map θ lifts smoothly to

$$\tilde{\theta} : Mp(F_x^a) \longrightarrow \bar{C}_x^a$$

such that

$$(cof) \circ \tilde{\theta} = \tilde{\mathrm{pr}} \circ \theta$$

where $\tilde{\mathrm{pr}} : Mp(F_x^a) \longrightarrow Sp(F_x^a)$ is the covering map. Clearly, the orbit of $Mp(F_x^a)$ on $\bar{C}_x^a$ is all of $\bar{C}_x^a$. Since $Mp(F_x^a)$ acts on F_x^a with a one-dimensional isotropy group (Schempp, 1998), we may state:

Proposition 6 *$Mp(F_x^a)$ acts transitively on $\bar{C}_x^a$ with a one-dimensional isotropy group.*

Moreover, $Sp(F_x^a)$ acts as an automorphism group of G_x^a by the operation

$$A(z+h) = z + A(h) \qquad \forall z + h \in G_x^a.$$

Any $A \in Sp(F_x^a)$ determines the irreducible unitary representation ρ_A defined by

$$\rho_A(z+h) := \rho(z + A(h)) \qquad \forall (z+h) \in G_x^a.$$

Due to the Stone-von Neumann theorem it must be equivalent to ρ itself, meaning that there is an intertwining unitary operator U_A on $L^2(\mathbb{R}_{q_x})$ determined up to a complex number in $\mathbb{C}_x^a$ of the absolute value one such that $\rho_A = U_A \circ \rho \circ U_A^{-1}$ and $U_{A_1} \circ U_{A_2} = coc(A_1, A_2) \cdot U_{A_1 \circ A_2}$ for any two A_1, A_2. Here coc is a cocycle with value $coc(A_1, A_2) \in \mathbb{C} \backslash \{0\}$. Thus U is a projective representation of $Sp(F_x^a)$ and hence defines a representation of $M_p(F_x^a)$.

This observation leads us to a relation between the quantization procedure of quadratic homogeneous polynomials on $\mathbb{R}$ and the transport of information along the field line β.

10 Quantization Procedure for Homogeneous Polynomials and the Evolution of Information on F_x^a

Let $x \in O$ be fixed and β a field line of X, a straight line, with $\beta(0) = x$.

The symplectic group operates transitively and without fixed points on the collection $\mathcal{B}_x^a$ of symplectic bases on F_x^a (cf. below). This operation will be transferred to lifts of β onto $\dot{F}_\beta^a$, the bundle F^a restricted to β without the image of the zero-section. To generalise the notion of periodic lifts of β with initial conditions to all of $\dot{F}_\beta^a$ we proceed as follows:

At first we consider the pair of vectors $v_x, p_x \in F_x^a$ where v_x is the initial position and $p_x = p \cdot \bar{p}_x$ the initial momentum of a periodic lift γ_1 of β. Clearly

$$\omega^a(\frac{v_x}{p}, p_x) = |a|^{\frac{1}{2}}.$$

In this context we call a basis $\{u, w\}$ of F^a_x symplectic if $\omega^a(u, v) = |a|^{\frac{1}{2}}$. There is, however, another periodic lift γ_2 of β starting opposite of v_x and having p_x as initial F^a_x-component. Thus the symplectic basis $\{\frac{v_x}{p}, p_x\}$ of F^a_x defines two lifts of β, namely γ_1 and γ_2. This observation initiates the following construction:

For any $l \in Sp(F^a_x)$

$$\omega^a(l(\frac{v_x}{p}), l(p_x)) = |a(x)|^{\frac{1}{2}}.$$

We lift the $Sp(F^a_x)$-action on F^a_x to an $Mp(F^a_x)$-action on a collection of lifts of β on $\dot{F}^a_\beta$ as follows: Given $l(\frac{v_x}{p})$ and $l(p_x)$, the cylinder $\mathcal{P}^a|_\beta$ is shifted in E such that it stays perpendicular to F^a_x, passes through $l(v_x)$ and admits $l(p_x)$ as tangent vector at $l(v_x)$. This shifted surface of rotation is denoted by $\mathcal{P}^l_1$. Hence there is a lift $\gamma'_{l(v_x),p_x}$ of β on $\mathcal{P}^l_1$ starting at $l(v_x)$ and having $l(p_x)$ as its initial F^a_x-component of velocity. However, there is another surface $\mathcal{P}^l_2$, say, passing through $-l(v_x)$ and to which $l(p_x)$ is tangent in $-l(v_x)$ such that $l(p_x)$ defines an orientation on $\mathcal{P}^l_2$ opposite to the one of $\mathcal{P}^l_1$, if $p_x \neq 0$. Hence there is another periodic lift $\gamma''_{-l(v_x),p_x}$ of β on $\mathcal{P}^l_2$ with $l(p_x)$ as F^a_x-component of its initial velocity. Thus the same basis $\{l(\frac{v_x}{p}), l(p_x)\}$ defines two periodic lifts $\gamma'_{l(v_x),p_x}$ and $\gamma''_{-l(v_x),p_x}$, called conjugate symplectic periodic lifts of β. These are isochromatic since both have the same momentum p.

Let $\tilde{C}^a_x$ be the collection of periodic lifts of β constructed as above for any basis in $\mathcal{B}^a_x$. By construction $Mp(F^a_x)$ operates transitively on $\tilde{C}^a_x$ and without fixed points, turning $\tilde{C}^a_x$ into a manifold diffeomorphic to $Mp(F^a_x)$.

Therefore we may reformulate:

Proposition 7 *There is a one-to-one correspondence between the collection $\tilde{C}^a_x$ of symplectic periodic lifts of β and the elements in $Mp(F^a_x)$. Thus $\tilde{C}^a_x$ inherits a manifold structure making $\tilde{C}^a_x$ diffeomorphic to $Mp(F^a_x)$. Moreover, for each $A \in Mp(F^a_x)$ the operator $U(A)$ characterises a lift of β in $\tilde{C}^a_x$ and vice versa.*

The Lie algebra $mp(F^a_x)$ of $Mp(F^a_x)$ is isomorphic to Q^a_x, the collection of all homogeneous polynomials in two variables p and q defined on $\mathbb{R} \cdot v_x \subset F^a_x$ with the Poisson bracket as Lie bracket (Guillemin, Sternberg, 1991). Given any $A \in Mp(F^a_x)$, let dU_A be the infinitesimal representation of U_A defined on Q^a_x. Therefore we have:

Theorem 8 *Given $\gamma \in \tilde{C}^a_x$, for each tangent vector $k \in T_\gamma\tilde{C}^a_x$ the operator $dU_A(k)$ corresponds to a quantization of a homogeneous quadratic polynomial and vice versa.*

Since Q^a_x and $sp(F^a_x)$ are isomorphic as Lie algebras, the representation dU of $sp(F^a_x)$ yields a quantization procedure. This is to say that the tangent bundle to the collection of lifts of β in $\tilde{C}^a_x$ visualises a quantization procedure of the Poisson algebra Q^a_x.

As a remark it should be emphasized that the fact that $Mp(2, \mathbb{R})$ forms a two-fold covering of the symplectic group $SL(2, \mathbb{R})$ implies the non-cloning theorem in quantum teleportation (Binz, Schempp, 1999, 2000c). Moreover, the metaplectic group introduces the aspect of undecidability into quantum physics (Binz, Schempp, 2000c).

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